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# A SURVEY OF THE RIEMANN ZETA FUNCTION WITH ITS APPLICATIONS

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## Abstract

This paper explicates the Riemann hypothesis and proves its validity; it explains why the non-trivial zeros of the Riemann zeta function $\zeta$ will always be on the critical line $\mathrm{Re}(s) = 1/2$ and not anywhere else on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$. Much exact calculations are presented, instead of approximations, for the sake of accuracy or precision, clarity and rigor.

## Contents



## 1. Introduction to the mysterious Non-Trivial Zeros of the Riemann Zeta Function $\zeta$

What is the role of the non-trivial zeros of the Riemann zeta function $\zeta$, which are mysterious and evidently not much understood?

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To understand what Riemann wanted to achieve with the non-trivial zeros, we need to understand the part played by the complex plane.

First, the terms in the Riemann zeta function $\zeta$:-

$$\zeta(s) = \sum_{n=1} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots \tag{1.1}$$

where $s$ is the complex number $1/2 + bi$

For the term $1/2^{1/2+bi}$ above, e.g., whether it would be positive or negative in value would depend on which part of the complex plane this term $1/2^{1/2+bi}$ would be found in, which depends on $2(n)$ and $b$ (it does not depend on $1/2 - 1/2$ and $2(n)$ only determine how far the term is from zero in the complex plane). This term could be in the positive half (wherein the term would have a positive value) or the negative half (wherein the term would have a negative value) of the complex plane. Thus, some of the terms in the Riemann zeta function $\zeta$ would have positive values while the rest have negative values (depending on the values of $n$ and $b$). The sum of the series in the Riemann zeta function $\zeta$ is calculated with a formula, e.g., the Riemann-Siegel formula, or, the Euler-Maclaurin summation formula.

Riemann evidently anticipated that there would be an equal, or, almost equal number of primes among the terms in the positive half and the negative half of the complex plane when there is a zero. In other words, he thought that the distribution of the primes would be statistically fair, the more terms are added to the Riemann zeta function $\zeta$, the fairer or "more equal" would be the distribution of the primes in the positive half and the negative half of the complex plane when there is a zero. (Compare: The tossing of a coin wherein the more tosses there are the "more equal" would be the number of heads and the number of tails.) That is, in the longer term, with more and more terms added to the Riemann zeta function $\zeta$, more or less 50% of the primes should be found in the positive half of the complex plane and the balance 50% should be found in the negative half of the complex plane, the more terms there are the fairer or "more equal" would be this distribution, when there is a zero.

A non-trivial zero indicates the point in the Riemann zeta function $\zeta$ wherein the total value of the positive terms equals the total value of the negative terms. There is an infinitude of such points, i.e., non-trivial zeros. Riemann evidently thought that for the case of a zero the number of primes found among the positive terms would be more or less equal to the number of primes found among the negative terms, which represents statistical fairness. It is evident that through a zero the order or pattern of the distribution of the primes could be observed.

Next, the error term in the following $J$ function for calculating the number of primes less than a given quantity:-

$$J(n) = Li(n) - \sum_{p} Li(n^{p}) - \log 2 + \int_{n}^{\infty} dt/(t(t^{2} - 1)\log t) \tag{1.2}$$

where the 1st term $Li(n)$ is generally referred to as the "principal term" and the 2nd term $\sum_{p} Li(n^{p})$ had been called the "periodic terms" by Riemann, $Li$ being the logarithmic integral

$\sum_{p} Li(n^{p})$, the secondary term of the function, the error term, represents the sum taken over all the non-trivial zeros of the Riemann zeta function $\zeta$. n here is a real number raised to the power of $p$, which is in this instance a complex number of the form $1/2 + bi$, for some real number $b$, $n^{1/2}$ being $\sqrt{n}$. If the Riemann hypothesis is true, for a given number $n$, when computing the values of $n^{p}$ for a number of different zeta zeros $p$, the numbers we obtain are scattered round the circumference of a circle of radius $\sqrt{n}$ in the complex plane, centered on zero, and are either in the positive half or negative half of the complex plane.

To evaluate $\sum_{p} Li(n^{p})$ each zeta zero has to be paired with its mirror image, i.e., complex conjugate, in the south half of the argument plane. These pairs have to be taken in ascending order of the positive imaginary parts as follows:-

zeta zero: $1/2 + 14.134725i$ & its complex conjugate: $1/2 - 14.134725i$

zeta zero: $1/2 + 21.022040i$ & its complex conjugate: $1/2 - 21.022040i$

zeta zero: $1/2 + 25.010858i$ & its complex conjugate: $1/2 - 25.010858i$

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(Note: The complex conjugates are all also zeros.)

If, e.g., we let $n = 100$, then the error term for $n = 100$ would be $\sum_p Li(100^p)$. To calculate this error term, we have to first raise 100 to the power of a long list of zeta zeros in ascending order of the positive imaginary parts (the 1st 3 zeta zeros are shown above), the longer the list of zeta zeros the better, e.g., 100,000 zeta zeros, in order to achieve the highest possible accuracy in the error term. Then we take the logarithmic integrals of the above powers (100,000 pairs of zeta zeros and their complex conjugates) and add them up, which is as follows:-

$$100^{1/2+14.134725i} + 100^{1/2-14.134725i}$$

$$+ 100^{1/2+21.022040i} + 100^{1/2-21.022040i}$$

$$+ 100^{1/2+25.010858i} + 100^{1/2-25.010858i}$$

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The imaginary parts of the zeta zeros would cancel out the imaginary parts of their complex conjugates, leaving behind their respective real parts. For example, for the 1st zeta zero $1/2 + 14.134725i$, its imaginary part $+14.134725i$ would cancel out the imaginary part $-14.134725i$ in its complex conjugate $1/2 - 14.134725i$, leaving behind only the real parts $100^{1/2}$ for each of them. That is, for $100^{1/2+14.134725\,i} + 100^{1/2-14.134725\,i}$, we only have to add together the logarithmic integral of $100^{1/2}$ (from $100^{1/2+14.134725i}$) and the logarithmic integral of $100^{1/2}$ (from $100^{1/2-14.134725i}$) to get the 1st term. The same is to be carried out for the next 99,999 powers in ascending order of the positive imaginary parts, giving altogether a total of 200,000 logarithmic integrals (of both the zeta zeros and their complex conjugates) to be added together to give the 100,000 terms. These terms have either positive or negative values, an equal or almost equal number of positive and negative values, which depend on whether they are in the positive or negative half of the complex plane, as is described above. The positive values and the negative values of these 100,000 terms are added together and should cancel out each other, slowly converging. The difference between the positive values and the negative values of these 100,000 terms constitutes the error term. (Note that the Riemann hypothesis asserts that the difference between the true number of primes $p(n)$ and the estimated number of primes $q(n)$ would be not much larger than $\sqrt{n}$ – not much larger than $\sqrt{100}$ ($\sqrt{100}$ is also expressed as $100^{1/2}$) in the above case. Like the case of tossing a coin wherein the statistical probability is that in the long run the number of heads would practically equal the number of tails, there should be equal or almost equal quantities of positive terms and negative terms, i.e., 50,000 or thereabout positive terms and 50,000 or thereabout negative terms, which would be statistically fair, the discrepancy if any being the error.

All this is evidently a laborious process, though the ingenuity of the ideas behind the Riemann hypothesis should be acknowledged.

It may be compared to the sieve of Eratosthenes, which could find the exact number of primes less than a given quantity without any error at all.

## 2. The Riemann Hypothesis, the Prime Number Theorem and Prime Counting

The Riemann hypothesis is an important outstanding problem in number theory as its validity will affirm the manner of the distribution of the prime numbers. It posits that all the non-trivial zeros of the zeta function $\zeta$ lie on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ at the critical line $\mathrm{Re}(s) = 1/2$. The important question is whether there would be zeros appearing at other locations on this critical strip, e.g., at $\mathrm{Re}(s) = 1/4, 1/3, 3/4,$ or, $4/5,$ etc., which would disprove the Riemann hypothesis. We probe into this here.

According to the precepts of fractal geometry, phenomena which appear random when viewed en masse display some orderliness and pattern which could be regarded as a fractal characteristic. For instance, the prime numbers are very random and haphazard entities, yet, when viewed en masse they display a regularity in the way they thin out, whereby it is affirmed that the number of primes not exceeding a given natural number $n$ is approximately $n/\log n,$ in the sense that the ratio of the number of such primes to $n/\log n$ eventually approaches 1 as $n$ becomes larger and larger, $\log n$ being the natural logarithm (to the base $e$) of $n$ (vide the prime number theorem proved in 1896 by Hadamard and de la Vallee-Poussin). In other words, the prime number theorem, which is the direct outcome of the Riemann hypothesis, states that the limit of the quotient of the 2 functions $\pi(n)$ and $n/\log n$ as $n$ approaches infinity is 1, which is expressed by the formula:-

$$\lim_{n\to\infty} \pi(n)/(n/\log n) = 1 \tag{2.1}$$

the larger the number $n$ is, the better is the approximation of the quantity of primes, as is implied by the above formula where $\pi(n)$ is the prime counting function ($\pi$ here is not the $\pi$ which is the constant 3.142 used to compute perimeters and areas of circles, but is only a convenient symbol adopted to denote the prime counting function)

All this is in spite of the fact that the primes are scarcer and scarcer as $n$ is larger and larger.

The prime number theorem could in fact be regarded as a weaker version of the Riemann hypothesis which posits that all the non-trivial zeros of the zeta function $\zeta$ on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ would be at the critical line $\mathrm{Re}(s) = 1/2.$ For a better understanding of the close connection between the prime number theorem and the Riemann hypothesis, it should be noted that Hadamard and de la Vallee Poussin had in 1896 independently proven that none of the non-trivial zeros lie on the very edge of the critical strip, on the lines $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) = 1$ – this was enough for deducing the prime number theorem. The locations of these non-trivial zeros on the critical strip could be described by a complex number $1/2 + bi$ where the real part is $1/2$ and $i$ represents the square root of $-1$. It had already been proven that there is an infinitude of non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$. The moot question is whether there would be any zeros off the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) = 1/4, 1/3, 3/4,$ or, $4/5,$ etc., the presence of any of which would disprove the Riemann hypothesis. So far, no such "off-the-critical-line" zeros has been found.

The validity of the Riemann hypothesis would evidently imply the validity of the prime number theorem (which as described above is the offspring and weaker version of the Riemann hypothesis) though the validity of the prime number theorem does not imply the former. Nevertheless, both of them have one thing in common in that they are both concerned with the estimate of the quantity of primes less than a given number, with the Riemann hypothesis positing a more exact estimate of the quantity of primes less than a given number. But, on the other hand, what would be the result if the Riemann hypothesis were false? We will come back to this later.

Meanwhile, more about the non-trivial zeros of the zeta function $\zeta(s)$ defined by a power series shown below:-

$$\zeta(s) = \sum_{n=1} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \dots$$

At the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ all the non-trivial zeros would be found on an oscillatory sine-like wave which oscillates in spirals, there being an infinitude of these spirals (representing the so-called complex plane). All the properties of the prime counting function $\pi(n)$ are in some way coded in the properties of the zeta function $\zeta$, evidently resulting in the primes and the non-trivial zeros being some sort of mirror images of one another – the regularity in the way the primes progressively thin out and the progressively better approximation of the quantity of primes towards infinity by the prime counting function $\pi(n)$ mirror or reflect the regularity in the way the non-trivial zeros of the zeta function ζ line up at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, the non-trivial zeros becoming progressively closer together there, with no zeros appearing anywhere else on the critical strip, and, all this has been found to be true for the 1st $10^{13}$ non-trivial zeros.

Riemann had posited that the margin of error in the estimate of the quantity of primes less than a given number with the prime counting function $\pi(n)$ could be eliminated by utilizing the following $J$ function which is a step function involving the non-trivial zeros expressed in terms of the zeta function ζ, which has been shown to be effective (2 steps are involved here – first, the prime counting function $\pi(n)$ is expressed in terms of the $J(n)$ function, then the $J(n)$ function is expressed in terms of the zeta function ζ, with the $J(n)$ function forming the link between the counting of the prime counting function $\pi(n)$ and the measuring (involving analysis and calculus) of the zeta function ζ, which would result in the properties of the prime counting function $\pi(n)$ somehow encoded in the properties of the zeta function ζ):-

$$J(n) - Li(n) - \sum_{p} Li(n^{p}) - \log 2 + \int_{n}^{\infty} dt/(t(t^{2} - 1)\log t)$$

where the 1st term $Li(n)$ is generally referred to as the “principal term” and the 2nd term $\sum_{p} Li(n^{p})$ had been called the “periodic terms” by Riemann, $Li$ being the logarithmic integral.

The above formula may look intimidating but is actually not. The 3rd term $\log 2$ is a number which is 0.69314718055994… while the 4th term $\int_{n}^{\infty} dt/(t(t^{2} - 1)\log t)$ which is an integral representing the area under the curve of a certain function from the argument all the way out to infinity can only have a maximum value of 0.1400101011432869…. Since these 2 terms taken together (and minding the signs) are limited to the range from – 0.6931… to – 0.5531…, and since the prime counting function $\pi(n)$ deals with really large quantities up to millions and trillions they are much inconsequential and can be safely ignored. The 1st term or principal term $Li(n)$, where $n$ is a real number, should also be not much of a problem as its value can be obtained from a book of mathematical tables or computed by some math software package such as Mathematica or Maple. However, special attention should be given to the 2nd term $\sum_{p} Li(n^{p})$ which concerns the sum of the non-trivial zeros of the zeta function ζ ($p$ in this 2nd term is a "rho", which is the 17th letter of the Greek alphabet, and it means "root" – a root is a non-trivial zero of the Riemann zeta function ζ – a root here is a solution or value of an unknown of an equation which could be factorized). Riemann had evidently called the 2nd term "periodic terms" as the components there vary irregularly.

The prime number theorem asserts that $\pi(n) \sim n/Li(n)$ (technically $Li(n) = \int_{2}^{n} dx/\log(x))$ which also implies the weaker result that $\pi(n) \sim n/\log n.$ However, with $Li(n)$ the prime count estimate would have a margin of error. The Riemann hypothesis asserts that the difference between the true number of primes $p(n)$ and the estimated number of primes $q(n)$ would be not much larger than $\sqrt{n}$. With the above $J(n)$ function we could eliminate this margin of error and obtain an exact estimate of the quantity of primes less than a given number:-

$$J(n) = \text{ exact quantity of primes less than a given number}$$

Since the 3rd and 4th terms of the $J(n)$ function are inconsequential and can be safely ignored, as is described above, deducting the 2nd term from the 1st term should be sufficient:-

$$J(n) = Li(n) - \sum_{p} Li(n^{p}) = \text{ exact quantity of primes less than a given number}$$

The above in a nutshell shows the intimate relationship between the primes and the non-trivial zeros of the zeta function $\zeta$, the primes and the non-trivial zeros being some sort of mirror images of one another as is described above, with the distribution of the non-trivial zeros being regarded as the music of the primes by mathematicians.

We return to the question of the consequence of the falsity of the Riemann hypothesis. Let's here assume that the Riemann hypothesis is false, i.e., there are also zeros found off the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) = 1/4, 1/3, 3/4,$ or, $4/5,$ etc., and see the consequence. What would be the significant implication of this assumption? The falsity of the Riemann hypothesis would imply that the distribution of the zeros of the zeta function $\zeta$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ has lost the regularity of pattern which is characteristic of the non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ and which is described above, and is now disorderly and irregular. This would in turn imply that the distribution of the primes is also similarly disorderly and irregular since the primes and the non-trivial zeros of the zeta function $\zeta$ are intimately linked and are some sort of mirror images of one another – any changes in one of them would be reflected in the other on account of their intimate link – note that the zeta function $\zeta$ has the property of prime sieving (compare: sieve of Eratosthenes; see Appendix A below) encoded within it, the properties of the prime counting function $\pi(n)$ being somehow encoded in the properties of the zeta function $\zeta$, so that if the zeros generated were disorderly and irregular it would mean that the distribution of the primes were also similarly disorderly and irregular – the characteristic of the primes on the input side of the function determines the characteristic of the zeros on the output side of the function (i.e., the distribution of the primes determines the distribution of the zeros, so that from a study of the distribution of the zeros the distribution of the primes could be deduced and vice versa), which is expected for a function. The overall result would be that the more orderly the distribution of the zeros is the more orderly would be the corresponding distribution of the primes, the more disorderly the distribution of the zeros is the more disorderly would be the corresponding distribution of the primes, and, vice versa. But, according to the prime number theorem, or, prime counting function $\pi(n)$, which is closely connected with the Riemann hypothesis itself being an offspring and weaker version of it as is described above, there is instead actually a regularity in the way the primes thin out, with the prime counting function $\pi(n)$ even providing a progressively better estimate of the quantity of primes towards infinity - this progressively better estimate would not be possible if the primes behave really badly and are really highly disorderly and irregular – there is no such really great disorder or irregularity among the primes, a state of affair which is evidently affirmed by the fact that the corresponding non-trivial zeros at the critical line $\text{Re}(s) = 1/2$ on the critical strip between $\text{Re}(s) = 0$ and $\text{Re}(s) = 1$ display regularity in the way they line up at the critical line $\text{Re}(s) = 1/2$, the non-trivial zeros becoming progressively closer together there with no zeros appearing anywhere else on the critical strip (all of which has been found to be true for the 1st $10^{13}$ non-trivial zeros – an important point to note is that though the non-trivial zeros at the critical line $\text{Re}(s) = 1/2$ become more and more closely packed together the farther along we move up this critical line while the primes occur farther and farther along the number line, the density of the one is approximately the reciprocal of the density of the other wherein the complementariness, regularity, symmetry is evident), this regularity of the distribution of the non-trivial zeros mirroring the regularity of the distribution of the primes as is explained above. Our assumption of the falsity of the Riemann hypothesis has thus resulted in a contradiction of the actual distribution of the primes and the actual distribution of the corresponding non-trivial zeros at the critical line $\text{Re}(s) = 1/2$ on the critical strip between $\text{Re}(s) = 0$ and $\text{Re}(s) = 1$. If our assumption that the Riemann hypothesis is false is correct, the prime number theorem would be false as there would be great disorder and irregularity among the primes with no regularity in the way the primes thin out and without the prime counting function $\pi(n)$ providing a progressively better estimate of the quantity of primes towards infinity (this progressively better estimate of the quantity of primes actually implies some regularity in the distribution of the primes). However, as is explained just above the prime number theorem is not false; it had in fact been proven through both non-elementary methods (by Hadamard and de la Vallee Poussin) and elementary methods (by Erdos and Selberg later) and is indubitably true. Therefore, our assumption of the falsehood of the Riemann hypothesis is at fault. This implies that the Riemann hypothesis is true, since the hypothesis cannot be false; this is a proof by contradiction which may be interesting but may not be viewed a very strong or convincing proof as the reasoning may be too subtle to be fully grasped (even possibly causing misunderstanding) and make great sense (though, at least, it shows the close connection between the Riemann hypothesis and the prime number theorem); the very strong proof will be presented below. The close link between the Riemann hypothesis and the prime number theorem is thus evident.

**Appendix A**

**The Riemann Zeta Function and the Prime Numbers**

The Riemann zeta function $\zeta(s)$, shown below, is the sum over all natural numbers $n$:

$$\zeta(s) = \sum_{n=1} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \dots$$

The function could also be written in the following way (using Euler's product formula) showing its connection with the prime numbers:

$$\zeta(s) = \prod_{p\, prime} p^s/p^s - 1 = 2^s/2^s - 1 \times 3^s/3^s - 1 \times 5^s/5^s - 1 \times 7^s/7^s - 1 \times \dots \qquad \text{(A.1)}$$

where the product is over the consecutive prime numbers $p$, providing the first hint that the Riemann zeta function $\zeta(s)$ is closely linked to the prime numbers.

## 3. The Non-Trivial Zeros will always be on the Critical Line $\mathbf{Re}(s) = 1/2$

The Riemann hypothesis posits that all the non-trivial zeros of the zeta function $\zeta$ (shown below) on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ will always be at the critical line $\mathrm{Re}(s) = 1/2$ :-

$$\zeta(s) = \sum_{n=1} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \dots$$

This has been found to be true for the 1st $10^{13}$ non-trivial zeros. The locations of these non-trivial zeros on the critical strip are described by a complex number $1/2 + bi$ where the real part is $1/2$ and $i$ represents the square root of – 1. It should be noted that the mathematical operations and logic of the complex numbers $a + bi$, where $a$ and $b$ are real numbers and $i$ is the imaginary number square root of – 1, are practically the same as for the real numbers and are even more versatile. For the zeta function $\zeta(s)$ shown above to be zero, its series would have to have both the positive terms and negative terms cancelling each other out, though the positive or "+" signs in the series may indicate positive values only which is misleading. The sum of this series is calculated with a formula, e.g., the Riemann-Siegel formula, or, the Euler-Maclaurin summation formula. Is there a possibility of any non-trivial zeros being off the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) = 1/4, 1/3, 3/4,$ or, $4/5,$ etc., the presence of any of which would disprove the Riemann hypothesis?

It had already been proven that there will not be zeros at $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1.$ The 1st $10^{13}$ non-trivial zeros are found only at the critical line $\mathrm{Re}(s) = 1/2.$ Nature appears to dictate that these zeros must appear only at $\mathrm{Re}(s) = 1/2,$ exactly mid-way in the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ where in the symmetry is perfect. "$1/2$" in the complex number $1/2 + bi,$ which is "square root", also appears to be compatible with and work fine with "$i$", which is "square root of –1" – both of them are square roots. $1/2 + bi$ has what is called a complex conjugate $1/2 - bi$ so that when $1/2 + bi$ and $1/2 - bi$ are added together the terms $bi$ in both $1/2 + bi$ and $1/2 - bi$ will cancel out one another – in this way the troublesome $i$ which does not actually make mathematical sense will be out of the way. $1/2$ is also the reciprocal of the smallest prime and the smallest even number 2, which is significant. But there is a much more compelling reason why all the non-trivial zeros must lie on the critical line $\mathrm{Re}(s) = 1/2$ and it is due to some important similarity to Fermat's last theorem.

The reasoning which follows will be reasoning by analogy, with Fermat's last theorem taken as an analogue, whereby the reasoning is that if it is true for Fermat's last theorem it will be true for something comparable in the Riemann hypothesis.

**Fermat's Last Theorem**

2 square numbers can be added together to form a 3rd square, e.g., $3^2 + 4^2 = 5^2$ and $5^2 + 12^5 = 13^2$. Fermat's last theorem states that for any 4 whole numbers $x, y, z$ and $n$, there are no solutions to the equation $x^n + y^n = z^n$ when $n > 2$. (Such an equation involving whole numbers is known as a Diophantine equation.)

Fermat's last theorem is connected with Pythagoras' theorem which states that if $x, y, z$ represent the lengths of the 3 sides of a right-angled triangle, $x$ and $y$ being the adjacent sides and $z$ being the hypotenuse (the side opposite the right angle), then $x^2 + y^2 = z^2$. (Here $x, y, z$ need not and may not be whole numbers, i.e., this equation needs not be a Diophantine equation.)

To put it another way, according to Fermat's last theorem, the following Diophantine equation which has power $n = 2$ is the only Diophantine equation with zeros or solutions (zeros and solutions are synonymous):-

$$x^2 + y^2 = z^2 \tag{3.1}$$

The following is a partial list of Diophantine equations with their zeros:-

[1] $3^2 + 4^2 = 5^2$

$3^2 + 4^2 - 5^2 = 0$

[2] $5^2 + 12^2 = 13^3$

$5^2 + 12^2 - 13^2 = 0$

[3] $7^2 + 24^2 = 25^2$

$7^2 + 24^2 - 25^2 = 0$

[4] $8^2 + 15^2 = 25^2$

$8^2 + 15 - 17^2 = 0$

[5] $9^2 + 40^2 = 41^2$

$9^2 + 40^2 - 41^2 = 0$

[6] $11^2 + 60^2 = 61^2$

$11^2 + 60^2 - 61^2 = 0$

[7] $12^2 + 35^2 = 37^2$

$12^2 + 35^2 - 37^2 = 0$

[8] $13^2 + 84^2 = 85^2$

$13^2 + 84^2 - 85^2 = 0$

[9] $16^2 + 63^2 = 65^2$

$16^2 + 63^2 - 65^2 = 0$

[10] $20^2 + 21^2 = 29^2$

$20^2 + 21^2 - 29^2 = 0$

[11] $28^2 + 45^2 = 53^2$

$28^2 + 45^2 - 53^2 = 0$

[12] $33^2 + 56^2 = 65^2$

$33^2 + 56^2 - 65^2 = 0$

[13] $36^2 + 77^2 = 85^2$

$36^2 + 77^2 - 85^2 = 0$

[14] $39^2 + 80 = 89^2$

$39 + 80^2 - 89^2 = 0$

[15] $48^2 + 55^2 = 73^3$

$48^2 + 55^2 - 73^2 = 0$

[16] $65^2 + 72^2 = 97^2$

$$65^{21} + 72^2 - 97^2 = 0$$

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There is some important similarity between Fermat's last theorem and the Riemann hypothesis, both of them being involved with series, which will be dealt with.

Like the series of the Riemann zeta function $\zeta(1/2)$, the above Diophantine equations (a few equations with terms that are duplicative are omitted) could be turned into a long series (in fact, an infinitely long series like the series of the Riemann zeta function $\zeta(1/2)$) of positive and negative terms which give a zero, by adding them together as follows:-

$$3^2 + 4^2 - 5^2 + 7^2 + 24^2 - 25^2 + 8^2 + 15^2 - 17^2 + 9^2 + 40^2 - 41^2 + 11^2 + 60^2$$
$$- 61^2 + 12^2 + 35^2 - 37^2 + 13^2 + 84^2 - 85^2 + 20^2 + 21^2 - 29^2 + 28^2 + 45^2$$
$$- 53^2 + 39^2 + 80^2 - 89^2 + 48^2 + 55^2 - 73^2 + 65^2 + 72^2 - 97^2 = 0$$

or, with the same terms re-arranged in numerically ascending order, as follows:-

$$3^2 + 4^2 - 5^2 + 7^2 + 8^2 + 9^2 + 11^2 + 12^2 + 13^2 + 15^2 - 17^2 + 20^2 + 21^2 + 24^2$$
$$- 25^2 + 28^2 - 29^2 + 35^2 - 37^2 + 39^2 + 40^2 - 41^2 + 45^2 + 48^2$$
$$- 53^2 + 55^2 + 60^2 - 61^2 + 65^2 + 72^2 - 73^2 + 80^2$$
$$+ 84^2 - 85^2 - 89^2 - 97^2 = 0.$$

The long series above show the uncanny likeness between Fermat's last theorem and the Riemann hypothesis.

In the above list of Diophantine equations, the regularity of the powers $n = 2$ is evident. If any of these equations are raised to powers $n > 2$ the regularity will be lost, as is explained below.

We will explain why there are no zeros for the Riemann zeta function $\zeta$ for $s < 1/2$ and $s > 1/2$ by bringing up the common underlying principle behind it and Fermat's last theorem, $s = 1/2$ being evidently the optimum or equilibrium power, the only power which brings equilibrium, balance or regularity and thereby the zeros to the Riemann zeta function $\zeta$.

For the case for $x^n + y^n = z^n$ above for Fermat's last theorem which asserts that there are no solutions for $n > 2$, we first get some mathematical insight on why there are no solutions for $n > 2$. We commence by selecting example [1] from the list of Diophantine equations above, which has the smallest odd prime number 3 and the smallest composite number 4 (which is the square of the smallest prime number 2) in the series on the left, i.e., the smallest Diophantine equation which has 2 as the power, for illustration:-

$$3^2 + 4^2 = 5^2$$

If the power of 2 in the series on the left above were increased to 3, which is the next, consecutive whole number, e.g., the sum on the right would not be a whole number anymore, which is in accordance with Fermat's last theorem:-

$$3^3 + 4^3 = 4.49795^3$$

The regularity of the power of 2 is now lost. And this is for the smallest Diophantine equation which initially had 2 as the power. For the larger Diophantine equations with initial powers of 2 the irregularity after increasing their powers to 3, which is the next, consecutive whole number, or, higher powers, could be expected to be worse.

Next we bring up the values of, say, 100, of consecutive whole number powers $n$, say, 2 to 5, this quantity 100 being representative of the terms of the equation $x^n + y^n = z^n$ as per Fermat's last theorem, to explain the reason for this irregularity, which is as follows:-

[1] $100^2 = 10{,}000$ (The terms of the series of Fermat's last theorem fall under this category. All zeros will be found under this category only.)

[2] $100^3 = 1{,}000{,}000$ (This quantity represents an increase of 9,900% compared to [1] above while the increase in power from $n = 2$ to $n = 3$ is only 50%.)

[3] $100^4 = 100{,}000{,}000$ (This quantity represents an increase of 999,900% compared to [1] above while the increase in power from $n = 2$ to $n = 4$ is only 100%.)

[4] $100^5 = 10{,}000{,}000{,}000$ (This quantity represents an increase of 99,999,900% compared to [1] above while the increase in power from $n = 2$ to $n = 5$ is only 150%.)

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The quantities from the consecutive whole number powers $n > 2$ above increase progressively compared to [1], the larger the power $n$ is the larger the percentage of increase in the quantity is. The increases in the respective quantities and powers are also disproportionate when compared to one another, with the increases in the respective quantities being evidently much too quick. All this implies that the equilibrium, balance or regularity of $x^n + y^n = z^n$ when $n = 2$ as per Fermat's last theorem cannot be maintained when $n > 2$, when disproportionateness between the increases in the respective quantities and powers sets in as is described above, as the increase in quantity is too quick, and, when $n < 2$, e.g., $n = 5/4, 3/2, 7/4,$ etc., as the increase in quantity is too slow as could be extrapolated from the above example. (Refer to Appendix B below for analogous example.) For Fermat's last theorem, $n = 2$ can be regarded as the optimum or equilibrium power, the only power wherein $x^n + y^n = z^n$ is possible. There is also the question of the easier solubility of equations with whole number powers $n = 2$ as compared to equations with powers $n > 2$, e.g., $n = 3, 4, 5,$ etc., and $n > 2$, e.g., $n = 5/4, 3/2, 7/4,$ etc., which will be explained below.

For the case of the Riemann zeta function $\zeta$ wherein there are no zeros for powers $s < 1/2$ and $s > 1/2$ we bring up the values of the reciprocals of, say, 100, with consecutive fractional powers $s$, say, $1/2$ to $1/5$ these reciprocals being representative of the terms of the Riemann zeta function $\zeta$, to explain the reason for the irregularity for powers $s < 1/2$ and $s > 1/2$, which is as follows:-

[1] $1/100^{1/2} = 1/10 = 0.100$ (The terms of the series of the Riemann zeta function $\zeta$ as per the Riemann hypothesis fall under this category. $10^{13}$ zeros have been found under this category only.)

[2] $1/100^{1/3} = 1/4.6416 = 0.215$ (This quantity represents an increase of 115% compared to [1] above while the decrease in power from $s = 1/2$ to $s = 1/3$ is only 33.33%.)

[3] $1/100^{1/4} = 1/3.1623 = 0.316$ (This quantity represents an increase of 216% compared to [1] above while the decrease in power from $s = 1/2$ to $s = 1/4$ is only 50%.)

[4] $1/100^{1/5} = 1/25119 = 0.398$ (This quantity represents an increase of 298% compared to [1] above while the decrease in power from $s = 1/2$ to $s = 1/5$ is only 60%.)

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As can be seen above, the smaller the power of the reciprocal/denominator is the larger will be the result after division with 1 (or, the larger the power of the reciprocal/denominator is the smaller will be the result after division with 1). The quantities from the reciprocals with consecutive fractional powers $s = 1/2$ above increase progressively compared to [1], the smaller the power $s$ is the larger the percentage of increase in the quantity is, the increases in the quantities being similar to the case above for Fermat's last theorem-this indicates a similarity between Fermat's last theorem and the Riemann hypothesis. The increases in the respective quantities and the decreases in the respective powers are also disproportionate when compared to one another, with the increases in the respective quantities being evidently much too quick, which is similar to the case above for Fermat's last theorem – this indicates another similarity between Fermat's last theorem and the Riemann hypothesis. All this implies that the equilibrium, balance or regularity of the Riemann zeta function $\zeta$ when $s = 1/2$ cannot be maintained when $s = 1/2$ when disproportionateness between the increases and decreases in the respective quantities and powers sets in as is described above, as the increase in quantity is too quick, and, when $s = 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., as the increase in quantity is too slow as could be extrapolated from the above example. (Refer to Appendix B below for full details.) For these reciprocals, $s = 1/2$ can be regarded as the optimum or equilibrium power, the only power wherein zeros for the Riemann zeta function $\zeta$ are possible. Like the case for Fermat's last theorem above, there is also the question of the easier solubility of equations with fractional powers $s = 1/2$ as compared to equations with fractional powers $s = 1/2$, e.g., $s = 1/3, 1/4, 1/5$, etc., and $s = 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., which will be explained below.

The following list of the 1st 10 terms of the series of the Riemann zeta function $\zeta$ with consecutive fractional powers $s = 1/2$ also shows that the sums with smaller powers increase progressively, i.e., the smaller the power $s$ is the larger the percentage of increase in the quantity is:-

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + 1/6^{1/2} + 1/7^{1/2} + 1/8^{1/2} + 1/9^{1/2} + 1/10^{1/2} + \ldots = 5.03$ (The Riemann hypothesis asserts that all zeros will be found in this series only.)

[2] $\zeta(1/3) = 1 + 1/2^{1/3} + 1/3^{1/3} + 1/4^{1/3} + 1/5^{1/3} + 1/6^{1/3} + 1/7^{1/3} + 1/8^{1/3} + 1/9^{1/3} + 1/10^{1/3} + \ldots = 6.20$ (The sum 6.20 here represents an increase of 23.26% compared to the sum 5.03 in [1] above while the percentage of decrease in power from $s = 1/2$ to $s = 1/3$ is 33.33%.)

[3] $\zeta(1/4) = 1 + 1/2^{1/4} + 1/3^{1/4} + 1/4^{1/4} + 1/5^{1/4} + 1/6^{1/4} + 1/7^{1/4} + 1/8^{1/4} + 1/9^{1/4} + 1/10^{1/4} + \ldots = 6.97$ (The sum 6.97 here represents an increase of 38.57% compared to the sum 5.03 in [1] above while the percentage of decrease in power from $s = 1/2$ to $s = 1/4$ is 50%.)

[4] $\zeta(1/5) = 1 + 1/2^{1/5} + 1/3^{1/5} + 1/4^{1/5} + 1/5^{1/5} + 1/6^{1/5} + 1/7^{1/5} + 1/8^{1/5} + 1/9^{1/5} + 1/10^{1/5} + \ldots = 7.46$ (The sum 7.46 here represents an increase of 48.31% compared to the sum 5.03 in [1] above while the percentage of

decrease in power from $s = 1/2$ to $s = 1/5$ is 60%.)

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**Note:** Though the respective percentages of increase in quantity above, namely, 23.26%, 38.57% and 48.31%, are disproportionate with and lower than the respective percentages of decrease in power, namely, 33.33%, 50% and 60%, at a later stage when there are more and more terms in the series, there being an infinitude of terms, when the sums get larger and larger, the percentages of increase in quantity will all be infinitely higher than the percentages of decrease in power, as is evident from the tabulation below. The same will apply for the quantities when the powers $s = 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., as could be extrapolated from the above list (and evident from Appendix C below).

(The series of the Riemann zeta function $\zeta$ with powers $s > 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., will have sums which are all smaller than the sums shown in the above list for powers $s \leq 1/2$ as could be extrapolated from the above list. For the largest power in the critical strip $s = 1$, which has no zeros, the sum of the 1st 10 terms is a mere 2.93. Refer to Appendix B below for analogous example.)

It is clear from all the above that when the sum of the series in the Riemann zeta function $\zeta$ increases too quickly as is the case when the powers $s < 1/2$, when disproportionateness between the increases and decreases in the respective quantities and powers sets in as is described above, or, too slowly as is the case when the powers $s > 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., as could be extrapolated from the above list, the equilibrium, balance or regularity will be lost and there will not be zeros. (Refer to Appendix B below for analogous example.) As is in the case of Fermat's last theorem wherein all the zeros will be at the optimum or equilibrium power $n = 2$ only, all the zeros of the Riemann zeta function $\zeta$ will be at the optimum or equilibrium powers $= 1/2$ only. (The analogue of this optimum or equilibrium power could be that of a shirt or pants that exactly fits a person, e.g., size $A$ could be too small for the person, size $C$ too large, while size $B$ fits just fine.) At least $10^{13}$ zeros have been found at $s = 1/2$ while none has been found for $s < 1/2$ and $s > 1/2$.

An important point will be added here. If more and more terms are added to the series in the list of the sums of the Riemann zeta function $\zeta$ above where the consecutive fractional powers $s \leq 1/2$, which presently have 10 terms each, the differences in the sums between that for powers $= 1/2$ and that for powers $s < 1/2$, e.g., $s = 1/3, 1/4, 1/5$, etc., and, that for power $s = 1/2$ and that for powers $s > 1/2$, e.g., $s = 3/4, 4/5, 5/6$, etc., will be greater and greater, i.e., the differences between these sums will be more pronounced the more terms are added to the series. We can see this point by comparing, e.g., the sums of the 1st 5 terms of the Riemann zeta function $\zeta$ for consecutive fractional powers $s \leq 1/2$ and the sums of the 1st 10 terms of the Riemann zeta function $\zeta$ for consecutive fractional powers $s \leq 1/2$, which is as follows, and extrapolating from there:-

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + \ldots = 3.24$ (The Riemann hypothesis asserts that all zeros will be found in this series only.)

[2] $\zeta(1/3) = 1 + 1/2^{1/3} + 1/3^{1/3} + 1/4^{1/3} + 1/5^{1/3} + \ldots = 3.69$ (The sum 3.69 here represents an increase of **13.89%** (the increase here is **23.26%** for the 1st 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

[3] $\zeta(1/4) = 1 + 1/2^{1/4} + 1/3^{1/4} + 1/4^{1/4} + 1/5^{1/4} + \ldots = 3.98$ (The sum 3.98 here represents an increase of **22.84%** (the increase here is **38.57%** for the 1st 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

[4] $\zeta(1/5) = 1 + 1/2^{1/5} + 1/3^{1/5} + 1/4^{1/5} + 1/5^{1/5} + \ldots = 4.15$ (The sum 4.15 here represents an increase of **28.09%** (the increase here is **48.31%** for the 1st 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

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The tabulation below of the above-mentioned percentage increases for the sums for the 1st 2 terms to the 1st 10 terms for $\zeta(1/3)$, $\zeta(1/4)$ and $\zeta(1/5)$ will provide a clearer picture:-

| | 1st. 2 Terms | 1st. 3 Terms | 1st. 4 Terms | 1st. 5 Terms | 1st. 6 Terms | 1st. 7 Terms | 1st. 8 Terms | 1st. 9 Terms | 1st. 10 Terms | 1st. 11 Terms … |
|---|---|---|---|---|---|---|---|---|---|---|
| [1]$\zeta(1/2)$ | – | – | – | – | – | – | – | – | – | – |
| [2]$\zeta(1/3)$ | 4.68% | 8.30% | 11.47% | **13.89%** | 16.16% | 18.11% | 20.09% | 21.87% | **23.26%** | To Be Extrapolated |
| [3]$\zeta(1/4)$ | 7.60% | 13.54% | 18.28% | **22.84%** | 26.30% | 29.78% | 32.88% | 35.88% | **38.57%** | To Be Extrapolated |
| [4]$\zeta(1/5)$ | 9.36% | 16.59% | 22.94% | **28.09%** | 32.88% | 37.22% | 41.32% | 45.01% | **48.31%** | To Be Extrapolated |

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It is evident that the percentage increases shown above will go up in value continuously to infinity with the infinitude of the terms of the Riemann zeta function $\zeta$. All this indicates more and more bad news for the solubility of the Riemann zeta function $\zeta$ for powers $s < 1/2$, and, $s > 1/2$ (as could be extrapolated from the above; refer to Appendix B and Appendix C (which provides an example) below) when there are more and more terms in the Riemann zeta function $\zeta$, i.e., for powers $s < 1/2$ and $s > 1/2$, the more terms there are in the Riemann zeta function $\zeta$ the less soluble it will be. This is a serious irregularity and is another reason why there are no zeros for the Riemann zeta function $\zeta$ for powers $s < 1/2$ and $s > 1/2$.

The similarity between the Riemann hypothesis and Fermat's last theorem is striking – they each have an optimum or equilibrium power which is the only power where in zeros are possible – $s = 1/2$ in the case of the Riemann hypothesis and $n = 2$ in the case of Fermat's last theorem, powers which are all solely responsible for all the zeros. The fact that all these optimum or equilibrium powers are either square root ($s = 1/2$ for the Riemann hypothesis) or square ($n = 2$ for Fermat's last theorem) is significant as they seem some sort of images of 2 which is the smallest prime number and the smallest even number. $s = 1/2$ is the largest root among the roots with 1 as the numerator. As such $s = 1/2$ as a fractional power with 1 as the numerator gives the largest result as compared to the fractional powers with 1 as the numerator $s < 1/2$, e.g., $s = 1/3, 1/4, 1/5$, etc. (but this largest result brings the *smallest increase in quantity* as compared to the results of the fractional powers with 1 as the numerator $s < 1/2$, e.g., $s = 1/3, 1/4, 1/5$, etc., when divided by 1, e.g., $s = 1/2^{1/2} < 1/2^{1/3} < 1/2^{1/4} < 1/2^{1/5}$, etc. – *this is an important similarity to the case for $n = 2$ described below*) – equations with fractional powers $s = 1/2$ would evidently be easier to solve than equations with fractional powers $s < 1/2$ (e.g., in a computation $s = 1/2$ needs only 1 rooting step while $s = 1/5$ needs 4 rooting steps) and $s > 1/2$, e.g., $s = 2/3, 3/4, 4/5$, etc. (e.g., in a computation $s = 1/2$ needs only 1 rooting step, while $s = 4/5$ needs 7 steps – 3 squaring steps for $s = 4$ and 4 rooting steps for $s = 1/5$). $n = 2$ is the smallest whole number power which brings an increase in quantity. As such $n = 2$ is the whole number power which brings the *smallest increase in quantity* as compared to the whole number powers $n > 2$, e.g., $n = 3, 4, 5$, etc., for instance $2^2 < 2^3 < 2^4 < 2^5$, etc. – equations with whole number powers $n = 2$ would evidently be easier to solve than equations with powers $n > 2$ (with general equations with powers $n = 5$ having been proven unsolvable – $n = 2$ needs only 1 squaring step while $n = 5$ needs 4 squaring steps) and $n < 2$, e.g., $n = 5/4, 3/2, 7/4$, etc. (e.g., in a computation $n = 2$ needs only 1 squaring step, while $n = 7/4$ needs 9 steps – 6 squaring steps for $n = 7$ and 3 rooting steps for $n = 1/4$). $n = 2$ and its reciprocals $s = 1/2$ are the opposite of one another but despite this there appears to be complementariness and symmetry between them, as can be seen in the cases of Fermat's last theorem and the Riemann hypothesis which involve optimum or equilibrium powers $n = 2$ and its reciprocal $s = 1/2$, the only powers wherein zeros are possible for each of them. $n = 2$ and its reciprocal $s = 1/2$ are evidently important quantities which may be comparable to $\pi$ (3.14159265) or $e$ (2.71828).

It can be seen that the Riemann hypothesis is the analogue of Fermat's last theorem, which implies its validity.

Thus, for the Riemann zeta function $\zeta$, $s = 1/2$ is the optimum or equilibrium power wherein there will be zeros. There will be no zeros in the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ for $s < 1/2$ and $s > 1/2$ because if $s < 1/2$ the sum of the series in the zeta function $\zeta$ increases too quickly when more and more terms are added to the series and if $s > 1/2$ the sum of the series in the zeta function $\zeta$ increases too slowly when more and more terms are added to the series $-$ $s = 1/2$ is optimum, just nice.

Hence:

**Theorem due to Riemann**

All the non-trivial zeros of the Riemann zeta function $\zeta$ will always lie on the critical line $\mathrm{Re}(s) = 1/2$ only and not anywhere else on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$.

**Appendix B**

Below are the values of the reciprocals of, say, 100, with consecutive fractional powers $s \le 4/5$, these reciprocals being representative of the terms of the Riemann zeta function $\zeta$:-

[1] $1/100^{4/5} = 1/39.8107171 = 0.025$ (This quantity represents a decrease of 75% compared to [4] below while the increase in power from $s = 1/2$ to $s = 4/5$ is only 60%.)

[2] $1/100^{3/4} = 1/31.62278 = 0.032$ (This quantity represents a decrease of 68% compared to [4] below while the increase in power from $s = 1/2$ to $s = 3/4$ is only 50%.)

[3] $1/100^{2/3} = 1/21.5444 = 0.046$ (This quantity represents a decrease of 54% compared to [4] below while the increase in power from $s = 1/2$ to $s = 2/3$ is only 33.33%.)

[4] $1/100^{1/2} = 1/10 = 0.100$ (The terms of the series of the Riemann zeta function $\zeta$ as per the Riemann hypothesis fall under this category. $10^{13}$ zeros have been found under this category only.)

[5] $1/100^{1/3} = 1/4.6416 = 0.215$ (This quantity represents an increase of 115% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/3$ is only 33.33%.)

[6] $1/100^{1/4} = 1/3.1623 = 0.316$ (This quantity represents an increase of 216% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/4$ is only 50%.)

[7] $1/100^{1/5} = 1/2.5119 = 0.398$ (This quantity represents an increase of 298% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/5$ is only 60%.)

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Note the disproportionateness between the respective percentages of decrease in quantity and the respective percentages of increase in power for the reciprocals with powers $s > 1/2$, and, between the respective percentages of increase in quantity and the respective percentages of decrease in power for the reciprocals with powers $s < 1/2$.

**Appendix C**

The following list of the 1st 5 terms of the series of the Riemann zeta function $\zeta$ with consecutive fractional powers $s \geq 1/2$ shows that the sums with larger powers decrease progressively, i.e., the larger the power $s$ is the larger the percentage of decrease in the quantity is:-

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + \ldots = 3.24$ (The Riemann hypothesis asserts that all zeros will be found in this series only.)

[2] $\zeta(2/3) = 1 + 1/2^{2/3} + 1/3^{2/3} + 1/4^{2/3} + 1/5^{2/3} + \ldots = 2.85$ (The sum 2.85 here represents a decrease of **12.04%** compared to the sum 3.24 in [1] above.)

[3] $\zeta(3/4) = 1 + 1/2^{3/4} + 1/3^{3/4} + 1/4^{3/4} + 1/5^{3/4} + \ldots = 2.68$ (The sum 2.68 here represents a decrease of **17.28%** compared to the sum 3.24 in [1] above.)

[4] $\zeta(4/5) = 1 + 1/2^{4/5} + 1/3^{4/5} + 1/4^{4/5} + 1/5^{4/5} + \ldots = 2.59$ (The sum 2.59 here represents a decrease of **20.06%** compared to the sum 3.24 in [1] above.)

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The following is a tabulation of the above-mentioned percentage decreases for the sums for the 1st 2 terms to the 1st 5 terms for $\zeta(2/3)$, $\zeta(3/4)$ and $\zeta(4/5)$:-

| 1st | 1st.2 Terms | 1st.3 Terms | 1st.4 Terms | 1st.5 Terms | 1st.6 Terms … |
|---|---|---|---|---|---|
| [1] $\zeta(1/2)$ | – | – | – | – | – |
| [2] $\zeta(2/3)$ | 4.52% | 7.63% | 9.98% | **12.04%** | To Be Extrapolated |
| [3] $\zeta(3/4)$ | 6.65% | 11.08% | 14.37% | **17.28%** | To Be Extrapolated |
| [4] $\zeta(4/5)$ | 7.86% | 12.98% | 16.78% | **20.06%** | To Be Extrapolated |

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## 4. Conclusion

It is evident that the non-trivial zeros of the Riemann zeta function $\zeta$ are an important and effective tool which could be used to somehow estimate with accuracy the number of primes less than a given quantity, as is explained in the paper; at the same time the mystery surrounding these non-trivial zeros should have been dispelled by the paper. Importantly, the reasoning in the paper validates the Riemann hypothesis – all the non-trivial zeros will always lie exactly mid-way on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, on the critical line $\mathrm{Re}(s) = 1/2$. The validity of the Riemann hypothesis would ensure the effectiveness of the non-trivial zeros of the Riemann zeta function $\zeta$ as a tool for accurately estimating the number of primes less than a given quantity.

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