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COMBINATORIAL PROPERTIES OF CIRCULANT INFINITE ARRAY

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Abstract

In this paper, a Fibonacci sequence based infinite array is introduced and the concepts of Circulant infinite array, Secondary Symmetric, Bi-Symmetric, *N*-Symmetric, *SN*-Symmetric, *BN*-Symmetric of an array are also introduced. The definition of primary diagonal, Secondary diagonal and Secondary Pascal Symmetric of an array is defined. And also classified the properties of the Circulant infinite array based on their rotation. The properties of the Fibonacci array over the four-letter alphabet and binary alphabet are compared.

1. Introduction

Combinatorics on words are widespread topics in automata theory in the field of theoretical computer science [3, 6-8]. Two dimensional words and their combinatorial properties are used in the field of number theory, fractal geometry and pattern matching. The study of two dimensional (2D) languages play an important role in the theory of image analysis which have numerous applications [1]. N. Jansirani and Rajkumar Dare introduced the Fibonacci 2D structure and studied it in the two dimensional direction [4]. N. Jansirani, V. Subharani and V. R Dare introduced and studied the combinatorial properties of Fibonacci array based on dimension [5]. Toeplitz introduced the concepts of Symmetric matrices and Anna Lee introduced Secondary Symmetric and Secondary orthogonal matrices [2]. In signal processing, image processing, digital image disposal, linear forecast and error-correcting code theory the Circulant matrices have a wide range of

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applications. In this paper, a new kind of 2D rectangular infinite array called Fibonacci Sequence Based Infinite Array (FSA) and Circulant array are constructed and Secondary Symmetric of an array is introduced. In general, a Symmetric array need not be a Secondary Symmetric array, for that the Bi-Symmetric array concept is introduced. Based on the rotation the Circulant arrays are discussed. The complexity of an array is investigated. Also, the combinatorial properties and algebraic properties of Circulant array are studied. All the concepts are discussed with examples. Also, the properties of the Fibonacci array over the four-letter alphabet and binary alphabet are studied.

2. Basic Definitions and Preliminaries

Let Σ be a finite alphabet over $\{a, b\}$. The set of all empty and nonempty word over Σ is denoted as Σ^*. $\Sigma^+ = \Sigma^* - \{\lambda\} \cdot \lambda$ is denoted as an empty word. An infinite word w over an alphabet Σ is a mapping from a positive integer into Σ. The set of all infinite words over Σ is denoted by Σ^{ω}. An array $W = (w_{ij})$ over an alphabet Σ is a rectangular arrangement of symbols of Σ in r rows and s columns. The dimension of the array W is the ordered pair (r, s). The set of all arrays over Σ is denoted by Σ^{**}. We adopt the convention that for an array $W = (w_{ij})_{(r\times s)}$, the bottom-most row is the first row and the left-most column is the first column. $\Sigma^{++} = \Sigma^{**} - \{\lambda\}$. A factor of the subarray of an array W is also an array which is a part of W. The complexity function counts the number of distinct subarrays of a given array. An infinite array W has an infinite number of rows and an infinite number of columns, the collection of all infinite arrays over Σ is denoted by $\Sigma^{\omega\omega}$ [4].

If a suffix and a prefix of an array are equal then the array is said to be bordered. Column and row concatenation of arrays in Σ^{**} are partial operations. The Row concatenation of two arrays A and B exists only if the number of columns of A and B should be equal and it is denoted by $A \ominus B$. The Column concatenation of A and B exists only if the number of rows of A and B should be equal and it is denoted by $A \oslash B$ [4].

3. Combinatorial Properties of FSA

In this section, Fibonacci Sequence Based Infinite Array is constructed. Circulant array and Secondary Symmetric, Bi-Symmetric, N-Symmetric, SN-Symmetric, BN-Symmetric, Secondary Pascal Symmetric of an array are introduced. And also the definition of Primary diagonal and Secondary diagonal of an array are defined. Further, the combinatorial properties of such arrays are discussed.

Definition 3.1. Fibonacci Sequence Based Infinite Array (FSA) W is constructed as follows:

$$W = (w_{ij})_{i,\, j \geq 1} = \begin{Bmatrix} a \text{ if } & f_k & \text{is odd} \\ b \text{ if } & f_k & \text{is even} \end{Bmatrix},$$

where $\{f_k\}_{k \geq 1}$, $k = i + j - 1$ is the famous Fibonacci sequence defined by the recurrence relation $f_{n+1} = f_n + f_{n-1}$, $f_0 = 1 = f_1$, $n \geq 1$.

Structure of FSA

$$W = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \iddots \\ a & a & b & a & \dots \\ b & a & a & b & \dots \\ a & b & a & a & \dots \\ a & a & b & a & \dots \end{pmatrix}.$$

Definition 3.2. Let $w = \{a_1 a_2, \ldots, a_{n-1} a_n\}$, where $a_i \in \Sigma$ be an ordered alphabet. The Circulant vector of a word $w \in \Sigma^*$ is defined as $\mathfrak{Cir}(w) = (a_2, a_3, \ldots, a_n, a_1)$.

Let $w = abaab$, then $\mathfrak{Cir}(w) = baaba$. The Right Circulant vector of a word $w \in \Sigma^*$ is defined as $\mathfrak{RCir}(w) = (a_n, a_1, \ldots, a_{n-2}, a_{n-1})$. Let $w = abaab$ then $\mathfrak{RCir}(w) = babaa$.

Definition 3.3. A Circulant array is a special kind of an array, where each row vector is rotated one element to the left relative to the preceding row vector.

$$\mathfrak{C}ir(W) = \begin{pmatrix} a_n & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \dots & a_n & a_1 \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}.$$

Each row vector is rotated one element to the right relative to the preceding row vector is called a Right Circulant array.

$$\mathfrak{RC}ir(W) = \begin{pmatrix} a_2 & a_3 & \dots & a_n & a_1 \\ a_3 & a_4 & \dots & a_1 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_n & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix}.$$

Definition 3.4. Let $A = (a_{ij})$ be an array of dimension (m, n) over an alphabet Σ. The Primary diagonal of $A = (a_{ij})$ is defined as $i = j$ and the Secondary diagonal of A is defined as $i = m - j + 1$. The Secondary transpose of A is defined as $A^S = (a_{ij})^S = a_{(n-j+1,\ m-j+1)}$, for all $i \leq m,\ j \leq n$. A is said to be Secondary Symmetric if $a_{ij} = (a_{ij})^S$, for all $i \leq m,\ j \leq n$. A is said to be Bi-Symmetric then A is both Symmetric and Secondary Symmetric array. A is said to be Secondary Pascal Symmetric then the elements in the Secondary diagonal of A are the same. Every Symmetric array need not be a Secondary Symmetric array and vice versa.

Definition 3.5. Let $A = (a_{ij})$ be an array of dimension (n, n) over an alphabet Σ. Then A is said to be

(i) N-Symmetric, if $AA^T = A^T A(A^T$ is the transpose of an array A).

(ii) SN-Symmetric, if $AA^S = A^S A$.

(iii) BN-Symmetric, if $A(A^T)^S = (A^T)^S A$ or $A(A^S)^T = (A^S)^T A$.

Theorem 3.1. *Let $A = (a_{ij})$ be an array of dimension (n, n) over an alphabet Σ. Then A is a Palindrome array iff A is a Bi-Symmetric array.*

Proof. If part: The given array A is a Palindrome array then the left boundary and the right boundary are equal and Palindrome. Then the elements in A satisfy the condition $a_{ij} = (a_{ij}^T)^S$ or $a_{ij} = (a_{ij}^S)^T$ where $1 \leq i \leq m, 1 \leq j \leq n.$ Hence every Palindrome array is a Bi-Symmetric array and vice versa.

Theorem 3.2. *Let A be a Circulant array over an alphabet Σ. Then A^k (row/column concatenation), A^T and A^S are Circulant array, where $k \geq 2$.*

Proof. The proof of this theorem is by the Method of Mathematical Induction. Let us prove the theorem for $k = 2$. $A = \begin{pmatrix} a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \\ a_1 & a_2 & a_3 \end{pmatrix}$ then

$$A^2 = A \oslash A = \begin{pmatrix} a_3 & a_1 & a_2 & a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 & a_2 & a_3 & a_1 \\ a_1 & a_2 & a_3 & a_1 & a_2 & a_3 \end{pmatrix}.$$ Hence, this theorem is true for $k = 2$. Assume that, it is true for m. We have to prove for $m + 1$.

$$A^m \oslash A = \begin{pmatrix} a_3 & a_1 & a_2 & \dots & a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 & \dots & a_2 & a_3 & a_1 \\ a_1 & a_2 & a_3 & \dots & a_1 & a_2 & a_3 \end{pmatrix}_{(3,\ m+1)}.$$

Hence, this theorem is true for all $k \geq 2$.

$$A = \begin{pmatrix} a_n & a_1 & \dots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & \dots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_3 & \dots & a_n & a_1 \\ a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix} = A^T$$

and

$$A^S = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_3 & a_2 & a_1 \\ a_1 & a_n & a_{n-1} & \cdots & a_4 & a_3 & a_2 \\ a_2 & a_1 & a_n & \cdots & a_5 & a_4 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n-3} & a_{n-4} & a_{n-5} & \cdots & a_n & a_{n-1} & a_{n-2} \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & a_1 & a_n & a_{n-1} \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_2 & a_1 & a_n \end{pmatrix}$$

are satisfied the definition of Circulant array.

Theorem 3.3. *Let A be a Circulant array over an alphabet* Σ. *Then A is a Pascal Symmetric and N-Symmetric array.*

Proof. Let

$$A = \begin{pmatrix} a_n & a_1 & a_2 & \cdots & a_{n-3} & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-4} & a_{n-3} & a_{n-2} \\ a_{n-2} & a_{n-1} & a_n & \cdots & a_{n-5} & a_{n-4} & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_3 & a_4 & a_5 & \cdots & a_n & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n & a_1 \\ a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \end{pmatrix}.$$

The primary diagonal elements of A are the same. Hence, it is a Pascal Symmetric array. Furthermore, A and A^T are the same. In general every Symmetric array is N-Symmetric array. Hence the theorem.

Theorem 3.4. *Let A be a Circulant array over an alphabet* Σ. *Then every sub array of A of dimension* (i, i) *is a Pascal Symmetric array.*

Proof. Let $A = \begin{pmatrix} a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_1 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}$, then the sub arrays are

$$\begin{pmatrix} a_3 & a_4 & a_1 \\ a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \end{pmatrix}, \begin{pmatrix} a_4 & a_1 & a_2 \\ a_3 & a_4 & a_1 \\ a_2 & a_3 & a_4 \end{pmatrix}, \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_1 & a_2 \\ a_3 & a_4 & a_1 \end{pmatrix}, \begin{pmatrix} a_2 & a_3 \\ a_1 & a_2 \end{pmatrix}, \begin{pmatrix} a_3 & a_4 \\ a_2 & a_3 \end{pmatrix},$$
$$\begin{pmatrix} a_4 & a_1 \\ a_3 & a_4 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ a_4 & a_1 \end{pmatrix}.$$

From this we can prove that the sub arrays of A of dimension (i, i) are satisfied the property of Pascal Symmetric array.

Theorem 3.5. *Let A be a Right Circulant array over an alphabet* Σ. *Then*

(i) A^k (*row/column concatenation*), A^T *and* A^S *are a Right Circulant array*, *where* $k \geq 2$.

(ii) *A is a Secondary Pascal Symmetric and SN-Symmetric array*.

(iii) *Every sub arrays of A of dimension* (i, i) *are in Secondary Pascal Symmetric array*.

Theorem 3.6. *Let A be a FSA over an alphabet* Σ. *Then A has the following properties*

(i) *A is a Circulant array*.

(ii) *A is a Pascal Symmetric array*.

(iii) *Every sub-array of A of dimension* (i, i) *is a Symmetric array and Pascal Symmetric array*.

Proof. The proof of the theorem follows from the construction and definition of the array itself.

Lemma 3.1. *Let A be a FSA over an alphabet* Σ *and the dimension of A is in multiple of three. Then*

(i) *A is a Bi-Symmetric array.*

(ii) A^T *and* A^S *are also a Bi-Symmetric array.*

(iii) *A is a BN-Symmetric array.*

(iv) *Tandem of type A, type B occurred for A.*

(v) *A is a Boundary Palindrome array.*

Proof. The proof of this is by the contrapositive method. Let us assume that this theorem is true for all values. Let $m, n = 4$ then the array is $\begin{pmatrix} a & a & b & a \\ b & a & a & b \\ a & b & a & a \\ a & a & b & a \end{pmatrix}$. From this, the array is not a Circulant and Secondary Symmetric but it is Symmetric, the right boundary and the left boundary are the same but not a Palindrome. This contradicts our assumption. Hence the theorem is true only if, the dimension of an array is divisible by 3.

Theorem 3.7. *Let A be a FSA of dimension* (m, n) *over an alphabet* Σ. *Then the complexity function of A is bounded by three, where* $m, n \geq 2$.

Proof. Let consider the arbitrary array A of dimension is (m, n). Then the extension of A is $(m+1, n)$ or $(m, n+1)$ or $(m+1, n+1)$. FSA is a Pascal Symmetric array. By the definition of Pascal Symmetric the extension of A of any case will be decided by the single element $a_{m+1, n} | \, a_{m, n+1} \, | a_{m+1, n+1}$. But the possibility of the right boundaries of FSA are $aab \ldots ab$, $aab \ldots ba$, $aab \ldots aa$. Hence the complexity function of FSA is bounded by 3.

4. Comparision of Fibonacci Array over four Letters Alphabet and Binary Alphabet

In this section, the properties of the Fibonacci array over the four-letter alphabet and the two-letter alphabet are compared. Further, a formula for finding the number of occurrences of the tandem of type A and type B are computed [4].

Observation 4.1. For the Fibonacci array with four letters the following observations are noted

(i) Construction is based on Concatenation.

(ii) All types of tandems are existed.

(iii) Bordered properties of an array are satisfied with respect to the Row and Column of same width.

Observation 4.2. For the Fibonacci array with two letters the following observations are noted

(i) Construction is based on the Fibonacci sequence (FSA).

(ii) It has various types of array property such as Circulant array, Palindrome array, Pascal Symmetric array, Bi-Symmetric array, boundary Palindrome array.

(iii) Tandems of type A, type B and bordered are occurred.

Theorem 4.3. *Let A be a FSA of dimension (m, n) over an alphabet Σ. Then the number of occurrences of the tandem of type B is* $\sum_{2 \leq i \leq m} \left\lceil \frac{m}{i} \right\rceil$ *or* $\sum_{2 \leq i \leq n} \left\lceil \frac{n}{i} \right\rceil$.

Proof. Consider an array of dimension (4, 4). Let $A = \begin{pmatrix} a & a & b & a \\ b & a & a & b \\ a & b & a & a \\ a & a & b & a \end{pmatrix}$.

By the definition of tandem the number of occurrences of the tandem of type B is $\left\lceil \frac{4}{1} \right\rceil + \left\lceil \frac{4}{2} \right\rceil = 4 + 2 = 6.$ Since the theorem is true for A. Without loss of generality the theorem can be proved for any (m, n), where $m, n \in \mathbb{N}$.

Theorem 4.4. *Let A be a FSA of dimension (m, n) over an alphabet Σ.*

Then the number of occurrences of the tandem of type A is $\left\lceil \frac{m}{3} \right\rceil \left\lceil \frac{n}{3} \right\rceil$.

Proof. Consider the array of dimension (6, 6). Let
$$A = \begin{pmatrix} b & a & a & b & a & a \\ a & b & a & a & b & a \\ a & a & b & a & a & b \\ b & a & a & b & a & a \\ a & b & a & a & b & a \\ a & a & b & a & a & b \end{pmatrix}.$$
By definition, the number of occurrences of the tandem of type A is $\left\lceil \frac{6}{3} \right\rceil \left\lceil \frac{6}{3} \right\rceil = 2 \times 2 = 4$. By the above discussion the theorem is true for A. The result can be generalized for all FSA of dimension (m, n), where $m, n \in \mathbb{N}$.

5. Conclusion

A new kind of infinite array FSA is introduced and several properties based on Symmetric properties are established. The comparison between an array based on two types of alphabets is done. Future work focuses on structural and combinatorial properties of Pattern matching and DNA-computing.

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References

[1] Alberto Apostolico and E. Valentin, Brimkov, Fibonacci arrays and their two dimensional repetitions, Theoretical Computer Science 237 (2000), 263-273.

[2] Anna Lee, Secondary Symmetric Secondary skew Symmetric Secondary orthogonal

matrices, period, Math, Hungary 7 (1976), 63-70.

[3] Y. Huang and Z. Wen, The numbers of repeated Palindromes in the Fibonacci and Tribonacci words, Discrete Applied Mathematics 230 (2017), 78-90.

[4] N. Jansirani, V. Rajkumar Dare and K. G. Subramanian, Sturmian Arrays, Advances in Image Analysis and Applications, (Chapter 9) Research Publishing, Printed in Singapo ISBN-13:978-08-7923-5, ISBN-10:381-08-7923-7, May 2011.

[5] N. Jansirani and V. Rajkumar Dare, Special Properties of Fibonacci Array, Mathematical Sciences International Research Journal (2012), 560-569.

[6] M. Lothaire, Combinatorics on words, Addison Wesley Publishing Company, 1983.

[7] M. Lothaire, Algebraic Combinatorics on words, Encyclopedia of Mathematics and its Applications, vol. 90, Cambridge Uniersity Press, UK, 2002.

[8] B. Tan, Mirror Substitutions and Palindromic sequences, Theoret. Comput. Sci. 389 (2007), 118-124.