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FUZZY QUASI-Oz-SPACES

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Abstract

In this paper, a new class of fuzzy topological spaces, namely fuzzy quasi-Oz-spaces, is introduced in terms of fuzzy regular closed sets and fuzzy G_δ-sets. It is obtained that fuzzy extremally disconnected spaces are fuzzy quasi-Oz-spaces and fuzzy quasi-Oz-spaces are not fuzzy hyperconnected spaces. A condition under which fuzzy submaximal spaces become fuzzy quasi-Oz-spaces is obtained in terms of fuzzy residual sets.

1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets, fuzzy set operations was introduced by L. A. Zadeh [20] in 1965. The potential of fuzzy notion was realized by the researchers and has successfully been applied in all branches of Mathematics. In 1968, C. L. Chang [5] introduced the concept of fuzzy topological spaces and his work paved the way for the subsequent tremendous growth of the numerous fuzzy topological concepts. R. L. Blair [4], V. E. Scepin [10] and T. Terada [11] independently introduced the concept of Oz-spaces, also known as perfectly κ-normal spaces which is analogous to that of normality in classical topology. R. L. Blair studied the class of Oz-topological spaces in which every open set is z-embedded. A. Chigogidze [6]

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investigated weak Oz-spaces under the name of almost Scepin spaces. The notion of quasi-Oz-spaces was introduced by Chang Il Kim [8] which generalizes the concept of Oz-spaces.

In recent years, there has been a growing trend among many fuzzy topologists to introduce and study various types of fuzzy topological spaces. The notion of fuzzy Oz-spaces was introduced and studied in [19]. The purpose of this paper is to introduce the concept of fuzzy quasi-Oz-spaces and study its properties and applications.

2. Preliminaries

In order to make the exposition self-contained, some basic notions and results used in the sequel, are given. In this work by (X, T) or simply by X, we will denote a fuzzy topological space due to Chang [5]. Let X be a non-empty set and I, the unit interval $[0, 1]$. A fuzzy set λ in X is a mapping from X into I. The fuzzy set 0_X is defined as $0_X(x) = 0$, for all $x \in X$ and the fuzzy set 1_X is defined as $1_X(x) = 1$, for all $x \in X$.

Definition 2.1[5]**.** Let (X, T) be a fuzzy topological space and λ be any fuzzy set in (X, T). The interior, the closure and the complement of λ are defined respectively as follows:

(i) $\operatorname{int}(\lambda) = \vee\{\mu/\mu \leq \lambda, \mu \in T\}$.

(ii) $cl(\lambda) = \wedge\{\mu/\lambda \leq \mu, 1 - \mu \in T\}$.

(iii) $\lambda'(x) = 1 - \lambda(x)$, for all $x \in X$.

Note that for a family $\{\lambda_i / i \in I\}$ of fuzzy sets in (X, T), the union $\psi = \vee_i(\lambda_i)$ and the intersection $\delta = \wedge_i(\lambda_i)$, are defined respectively as

(iv) $\psi(x) = \sup_i\{\lambda_i(x)/x \in X\}$.

(v) $\delta(x) = \inf_i\{\lambda_i(x)/x \in X\}$.

Lemma 2.1[1]**.** *For a fuzzy set* λ *of a fuzzy topological space* X,

(i) $1 - \operatorname{int}(\lambda) = cl(1 - \lambda)$ *and* (ii) $1 - cl(\lambda) = \operatorname{int}(1 - \lambda)$.

Definition 2.2. A fuzzy set λ in a fuzzy topological space (X, T) is called a

(1) fuzzy regular-open if $\lambda = \operatorname{int} cl(\lambda)$ and fuzzy regular-closed if $\lambda = cl \operatorname{int}(\lambda)$ [1].

(2) fuzzy G_δ-set if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$ [2].

(3) fuzzy F_σ-set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where $1 - \lambda_i \in T$ for $i \in I$ [2].

(4) fuzzy semi-open if $\lambda \leq cl \operatorname{int}(\lambda)$ and fuzzy semi-closed if $\operatorname{int} cl(\lambda) \leq \lambda$ [1].

(5) fuzzy pre-open if $\lambda \leq \operatorname{int} cl(\lambda)$ and fuzzy pre-closed if $cl \operatorname{int}(\lambda) \leq \lambda$ [3].

Definition 2.3. A fuzzy set λ in a fuzzy topological space (X, T), is called a

(i) fuzzy dense set if there exists no fuzzy closed set μ in (X, T) such that $\lambda < \mu < 1$. That is, $cl(\lambda) = 1$, in (X, T) [12].

(ii) fuzzy nowhere dense set if there exists no non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $\operatorname{int} cl(\lambda) = 0$, in (X, T) [12].

(iii) fuzzy first category set if $\lambda = \vee_{i=1}^{\infty}(\lambda_i)$, where (λ_i)'s are fuzzy nowhere dense sets in (X, T). Any other fuzzy set in (X, T) is said to be of fuzzy second category [12].

(iv) fuzzy residual set if $1 - \lambda$ is a fuzzy first category set in (X, T) [14].

(v) fuzzy somewhere dense set if there exists a non-zero fuzzy open set μ in (X, T) such that $\mu < cl(\lambda)$. That is, $\operatorname{int} cl(\lambda) \neq 0$, in (X, T) [13].

Definition 2.4. A fuzzy topological space (X, T) is called a

(i) fuzzy hyperconnected space if every non-null fuzzy open subset of (X, T) is fuzzy dense in (X, T) [9].

(ii) fuzzy perfectly disconnected space if for any two non-zero fuzzy sets λ and μ defined on X with $\lambda \leq 1 - \mu$, $cl(\lambda) \leq 1 - cl(\mu)$, in (X, T) [16].

(iii) fuzzy fraction dense space if for each fuzzy open set λ in (X, T), $cl(\lambda) = cl(\mu)$, where μ is a fuzzy F_σ-set in (X, T) [18].

(iv) fuzzy Oz-space if each fuzzy regular closed set is a fuzzy G_δ-set in (X, T) [19].

(v) fuzzy extremally disconnected space if the closure of every fuzzy open set of (X, T) is fuzzy open in (X, T) [7].

Theorem 2.1[15]**.** *If* λ *is a fuzzy somewhere dense set in a fuzzy topological space* (X, T), *then there exists a fuzzy regular closed set* η *in* (X, T) *such that* $\eta \leq cl(\lambda)$.

Theorem 2.2[18]**.** *If* λ *is a fuzzy regular open set in a fuzzy fraction dense space* (X, T), *then there exists a fuzzy* G_δ*-set* θ *in* (X, T) *such that* $\operatorname{int}(\theta) \leq \lambda$.

Theorem 2.3[17]**.** *If* λ *is a fuzzy pre-closed set in a fuzzy perfectly disconnected space* (X, T), *then* $\operatorname{int}(\lambda)$ *is a fuzzy regular closed set in* (X, T).

Theorem 2.4[1]**.** *In a fuzzy topological space*,

(a) *the closure of a fuzzy open set is a fuzzy regular closed set*,

(b) *the interior of a fuzzy closed set is a fuzzy regular open set*.

Theorem 2.5[16]**.** *If* (X, T) *is a fuzzy perfectly disconnected space*, *then* (X, T) *is a fuzzy extremally disconnected space*.

Theorem 2.6[19]**.** *If* (X, T) *is a fuzzy extremally disconnected space*, *then* (X, T) *is a fuzzy Oz-space*.

3. Fuzzy Quasi-Oz-Spaces

Motivated by the works of Chang Il Kim [8] on quasi-Oz-spaces, the notion of quasi-Oz-space in fuzzy setting is defined as follows:

Definition 3.1. A fuzzy topological space (X, T) is called a fuzzy quasi-Oz-space if for a fuzzy regular closed set λ in (X, T), there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda = cl\operatorname{int}(\mu)$.

Example 3.1. Let $X = \{a, b, c\}$. Let α, β and γ be fuzzy sets in X defined as follows:

$\alpha : X \to [0, 1]$ is defined by $\alpha(a) = 0.4, \alpha(b) = 0.5, \alpha(c) = 0.6,$

$\beta : X \to [0, 1]$ is defined by $\beta(a) = 0.5, \beta(b) = 0.7, \beta(c) = 0.4,$

$\gamma : X \to [0, 1]$ is defined by $\gamma(a) = 0.6, \gamma(b) = 0.4, \gamma(c) = 0.5,$

$\lambda : X \to [0, 1]$ is defined by $\lambda(a) = 0.5, \lambda(b) = 0.5, \lambda(c) = 0.5.$

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \beta, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], \beta \vee [\alpha \wedge \gamma], \gamma \vee [\alpha \wedge \beta], \alpha \wedge [\beta \vee \gamma], \beta \wedge [\alpha \vee \gamma], \gamma \wedge [\alpha \vee \beta], \alpha \wedge \beta \wedge \gamma, \alpha \vee \beta \vee \gamma, 1\}$ is a fuzzy topology on X. By computation one can find that $cl(\alpha) = 1 - (\beta \wedge [\alpha \vee \gamma]) = \alpha \vee [\beta \wedge \gamma], cl(\beta) = 1; cl(\gamma) = 1 - (\alpha \wedge [\beta \vee \gamma]) = \gamma \vee (\alpha \wedge \beta); cl(\alpha \vee \beta) = 1; cl(\alpha \vee \gamma) = 1 - (\alpha \wedge \beta) = \alpha \vee \gamma; cl(\beta \vee \gamma) = 1; cl(\alpha \wedge \beta) = 1 - (\alpha \vee \gamma) = \alpha \wedge \beta; cl(\alpha \wedge \gamma) = 1 - (\gamma \vee [\alpha \wedge \beta]) = \alpha \wedge [\beta \vee \gamma); cl(\beta \wedge \gamma) = 1 - (\alpha \vee [\beta \wedge \gamma]) = \beta \wedge [\alpha \vee \gamma]; cl(\alpha \vee [\beta \wedge \gamma)) = 1 - (\beta \wedge [\alpha \vee \gamma]) = \alpha \vee [\beta \wedge \gamma]; cl(\beta \vee [\alpha \wedge \gamma]) = 1; cl(\gamma \vee [\alpha \wedge \beta]) = 1 - (\alpha \wedge [\beta \vee \gamma]) = \gamma \vee [\alpha \wedge \beta]; cl(\alpha \wedge [\beta \vee \gamma]) = 1 - (\gamma \vee [\alpha \wedge \beta]) = \alpha \wedge [\beta \vee \gamma]; cl(\beta \wedge [\alpha \vee \gamma]) = 1 - (\alpha \vee [\beta \wedge \gamma]) = \beta \wedge [\alpha \vee \gamma]; cl(\gamma \wedge [\alpha \vee \beta]) = 1 - (\gamma \wedge [\alpha \vee \beta]); cl(\alpha \vee \beta \vee \gamma) = 1; cl(\alpha \wedge \beta \wedge \gamma) = 1 - (\alpha \vee \gamma) = \alpha \wedge \beta.$

Also $\operatorname{int}(1 - \alpha) = \beta \wedge [\alpha \vee \gamma]; \operatorname{int}(1 - \beta) = 0; \operatorname{int}(1 - \gamma) = \alpha \wedge [\beta \vee \gamma]; \operatorname{int}(1 - [\alpha \vee \beta]) = 0; \operatorname{int}(1 - [\alpha \vee \gamma]) = \alpha \wedge \beta; \operatorname{int}(1 - [\beta \vee \gamma]) = 0; \operatorname{int}(1 - [\alpha \wedge \beta]) = \alpha \vee \gamma; \operatorname{int}(1 - [\alpha \wedge \gamma]) = (\gamma \vee [\alpha \wedge \beta]); \operatorname{int}(1 - [\beta \wedge \gamma]) = (\alpha \vee [\beta \wedge \gamma]); \operatorname{int}(1 - (\alpha \vee [\beta \wedge \gamma])) = (\beta \wedge [\alpha \vee \gamma]); \operatorname{int}(1 - (\beta \vee [\alpha \wedge \gamma])) = 0; \operatorname{int}(1 - (\gamma \vee [\alpha \wedge \beta])) = (\alpha \wedge [\beta \vee \gamma]); \operatorname{int}(1 - (\alpha \wedge [\beta \vee \gamma])) = (\gamma \vee [\alpha \wedge \beta]); \operatorname{int}(1 - (\beta \wedge [\alpha \vee \gamma])) = (\alpha \vee [\beta \wedge \gamma]); \operatorname{int}(1 - (\gamma \wedge [\alpha \vee \beta])) = (\gamma \wedge [\alpha \vee \beta]); \operatorname{int}(1 - (\alpha \vee \beta \vee \gamma)) = 0; \operatorname{int}(1 - (\alpha \wedge \beta \wedge \gamma)) = \alpha \vee \gamma.$

The fuzzy regular closed sets in (X, T) are $1 - [\alpha \vee \gamma], 1 - [\alpha \wedge \beta], 1 - (\alpha \vee [\beta \wedge \gamma]), 1 - (\gamma \vee [\alpha \wedge \beta]), 1 - (\alpha \wedge [\beta \vee \gamma]), 1 - (\beta \wedge [\alpha \vee \gamma])$ and $1 - (\gamma \wedge [\alpha \vee \beta])$.

Now for the fuzzy regular closed set $1-(\gamma \vee [\alpha \wedge \beta])$, there exists a fuzzy G_δ-set $\lambda = ((\alpha \vee \beta) \wedge (\alpha \vee \gamma) \wedge (\beta \vee \gamma))$ such that $1-(\gamma \vee [\alpha \wedge \beta]) = cl\,\text{int}(\lambda)$ [λ is not fuzzy open in (X, T)].

For the fuzzy regular closed set $1-(\gamma \wedge [\alpha \vee \beta])$, there exists a fuzzy G_δ-set $\gamma \wedge [\alpha \vee \beta] = ((\alpha \vee \beta) \wedge (\beta \vee \gamma) \wedge \gamma)$ such that $1-(\gamma \wedge [\alpha \vee \beta])$ $= cl\,\text{int}(\gamma \wedge [\alpha \vee \beta])$.

For the fuzzy regular closed set $1-(\alpha \wedge [\beta \vee \gamma])$, there exists a fuzzy G_δ-set $\gamma \vee [\alpha \wedge \beta] = ((\alpha \vee \beta \vee \gamma) \wedge (\beta \vee \gamma) \wedge (\alpha \vee \gamma))$ in (X, T) such that $1-(\alpha \wedge [\beta \vee \gamma]) = cl\,\text{int}(\gamma \vee [\alpha \wedge \beta])$.

For the fuzzy regular closed set $1-(\beta \wedge [\alpha \vee \gamma])$, there exists a fuzzy G_δ-set $\alpha \vee [\beta \wedge \gamma] = ((\alpha \vee \beta) \wedge \beta \wedge (\alpha \vee [\beta \wedge \gamma]))$ such that $1-(\beta \wedge [\alpha \vee \gamma])$ $= cl\,\text{int}(\alpha \vee [\beta \wedge \gamma])$.

For the fuzzy regular closed set $1-(\alpha \vee [\beta \wedge \gamma])$, there exists a fuzzy G_δ-set $\beta \wedge [\alpha \vee \gamma] = (\beta \wedge (\beta \vee [\alpha \wedge \gamma]) \wedge (\alpha \vee [\beta \wedge \gamma]))$ such that $1-(\alpha \vee [\beta \wedge \gamma]) = cl\,\text{int}(\beta \wedge [\alpha \vee \gamma])$.

For the fuzzy regular closed set $1-[\alpha \vee \gamma]$, there exists a fuzzy G_δ-set $\alpha \wedge \beta = (\alpha \wedge (\beta \wedge [\alpha \vee \gamma]) \wedge (\alpha \wedge [\beta \vee \gamma]))$ in (X, T) such that $1-[\alpha \vee \gamma]$ $= cl\,\text{int}(\alpha \wedge \beta)$.

For the fuzzy regular closed set $1-[\alpha \wedge \beta]$, there exists a fuzzy G_δ-set $\alpha \vee \gamma = ((\alpha \vee \beta \vee \gamma) \wedge (\alpha \vee \gamma) \wedge 1)$ in (X, T) such that $1-[\alpha \wedge \beta]$ $= cl\,\text{int}(\alpha \vee \gamma)$. Hence (X, T) is a fuzzy quasi-Oz-space.

Example 3.2. Let $X = \{a, b, c\}$. Let α, β and γ be fuzzy sets in X defined as follows:

$\alpha : X \to [0, 1]$ is defined by $\alpha(a) = 0.3; \alpha(b) = 0.4; \alpha(c) = 0.5$,

$\beta : X \to [0, 1]$ is defined by $\beta(a) = 0.4; \beta(b) = 0.5; \beta(c) = 0.6$,

$\gamma : X \to [0, 1]$ is defined by $\gamma(a) = 0.6; \gamma(b) = 0.7; \gamma(c) = 0.4$.

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \gamma, \beta \vee \gamma, \alpha \wedge \gamma, \beta \wedge \gamma, \alpha \vee [\beta \wedge \gamma], 1\}$ is a fuzzy topology on X. By computation, one can find that $cl(\alpha) = 1 - (\alpha \vee [\beta \wedge \gamma]);$ $cl(\beta) = 1 - (\beta \wedge \gamma); cl(\gamma) = 1; cl(\alpha \vee \gamma) = 1; cl(\beta \vee \gamma) = 1; cl(\alpha \wedge \gamma) = 1 - \beta; cl(\beta \wedge \gamma)$ $= 1 - \beta; cl(\alpha \vee [\beta \wedge \gamma]) = 1 - (\alpha \vee [\beta \wedge \gamma])$. Also $\text{int}(1 - \alpha) = \alpha \vee [\beta \wedge \gamma];$ $\text{int}(1 - \beta) = \beta \wedge \gamma; \text{int}(1 - \gamma) = 0; \text{int}(1 - [\alpha \vee \gamma]) = 0; \text{int}(1 - [\beta \vee \gamma]) = 0;$ $\text{int}(1 - [\alpha \wedge \gamma]) = \beta; \text{int}(1 - [\beta \wedge \gamma]) = \beta; \text{int}(1 - [\alpha \vee (\beta \wedge \gamma)]) = \alpha \vee (\beta \wedge \gamma).$

The fuzzy regular closed sets in (X, T) are $1 - \beta, 1 - (\beta \wedge \gamma)$ and $1 - [\alpha \vee (\beta \wedge \gamma)]$. By computation, one can find that $\alpha \wedge \gamma, \gamma, \beta \wedge \gamma$ are fuzzy G_δ-sets in (X, T). Now, for the fuzzy regular closed set $1 - [\beta \wedge \gamma]$ in (X, T), there exists no fuzzy G_δ-set δ in (X, T) such that $1 - [\beta \wedge \gamma] = cl\,\text{int}(\delta)$. Hence (X, T) is not a fuzzy quasi-Oz-space.

Proposition 3.1. *If γ is a fuzzy regular open set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy F_σ-set δ in (X, T) such that $\gamma = \text{int}\,cl(\delta)$.*

Proof. Let γ be a fuzzy regular open set in (X, T). Then, $1 - \gamma$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, there exists a fuzzy G_δ-set μ in (X, T) such that $1 - \gamma = cl\,\text{int}(\mu)$. Then, $\gamma = 1 - cl\,\text{int}(\mu) = \text{int}\,cl(1 - \mu)$, in (X, T). Let $\delta = 1 - \mu$ and then δ is a fuzzy F_σ-set in (X, T). Hence, for the fuzzy regular open set γ, there exists a fuzzy F_σ-set δ in (X, T) such that $\gamma = \text{int}\,cl(\delta)$.

Proposition 3.2. *If γ is a fuzzy regular open set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy F_σ-set δ in (X, T) such that $\gamma \leq cl(\delta)$.*

Proof. Let γ be a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, by Proposition 3.1, there exists a fuzzy F_σ-set δ in (X, T) such that $\gamma = \text{int}\,cl(\delta)$. Now $\text{int}\,cl(\delta) \leq cl(\delta)$ in (X, T). Hence for a fuzzy regular open set γ, there exists a fuzzy F_σ-set δ in (X, T) such that $\gamma \leq cl(\delta)$.

Proposition 3.3. *If λ is a fuzzy regular closed set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy G_δ-set μ in (X, T) such that $\text{int}(\mu) \leq \lambda$.*

Proof. Let λ be a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda = cl\,\mathrm{int}(\mu)$. Now $\mathrm{int}(\mu) \leq cl\,\mathrm{int}(\mu)$, in (X, T). Hence, for the fuzzy regular closed set λ, there exists a fuzzy G_δ-set μ in (X, T) such that $\mathrm{int}(\mu) \leq \lambda$.

Proposition 3.4. *If λ is a fuzzy regular closed set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy open set θ in (X, T) such that $\theta \leq \lambda$.*

Proof. By Proposition 3.3, for the fuzzy regular closed set λ, there exists a fuzzy G_δ-set μ in (X, T) such that $\mathrm{int}(\mu) \leq \lambda$. Let $\theta = \mathrm{int}(\mu)$. Thus, for the fuzzy regular closed set λ, there exists a fuzzy open set θ in (X, T) such that $\theta \leq \lambda$.

Corollary 3.1. *If γ is a fuzzy regular open set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy closed set η in (X, T) such that $\gamma \leq \eta$.*

Proof. By Proposition 3.2, for the fuzzy regular open set γ, there exists a fuzzy F_σ-set δ in (X, T) such that $\gamma \leq cl(\delta)$. Let $\eta = cl(\delta)$. Then, η is a fuzzy closed set in (X, T). Thus, for the fuzzy regular open set γ, there exists a fuzzy closed set η in (X, T) such that $\gamma \leq \eta$.

In a fuzzy topological space (X, T), if λ is a fuzzy somewhere dense set, then $\mathrm{int}\,cl(\lambda) \neq 0$, in (X, T). This implies that there exists a fuzzy open set η in (X, T) such that $\eta \leq cl(\lambda)$. The following proposition shows that in a fuzzy quasi-Oz-space that open set η is the interior of some fuzzy G_δ-set in (X, T).

Proposition 3.5. *If λ is a fuzzy somewhere dense set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy G_δ-set μ in (X, T) such that* $\mathrm{int}(\mu) \leq cl(\lambda)$.

Proof. Let λ be a fuzzy somewhere dense set in (X, T). Then, by Theorem 2.1, there exists a fuzzy regular closed set η in (X, T) such that $\eta \leq cl(\lambda)$. By Proposition 3.3, for the fuzzy regular closed set η, there exists a fuzzy G_δ-set μ in (X, T) such that $\mathrm{int}(\mu) \leq \eta$. Thus, $\mathrm{int}(\mu) \leq cl(\lambda)$, in (X, T).

Proposition 3.6. *If* λ *is a fuzzy regular closed set in a fuzzy quasi-Oz-space* (X, T), *then there exists a fuzzy* G_δ*-set* δ *in* (X, T) *such that* $cl(\lambda) \leq cl(\delta)$.

Proof. Let λ be a fuzzy regular closed set in (X, T). Then, $cl\,\text{int}(\lambda) = \lambda$, in (X, T). Now $cl[cl\,\text{int}(\lambda)] = cl(\lambda)$ and then $cl\,\text{int}(\lambda) = cl(\lambda)$ ... (A). Since $\text{int}(\lambda)$ is a fuzzy open set in (X, T), by Theorem 2.4, $cl\,\text{int}(\lambda)$ is a fuzzy regular closed set in (X, T). From (A), $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, there exists a fuzzy G_δ-set μ in (X, T) such that $cl(\lambda) = cl\,\text{int}(\mu)$ and then $cl(\lambda) = cl\,\text{int}(\mu) \leq cl(\mu)$ and hence $cl(\lambda) \leq cl(\delta)$.

Corollary 3.2. *If* λ *is a fuzzy regular closed set in a fuzzy quasi-Oz-space* (X, T), *then there exist fuzzy* G_δ*-sets* μ_1 *and* μ_2 *in* (X, T) *such that* $cl\,\text{int}(\mu_1) \leq cl\,\text{int}(\mu_2)$.

Proof. Let λ be a fuzzy regular closed set in (X, T). Then, as in the proof of Proposition 3.6, $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since λ and $cl(\lambda)$ are fuzzy regular closed sets in the fuzzy quasi-Oz-space (X, T), there exist fuzzy G_δ-sets μ_1 and μ_2 in (X, T) such that $\lambda = cl\,\text{int}(\mu_1)$ and $cl(\lambda) = cl\,\text{int}(\mu_2)$. Since $\lambda \leq cl(\lambda)$, $cl\,\text{int}(\mu_1) \leq cl\,\text{int}(\mu_2)$, in (X, T).

Corollary 3.3. *If* λ *is a fuzzy regular open set in a fuzzy quasi-Oz-space* (X, T), *then there exist fuzzy* F_σ*-sets* δ_1 *and* δ_2 *in* (X, T) *such that* $\text{int}\,cl(\delta_2) \leq \text{int}\,cl(\delta_1)$.

Proposition 3.7. *If* λ *is a fuzzy regular closed set in a fuzzy quasi-Oz-space* (X, T), *then there exist fuzzy* G_δ*-sets* μ *and* δ *in* (X, T) *such that* $\text{int}(\mu) \leq \lambda \leq cl(\delta)$.

Proof. The proof follows from Propositions 3.3 and 3.6.

Corollary 3.4. *If* γ *is a fuzzy regular open set in a fuzzy quasi-Oz-space* (X, T), *then there exist fuzzy* F_σ*-sets* η *and* θ *in* (X, T) *such that* $\text{int}(\eta) \leq \gamma \leq cl(\theta)$.

Proof. Let γ be a fuzzy regular open set in (X, T). Then, $1 - \gamma$ is a fuzzy regular closed set in the fuzzy quasi-Oz-space (X, T). By Proposition 3.7, there exist fuzzy G_δ-sets μ and δ in (X, T) such that $\operatorname{int}(\mu) \leq 1 - \gamma \leq cl(\delta)$. Then, $1 - \operatorname{int}(\mu) \geq 1 - (1 - \gamma) \geq 1 - cl(\delta)$. Then, $cl(1 - \mu) \geq \gamma \geq \operatorname{int}(1 - \delta)$. Let $\theta = 1 - \mu$ and $\eta = 1 - \delta$. Then, θ and η are fuzzy F_σ-sets in (X, T). Thus, for the regular open set γ in (X, T), there exist fuzzy F_σ-sets η and θ in (X, T) such that $\operatorname{int}(\eta) \leq \gamma \leq cl(\theta)$.

Proposition 3.8. *If λ is a fuzzy semi-open set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy G_δ-set μ in (X, T) such that $cl(\lambda) = cl\operatorname{int}(\mu)$.*

Proof. Let λ be a fuzzy semi-open set in (X, T). Then, $\lambda \leq cl\operatorname{int}(\lambda)$, in (X, T). This implies that $cl(\lambda) \leq cl[cl\operatorname{int}(\lambda)] = cl\operatorname{int}(\lambda) \leq cl\operatorname{int}cl(\lambda) \leq cl\,cl(\lambda)$ $= cl(\lambda)$ and thus $cl(\lambda) \leq cl\operatorname{int}cl(\lambda) \leq cl(\lambda)$. Then, $cl(\lambda) = cl\operatorname{int}cl(\lambda)$ and thus $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, there exists a fuzzy G_δ-set μ in (X, T) such that $cl(\lambda) = cl\operatorname{int}(\mu)$.

Proposition 3.9. *If λ is a fuzzy regular closed set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy somewhere dense set δ in (X, T) such that $\lambda \leq cl(\delta)$.*

Proof. Let λ be a fuzzy regular closed set in (X, T). Then, $cl\operatorname{int}(\lambda) = \lambda$, in (X, T). This implies that $\operatorname{int}(\lambda) \neq 0$. Since (X, T) is a fuzzy quasi-Oz-space, by Proposition 3.6, there exists a fuzzy G_δ-set δ in (X, T) such that $cl(\lambda) \leq cl(\delta)$. Then, $\operatorname{int}(\lambda) \leq cl(\lambda) \leq cl(\delta)$, in (X, T) and $\operatorname{int}cl(\delta) \neq 0$, in (X, T). Hence δ is a fuzzy somewhere dense set in (X, T) such that $\lambda \leq cl(\delta)$.

Proposition 3.10. *If λ is a fuzzy open set in a fuzzy quasi-Oz-space (X, T), then there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda \leq cl\operatorname{int}(\mu)$.*

Proof. Let λ be a fuzzy open set in (X, T). Then, by Theorem 2.4, $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, there exists a fuzzy G_δ-set μ in (X, T) such that $cl(\lambda) = cl\operatorname{int}(\mu)$. Now $\lambda \leq cl(\lambda)$ implies that $\lambda \leq cl\operatorname{int}(\mu)$, in (X, T).

Proposition 3.11. *If* δ *is a fuzzy closed set in a fuzzy quasi-Oz-space* (X, T), *then there exists a fuzzy* F_σ*-set* η *in* (X, T) *such that* $\operatorname{int} cl(\lambda) \le \delta$.

Proof. Let δ be a fuzzy closed set in (X, T). Then, $1 - \delta$ is a fuzzy open set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, by Proposition 3.9, there exists a fuzzy G_δ-set μ in (X, T) such that $1 - \delta \le cl \operatorname{int}(\mu)$. This implies that $1 - cl \operatorname{int}(\mu) \le \delta$ and then $\operatorname{int} cl(1 - \mu) \le \delta$, in (X, T). Let $\eta = 1 - \mu$ and then η is a fuzzy F_σ-set in (X, T). Thus, for the fuzzy closed set δ, there exists a fuzzy F_σ-set η in (X, T) such that $\operatorname{int} cl(\lambda) \le \delta$.

4. Fuzzy Quasi-Oz-Spaces and Other Fuzzy Topological Spaces

In this section we discuss some relationships among fuzzy quasi-Oz-spaces and other fuzzy spaces.

Proposition 4.1. *If a fuzzy topological space* (X, T) *is a fuzzy Oz-space, then* (X, T) *is a fuzzy quasi-Oz-space.*

Proof. Let λ be a fuzzy regular closed set in (X, T). Then, $\lambda = cl \operatorname{int}(\lambda)$, in (X, T). Since (X, T) is a fuzzy Oz-space, λ is a fuzzy G_δ-set in (X, T). Thus, for the fuzzy regular closed set λ in (X, T), $\lambda = cl \operatorname{int}(\lambda)$, where λ is a fuzzy G_δ-set in (X, T), implies that (X, T) is a fuzzy quasi-Oz-space.

Remark. The converse of the above proposition need not be true. That is, a fuzzy quasi-Oz-space need not be a fuzzy Oz-space. For, consider the following example.

Example 4.1. Let $X = \{a, b, c\}$. Let α, β and γ be fuzzy sets in X defined as follows:

$\alpha : X \to [0, 1]$ is defined by $\alpha(a) = 0.4; \alpha(b) = 0.6; \alpha(c) = 0.5$,

$\beta : X \to [0, 1]$ is defined by $\beta(a) = 0.6; \beta(b) = 0.7; \beta(c) = 0.8$,

$\gamma : X \to [0, 1]$ is defined by $\gamma(a) = 0.5; \gamma(b) = 0.4; \gamma(c) = 0.7$.

Then, $T = \{0, \alpha, \beta, \gamma, \alpha \vee \gamma, \alpha \wedge \gamma, 1\}$ is a fuzzy topology on X. By computation, one can find that $cl(\alpha) = 1 - (\alpha \wedge \gamma); cl(\beta) = 1; cl(\gamma) = 1;$

$cl(\alpha \vee \gamma) = 1; cl(\alpha \wedge \gamma) = 1 - \alpha; \text{int}(1 - \alpha) = \alpha \wedge \gamma; \text{int}(1 - \beta) = 0; \text{int}(1 - \gamma) = 0;$ $\text{int}(1 - [\alpha \vee \gamma]) = 0; \text{int}(1 - [\alpha \wedge \gamma]) = \alpha.$ Also $cl\,\text{int}(1 - \alpha) = 1 - \alpha$ and $cl\,\text{int}(1 - [\alpha \wedge \gamma]) = 1 - [\alpha \wedge \gamma].$ The fuzzy regular closed sets in (X, T) are $1 - \alpha$ and $1 - [\alpha \wedge \gamma].$ The fuzzy G_δ-sets in (X, T) are α and $\alpha \wedge \gamma.$ By computation, one can find that $1 - \alpha = cl\,\text{int}(\alpha \wedge \gamma)$ and $1 - [\alpha \wedge \gamma] = cl\,\text{int}(\alpha),$ in $(X, T).$ Hence (X, T) is a fuzzy quasi-Oz-space. But (X, T) is not a fuzzy Oz-space, since the fuzzy regular closed sets $1 - \alpha$ and $1 - [\alpha \wedge \gamma]$ are not fuzzy G_δ-sets in $(X, T).$

The following proposition gives a condition for fuzzy quasi-Oz-spaces to become fuzzy Oz-spaces.

Proposition 4.2. *If each fuzzy G_δ-set is a fuzzy regular closed set in a fuzzy quasi-Oz-space $(X, T),$ then (X, T) is a fuzzy Oz-space.*

Proof. Let λ be a fuzzy regular closed set in $(X, T).$ Since (X, T) is a fuzzy quasi-Oz-space, there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda = cl\,\text{int}(\mu).$ By hypothesis, the fuzzy G_δ-set μ is a fuzzy regular closed set in (X, T) and then $cl\,\text{int}(\mu) = \mu.$ This implies that $\lambda = \mu,$ in $(X, T).$ This implies that the fuzzy regular closed set λ is a fuzzy G_δ-set in $(X, T).$ Hence (X, T) is a fuzzy quasi-Oz-space.

Proposition 4.3. *If λ is a fuzzy regular open set in a fuzzy fraction dense and fuzzy quasi-Oz-space $(X, T),$ then there exists a fuzzy G_δ-set θ and a fuzzy F_σ-set δ in (X, T) such that* $\text{int}(\theta) \leq \lambda \leq cl(\delta).$

Proof. Let λ be a fuzzy regular open set in $(X, T).$ Since (X, T) is a fuzzy fraction dense space, by Theorem 2.2, there exists a fuzzy G_δ-set θ in (X, T) such that $\text{int}(\theta) \leq \lambda.$ Also since (X, T) is a fuzzy quasi-Oz-space $(X, T),$ by Proposition 3.2, for the fuzzy regular open set λ in $(X, T),$ there exists a fuzzy F_σ-set δ in (X, T) such that $\lambda \leq cl(\delta).$ Then, $\text{int}(\theta) \leq \lambda \leq cl(\delta),$ in $(X, T).$

Corollary 4.1. *If λ is a fuzzy regular open set in a fuzzy fraction dense* *and fuzzy quasi-Oz-space* (X, T), *then there exists a fuzzy open set* α *and a fuzzy closed set* β *in* (X, T) *such that* $\alpha \leq \lambda \leq \beta$.

Proof. Let λ be a fuzzy regular open set in (X, T). Since (X, T) is a fuzzy fraction dense and fuzzy quasi-Oz-space, by Proposition 4.3, there exists a fuzzy G_δ-set θ and a fuzzy F_σ-set δ in (X, T) such that $\text{int}(\theta) \leq \lambda \leq cl(\delta)$. Let $\alpha = \text{int}(\theta)$ and $\beta = cl(\delta)$. Then α and β are respectively fuzzy open and fuzzy closed sets in (X, T). Thus, there exists a fuzzy open set α and a fuzzy closed set β in (X, T) such that $\alpha \leq \lambda \leq \beta$.

Proposition 4.4. *If* (X, T) *is a fuzzy perfectly disconnected space, then* (X, T) *is a fuzzy quasi-Oz-space.*

Proof. Let (X, T) be a fuzzy perfectly disconnected space. Then, by Theorem 2.5, (X, T) is a fuzzy extremally disconnected space. By Theorem 2.6 (X, T) is a fuzzy Oz-space and then, by Proposition 4.1, (X, T) is a fuzzy quasi-Oz-space.

Corollary 4.2. *If* λ *is a fuzzy pre-closed set in a fuzzy perfectly disconnected space,* (X, T) *then there exists a fuzzy* G_δ*-set* μ *in* (X, T) *such that* $\text{int}(\mu) \leq \lambda$.

Proof. Let λ be a fuzzy pre-closed set in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, by Theorem 2.3, $\text{int}(\lambda)$ is a fuzzy regular closed set in (X, T). By Proposition 4.4, (X, T) is a fuzzy quasi-Oz-space and thus for the fuzzy regular closed set $\text{int}(\lambda)$ in (X, T), there exists a fuzzy G_δ-set μ in (X, T) such that $\text{int}(\lambda) = cl\,\text{int}(\mu)$ and then $cl[\text{int}(\lambda)] = cl[cl\,\text{int}(\mu)]$ $= cl\,\text{int}(\mu)$, in (X, T). By the pre-closedness of λ, $cl\,\text{int}(\lambda) \leq \lambda$ and thus $cl\,\text{int}(\mu) \leq \lambda$. Now $\text{int}(\mu) \leq cl\,\text{int}(\mu) \leq \lambda$ implies that $\text{int}(\mu) \leq \lambda$, in (X, T).

Corollary 4.3. *If* γ *is a fuzzy pre-open set in a fuzzy quasi-Oz- and fuzzy perfectly disconnected space* (X, T), *then there exists a fuzzy* F_σ*-set* θ *in* (X, T) *such that* $\gamma \leq cl(\theta)$.

Proof. Let γ be a fuzzy pre-open set in (X, T). Then, $1 - \gamma$ is a fuzzy pre-closed set in (X, T). Since (X, T) is a fuzzy perfectly disconnected space, by Corollary 4.3, there exists a fuzzy G_δ-set μ in (X, T) such that $\text{int}(\mu) \leq 1 - \gamma$. Then, $\gamma \leq 1 - \text{int}(\mu)$ and $\gamma \leq cl(1 - \mu)$. Let $\theta = 1 - \mu$. Then, θ is a fuzzy F_σ-set in (X, T) and $\gamma \leq cl(\theta)$, in (X, T).

Remark. It is to be noted that a fuzzy quasi-Oz-space need not be a fuzzy extremally disconnected space. For, in example 3.1, (X, T) is a fuzzy quasi-Oz-space but not a fuzzy extremally disconnected space, since $cl(\gamma \wedge [\alpha \vee \beta]) = 1 - (\gamma \wedge [\alpha \vee \beta])$ which is not fuzzy open in (X, T).

The following proposition gives a condition under which a fuzzy quasi-Oz -space becomes a fuzzy extremally disconnected space.

Proposition 4.5. *If* $cl\,\text{int}(\mu)$ *is a fuzzy open set for each fuzzy* G_δ*-set* μ *in a fuzzy quasi-Oz-space* (X, T), *then* (X, T) *is a fuzzy extremally disconnected space.*

Proof. Let λ be a fuzzy open set in (X, T). Then, by Theorem 2.4, $cl(\lambda)$ is a fuzzy regular closed set in (X, T). Since (X, T) is a fuzzy quasi-Oz-space, for the fuzzy regular closed set $cl(\lambda)$ in (X, T), there exists a fuzzy G_δ-set μ in (X, T) such that $cl(\lambda) = cl\,\text{int}(\mu)$. By hypothesis, $cl\,\text{int}(\mu)$ is a fuzzy open set in (X, T) and thus $cl(\lambda)$ is a fuzzy open set in (X, T). Hence (X, T) is a fuzzy extremally disconnected space.

The following proposition shows that fuzzy quasi-Oz-spaces are not fuzzy hyperconnected spaces.

Proposition 4.6. *If* (X, T) *is a fuzzy quasi-Oz-space*, *then* (X, T) *is not a fuzzy hyperconnected space*.

Proof. Let λ be a fuzzy open set in (X, T). Then, by Proposition 3.9, there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda \leq cl\,\text{int}(\mu)$. Then $\lambda \leq cl(\mu)$ and thus $cl(\lambda) \neq 1$, in (X, T). Hence (X, T) is not a fuzzy hyperconnected space.

Proposition 4.7. *If* (X, T) *is a fuzzy extremally disconnected space*, *then* (X, T) *is a fuzzy quasi-Oz-space*.

Proof. Let (X, T) be a fuzzy extremally disconnected space. Then, by

Theorem 2.5, (X, T) is a fuzzy Oz-space. By Proposition 4.1, the fuzzy Oz-space (X, T) is a fuzzy quasi-Oz-space. The following proposition gives a condition under which fuzzy submaximal spaces become fuzzy quasi-Oz-spaces.

Proposition 4.8. *If there exists a fuzzy residual set* μ *in a fuzzy submaximal space* (X, T) *such that* $\lambda = cl\,\text{int}(\mu)$, *for each fuzzy regular closed set* λ *in* (X, T), *then* (X, T) *is a fuzzy quasi-Oz-space.*

Proof. Let λ be a fuzzy regular closed set in (X, T). By hypothesis, there exists a fuzzy residual set μ in (X, T) such that $\lambda = cl\,\text{int}(\mu)$. Since (X, T) is a fuzzy submaximal space, by Theorem 2.6, the fuzzy residual set μ is a fuzzy G_δ-set in (X, T). Thus, for the fuzzy regular closed set λ, there exists a fuzzy G_δ-set μ in (X, T) such that $\lambda = cl\,\text{int}(\mu)$, implies that (X, T) is a fuzzy quasi-Oz-space.

Conclusion

In this paper, a new class of fuzzy topological spaces, namely fuzzy quasi-Oz-spaces, is introduced in terms of fuzzy regular closed sets and fuzzy G_δ-sets. It is established that the existence of fuzzy regular open sets in fuzzy quasi-Oz-spaces ensures the existence of somewhere dense fuzzy F_σ-sets and each fuzzy regular closed set in a fuzzy quasi-Oz-space contains a fuzzy open set. It is obtained that each fuzzy regular closed set in a fuzzy quasi-Oz-space lies between the interior of some fuzzy G_δ-set and the closure of some fuzzy G_δ-set and each fuzzy regular open set in a fuzzy quasi-Oz-space lies between the interior of some fuzzy F_σ-set and the closure of some fuzzy F_σ-set. Also it is obtained that each fuzzy regular closed set is contained in the closure of some fuzzy somewhere dense set in a fuzzy quasi-Oz-space. It is found that fuzzy Oz-spaces are fuzzy quasi-Oz-spaces and a condition under which a fuzzy quasi-Oz-space becomes a fuzzy Oz-space is obtained. It is established that fuzzy extremally disconnected spaces are fuzzy quasi-Oz-spaces and fuzzy perfectly disconnected spaces are fuzzy quasi-Oz-spaces. It is found that fuzzy quasi-Oz-spaces are not fuzzy hyperconnected spaces. A condition under which fuzzy submaximal spaces become fuzzy quasi-Oz-spaces is obtained in terms of fuzzy residual sets.

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