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FIXED POINT THEOREMS FOR *F*-CONTRACTIVE TYPE MAPPINGS ON A DISLOCATED QUASI *b*-METRIC SPACE

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Abstract

In this paper we define F-contractive type mappings in dislocated quasi-b-metric spaces and establish some fixed point theorems for F-contractive mappings. Supporting examples also are provided.

Introduction

In 1922, Banach [7] proved a fixed point theorem, which later on came to be known as the famous Banach contraction principle. Since then generalizations of the contraction principle in different directions as well as many new fixed point results with applications have been established by different researchers ([1], [2], [4]-[6], [8], [10], [12]-[16], [18], [19], [21]-[23], [26]-[29], [30]-[38]).

It is a well-known fact that every Banach contraction is continuous. In

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1968, Kannan [25] proved the following result for not necessarily continuous mappings.

Theorem 1.1 (Kannan [25]). *Let* (Y_0, d_0) *be a complete metric space. Suppose* $T_0 : Y_0 \to Y_0$ *is a mapping such that* $d_0(T_0x_0, T_0y_0) \leqslant \lambda_0\{d_0(x_0, T_0x_0) + d_0(y_0, T_0y_0)\}\ \forall x_0, y_0 \in Y_0$ *and for some* $0 \leqslant \lambda_0 < \frac{1}{2}$. *Then* T_0 *has one and only one fixed point* $z_0 \in Y_0$, *and for any* $x_0 \in Y_0$ *the sequence* $\{T_0^n x_0\}$ *converges to* z_0.

This result shows that there exist self mappings with only one fixed point which are not necessarily continuous. Incidentally, the result of Kannan also gave a characterization of the metric space (Y_0, d_0) in terms of the fixed point of T_0. This was shown by Subrahmanyam [41] in 1975, by proving that a metric space is complete if and only if every Kannan mapping has a unique fixed point.

In 2012, Wardowski [43] introduced a new type of contraction called F-contraction (also called Wardowski-contraction [24]) and proved a fixed point theorem concerning F-Contractions. Since then much work has been done on the fixed point theory of F-Contraction mappings and their extensions ([25], [32], [33], [39], [40], [42], [44]-[48]).

In this paper, we present result on fixed point theory in dislocated quasi b-metric spaces considering a new type of mappings which is a combination of F-Contraction by Wardowski [43] as well as Kannan contraction [25] mappings.

2. Preliminaries

We start by defining some of the terms used in this paper.

Definition 2.1 (P. Hitzler and A. K. Seda [23]). Let $Y_0 \neq \phi$. Suppose that the mapping $d_0 : Y_0 \times Y_0 \to [0, \infty)$ satisfies the following conditions:

(d_1) $d_0(x_0, y_0) = d_0(y_0, x_0) = 0$ implies $x_0 = y_0$ $x_0, y_0 \in Y_0$, and

(d_2) there exists $s \geqslant 1$ such that $d_0(x_0, y_0) \leqslant s(d_0(x_0, z_0) + d_0(z_0, y_0))$ $\forall x_0, y_0, z_0 \in Y_0$.

The triplet (Y_0, d_0, s) is called a dislocated quasi-b-metric space. The number s is called the coefficient of (Y_0, d_0).

Example 2.1. Let $Y_0 = \{1, 2, 3, \ldots\}$ we define $d_0 : Y_0 \times Y_0 \to \mathbb{R}$ by

$$d_0(x_0, y_0) = \begin{cases} 1 & \text{if } x_0 < y_0 \\ 0 & \text{if } x_0 = y \\ 2 & \text{if } x_0 > y_0. \end{cases}$$

Then (Y_0, d_0, s) is a dislocated quasi b-metric space with $s = 1$.

Example 2.2. Let $Y_0 = [0, 1]$ and define $d_0 : Y_0 \times Y_0 \to [0, \infty)$ by

$$d_0(x_0, y_0) = \begin{cases} (x_0^2 + y_0)^2 & \text{if } x_0 \neq y_0 \\ 0 & \text{if } x_0 = y_0. \end{cases}$$

Then (Y_0, d_0, s) is dislocated quasi b-metric space with $s = 2$.

Definition 2.2. Let (Y_0, d_0, s) be a dislocated quasi b-metric space. A sequence x_{0n} in Y_0 is said to dislocated quasi left converge to $x_0 \in Y_0$ if $\lim_{n\to\infty} d_0(x_{0n}, x_0) = 0$.

Definition 2.3. Let (Y_0, d_0, s) be a dislocated quasi b-metric space. A sequence $\{x_{0n}\}$ in Y_0 is said to dislocated quasi right converge to $x_0 \in Y_0$ if $\lim_{n\to\infty} d_0(x_0, x_{0n}) = 0$.

Definition 2.4 (F. M. Zeyada, G. H. Hassan and M. A. Ahmed [46])**.** Let (Y_0, d_0, s) be a dislocated quasi b-metric space. A sequence $\{x_{0n}\}$ in Y_0 is said to converge to $x_0 \in Y_0$ if it is both dislocated quasi left and right converge to x_0 i.e., $\lim_{n\to\infty} d_0(x_{0n}, x_0) = \lim_{n\to\infty} d_0(x_0, x_{0n}) = 0$.

x_0 is called a dislocated quasi b-limit (or simply limit) of $\{x_{0n}\}$.

Definition 2.5 (F. M. Zeyada, G. H. Hassan and M. A. Ahmed [46])**.** A sequence $\{x_{0n}\}$ in a dislocated quasi b-metric space (Y_0, d_0, s) is called Cauchy if for each $\epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $\forall m, n \geqslant n_0, d_0(x_{0m}, x_{0n}) < \epsilon$ and $d(x_{0n}, x_{0m}) < \epsilon$.

Definition 2.6 (F. M. Zeyada, G. H. Hassan and M. A. Ahmed [46]). A dislocated quasi b-metric space (Y_0, d_0, s) is called complete if every Cauchy sequence in it is dislocated quasi b-convergent.

Definition 2.7. Let $Y_0 \neq \phi$. Suppose $T_0 : Y_0 \to Y_0$ be a self map of Y_0. An element $x_0 \in Y_0$ is called a fixed point of T_0 if $T_0 x_0 = x_0$.

Definition 2.8 (P. Hitzler and A. K. Seda [23]). Let (Y_0, d_0, s) be a dislocated quasi b-metric space. A map $T_0 : Y_0 \to Y_0$ is called a contraction if there exists $0 \leqslant \lambda_0 < 1$ such that $d_0(T_0 x_0, T_0 y_0) \leqslant \lambda_0 d_0(x_0, y_0)\ \forall x_0, y_0 \in Y_0$.

Lemma 2.1[23]. *Let* (Y_0, d_0, s) *be a dislocated quasi b-metric space. Let* $\{x_{0n}\}$ *be a sequence in* Y_0 *that converges to* x_0 *in* Y_0. *Then* $d_0(x_0, x_0) = 0$.

Lemma 2.2[23]. *Let* (Y_0, d_0, s) *be a dislocated quasi b-metric space. Let* $\{x_{0n}\}$ *be a sequence which converges to* x_0 *and* $y_0 \in Y_0$. *Then* $x_0 = y_0$.

Definition 2.9 (Wardowski [44]). Let $\mathcal{F} = \{F : (0, \infty) \to \mathbb{R}/F$ has the following properties$\}$

$F_1 : F$ is strictly increasing.

$F_2 :$ For each sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,

$F_3 :$ There exists $k \in (0, 1)$ such that $\lim_{\alpha \to 0^+} \alpha^k F(\alpha) = 0$.

For a metric space (Y_0, d_0), a mapping $T : Y_0 \to Y_0$ is said to be Wardowski F-contraction if there exists $\tau > 0$ such that $d_0(T_0 x_0, T_0 y_0) > 0$ implies $\tau + F(d_0(T_0 x_0, T_0 y_0)) \leqslant F(d_0(x_0, y_0))\ \forall x_0, y_0 \in Y_0$.

In 2015, Cosentino et al. [11] introduced the following condition in definition 2.9 to obtain some fixed point results in b-metric spaces. This extended definition with the following condition added to definition 2.9.

Suppose $s \geqslant 1$ then $\mathcal{F}_s = \{F \in \mathcal{F} \ni F$ satisfies the following property$\}$

$F_4 :$ For any sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ of positive numbers if $\tau + F(s\alpha_n) \leqslant F(\alpha_{n-1})$ for some $\tau > 0$, then $\tau + F(s^n \alpha_n) \leqslant F(s^{n-1}\alpha_{n-1})$.

We observe $F = \mathcal{F}_1$."

In 2015, Alsulami et al. [3] defined a generalized F-Suzuki type contractions in a b-metric space (Y_0, d_0, s) as a mapping $T_0 : Y_0 \to Y_0$ such that there exists $\tau > 0$ and for all $\forall x_0, y_0 \in Y_0$ with $x_0 \neq y_0$, $\frac{1}{2s} d_0(T_0x_0, T_0y_0) < d_0(x_0, y_0)$ implies

$$\tau + F(d_0(T_0x_0, T_0y_0))$$

$$\leqslant \alpha(F(d_0(x_0, y_0))) + \beta(F(d_0(x_0, T_0x_0))) + \gamma(F(d_0(y_0, T_0y_0))),$$

where $\alpha, \beta \in [0, 1]$ and $\gamma \in [0, 1)$ with $\alpha + \beta + \gamma = 1$ and F satisfies conditions F_1 and F_2.

In 2019, Goswami et al. [21], defined a new type of F-contractive mappings on b-metric spaces with F satisfying conditions F_1, F_2, F_3 and F_4.

Definition 2.10 (Nilakshi Goswami et al. [21])**.** For a b-metric space (Y_0, d_0, s) and $F \in \mathcal{F}_s$, a mapping $T_0 : Y_0 \to Y_0$ is said to be an F-contractive type mapping if there exists $r > 0$ such that $d_0(x_0, T_0x_0)d_0(x_0, T_0y_0) \neq 0$ implies

$$\tau + F(s\, d_0(T_0x_0, T_0y_0)) \leqslant \frac{1}{3}\{F(d_0(x_0, y_0)) + F(d_0(x_0, Tx_0)) + F(d_0(y_0, T_0y_0))\}$$

and $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ implies

$$\tau + F = (sd_0(T_0x_0, T_0y_0))$$

$$\leqslant \frac{1}{3}\{F(d_0(x_0, y_0)) + F(d_0(x_0, T_0y_0)) + F(d_0(y_0, Tx_0))\} \;\forall\, x_0, y_0 \in X_0.$$

3. Main results

In this section, we introduce the notion of F-contractive type mappings on a dislocated quasi b-metric space and obtain fixed point results for such mappings.

Definition 3.1. For a dislocated quasi b-metric space (Y_0, d_0, s), a mapping $T_0 : Y_0 \to Y_0$ is said to be an F-Contractive type mapping if there exists $\tau > 0$ and $F_0 \in \mathcal{F}_s$ such that (i) $d_0(x_0, T_0x_0)d_0(x_0, T_0y_0) \neq 0$ implies

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\})$$

$$+F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\} + F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

whenever $1 \times 2 \times 3 \times 4 \neq 0$, where $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$,

$2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$,

$3 : \max\{d_0(x_0, Tx_0), d_0(T_0x_0, x_0)\}$,

$4 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$,

and (ii) $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ implies

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\})$$

$$+F(\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}) + F(\max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\})\}$$

whenever $1 \times 2 \times 5 \times 6 \neq 0$, where $1 : \max\{d_0(T_0x_0, T_0x_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $5 : \max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}$, $6 : \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}$.

We first prove a lemma which is useful for later development.

Lemma 3.1. *Let* $F, G : (0, \infty) \to \mathbb{R}$ *be such that F and G satisfy* F_1 *and F or G satisfies* F_2 *Then* $(F + G)$ *satisfies both* F_1 *and* F_2.

Proof. Suppose $\alpha_n \to 0$ and F has F_2.

Then $F(\alpha_n) \to -\infty$.

Therefore $F(\alpha_n) + G(\alpha_n) \to -\infty$.

Therefore $(F + G)(\alpha_n) \to -\infty$.

Conversely suppose that $(F+G)(\alpha_n) \to -\infty$.

Then either $F(\alpha_n) \to -\infty$ or $G(\alpha_n) \to -\infty$.

Therefore $\alpha_n \to 0$.

Therefore $(F+G)$ has both F_1 and F_2. □

In view of this lemma, by taking $F(x_0) = \log x_0$ and $G(x_0) = x_0$ for $x_0 > 0$, then $F + G \in \mathcal{F}$. Similarly, by taking $F(x_0) = \log x_0$ and $G(x_0) = \log(x_0 + 1)$ for $x_0 > 0$, the $F + G \in \mathcal{F}_n$.

Now we state and prove our first main theorem.

Theorem 3.1. *Suppose* $s > 1$. *Let* (Y_0, d_0, s) *be a complete dislocated quasi b-metric space and let* $T_0 : Y_0 \to Y_0$ *be an F-contractive type mapping i.e., for some* $\tau > 0$ *and* $F_0 \in \mathcal{F}_s$ *such that* $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$ *implies* $\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}) + F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\})$$

$+F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$ *whenever* $1 \times 2 \times 3 \times 4 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, Tx_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $3 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $4 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, and (ii) $d_0(x_0, T_0x_0)d_0(y, T_0y_0) = 0$ *implies*

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_0, y_0), d_0(y_0x_0)\})$$

$$+F(\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}) + F(\max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\})\}$$

whenever $1 \times 2 \times 5 \times 6 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $3 : \max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}$, $4 : \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}$. *Then* T_0 *has one and only one fixed point.*

Proof. Let $x_0 \in Y_0, \tau > 0$ be as in definition (3.1) and consider the sequence $\{x_{0n}\}$ where $x_{0n+1} = T_0 x_{0n}$, $n = 0, 1, 2, \ldots$.

Denote $d_0(x_{0n}, x_{0n+1})$ by u_n and $d_0(x_{0n+1}, x_{0n})$ by v_n and we may suppose that either $u_n > 0$ or $v_n > 0$ $\forall\, n \in \mathbb{N}$.

Hence we may suppose that $u_n > 0$ for infinitely many n.

Then since T_0 is an F-Contractive type mapping, when $T_0 x_{0n} \neq x_{0n}$, we have

$$\tau + F(s \max\{d_0(T_0 x_{0n}, T_0 x_{0n+1}), d_0(T_0 x_{0n+1}, T_0 x_{0n})\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_{0n}, x_{0n+1}), d(x_{0n+1} x_{0n})\})$$

$$+F(\max\{d_0(x_{0n}, T_0 x_{0n}), d_0(T_0 x_{0n}, x_{0n})\})$$

$$+F(\max\{d_0(x_{0n+1}, T_0 x_{0n+1}), d_0(T_0 x_{0n+1}, x_{0n+1})\})\}$$

$$\tau + F(s \max\{d_0(T_0 x_{0n+1}, x_{0n+2}), d_0(x_{0n+2}, x_{n+1})\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_{0n}, x_{0n+1}), d(x_{0n+1} x_{0n})\})$$

$$+F(\max\{d_0(x_{0n}, x_{0n+1}), d(x_{0n+1}, x_{0n})\})$$

$$+F(\max\{d_0(x_{0n+1}, x_{0n+1}), d(x_{0n+2}, x_{0n+1})\})\}$$

$$\tau + F(s \max\{u_{n+1}, v_{n+1}\}) \leqslant \frac{1}{3}\{F(\max\{u_n, v_n\}) + F(\max\{u_n, v_n\})$$

$$+F(\max\{u_{n+1}, v_{n+1}\})\}$$

$$\tau + F(\max\{u_{n+1}, v_{n+1}\}) - \frac{1}{3} F(\max\{u_{n+1}, v_{n+1}\} \leqslant \frac{2}{3} F(\max\{u_n, v_n\})$$

$$\tau + \frac{2}{3} F(s \max\{u_{n+1}, v_{n+1}\}) < \tau + F(s \max\{u_{n+1}, v_{n+1}\})$$

$$-\frac{1}{3} F(\max\{u_{n+1}, v_{n+1}\}) \leqslant \frac{2}{3} F(\max\{u_n, v_n\})$$

$$\tau + \frac{2}{3} F(s \max\{u_{n+1}, v_{n+1}\}) \leqslant \frac{2}{3} F(\max\{u_n, v_n\})$$

Multiplying on both sides with $\frac{3}{2}$, we get

$$\frac{3}{2}\tau + F(s\max\{u_{n+1}, v_{n+1}\}) \leqslant F(\max\{u_n, v_n\})$$

write $w_n = \max\{u_n, v_n\}$

$$\frac{3}{2}\tau + F(sw_{n+1}) \leqslant F(w_n)$$

$$\frac{3}{2}\tau + F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) \text{ for } n = 0, 1, 2, \ldots \text{ (from } F_4)$$

$$F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) - \frac{3}{2}\tau$$

By induction, from F_4 we have

$$F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) - \frac{3}{2}\tau$$

and hence by induction

$$F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) - \frac{3}{2}\tau \leqslant \ldots \leqslant F(w_0) - \frac{3}{2}n\tau \tag{3.1}$$

In the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} F(s^{n+1}w_{n+1}) = -\infty,$$

so that $\lim_{n\to\infty} s^{n+1}w_{n+1} = 0$. (from F_2)

From condition F_3, there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty}(s^{n+1}w_{n+1})^k F(s^{n+1}w_{n+1}) = 0.$$

Multiplying (3.1) with $(s^{n+1}w_{n+1})^k$ yields

$$(s^{n+1}w_{n+1})^k F(s^{n+1}w_{n+1}) + \frac{3}{2}n(s^{n+1}w_{n+1})^k \tau \leqslant (s^{n+1}w_{n+1})^k F(w_0). \tag{3.2}$$

on taking the limit as $n \to \infty$ from (3.2), we get

$$\lim_{n\to\infty} n(s^{n+1}w_{n+1})^k = 0.$$

This inequality implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^{n+1}w_{n+1})^k \leqslant 1, \forall n \geqslant n_1$. Thus

$$s^{n+1}w_{n+1} \leqslant \frac{1}{n^{\frac{1}{k}}} \tag{3.3}$$

$\forall n \geqslant n_1$.

Now we prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

For $n = 0, 1, 2, \ldots$ and $p = 1, 2, 3, \ldots,$ the following chain of inequalities holds:

$$d_0(x_{0n}, x_{0n+p}) \leqslant s\{d_0(x_{0n}, x_{0n+1}) + d_0(x_{0n+1}, x_{0n+p})\}$$

$$= su_n + sd_0(x_{0n+1}, x_{0n+p})$$

$$\leqslant su_n + s\{sd_0(x_{0n+1}, x_{0n+2}) + sd_0(x_{0n+2}, x_{0n+p})\}$$

$$= su_n + s^2u_{n+1} + s^2d_0(x_{0n+2}, x_{0n+p})$$

$$\leqslant su_n + s^2u_{n+1} + s^3u_{n+2} + s^3d(d_{n+3}, x_{0n+p})$$

$$\vdots$$

$$\leqslant su_n + s^2u_{n+1} + s^3u_{n+2} + \ldots + s^{p-1}u_{n+p-2} + s^{p-1}u_{n+p-1}$$

$$\leqslant su_n + s^2u_{n+1} + s^3u_{n+2} + \ldots + s^{p-1}u_{n+p-2} + s^pu_{n+p-1}$$

$$= \frac{1}{s^{n-1}}\sum\nolimits_{i=1}^{n+p-1} s^iu_i.$$

Hence $\forall n \geqslant n_1$ and $p \geqslant 1$ inequality (3.3) implies

$$d(x_{0n}, x_{0n+p}) \leqslant \frac{1}{s^{n-1}}\sum\nolimits_{i=n}^{\infty} s^iu_i \leqslant \frac{1}{s^{n-1}}\sum\nolimits_{i=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \to 0. \tag{3.4}$$

Similarly

$$d(x_{0n+p}, x_{0n}) \leqslant \frac{1}{s^{n-1}} \sum_{i=n}^{\infty} s^i v_i \leqslant \frac{1}{s^{n-1}} \sum_{i=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \to 0. \tag{3.5}$$

From (3.4) and (3.5), $\{x_{0n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Since (Y_0, d_0, s) is complete, there exists $z_0 \in Y_0$ such that $\lim_{n\to\infty} x_{0n} = z_0$. Suppose $d_0(T_0 z_0, z_0) \neq 0$, Also $u_0 > 0$, for infinitely many n, by our assumption.

Then $\tau + F(s \max\{d_0(T_0 z_0, T_0 x_{0n+1}), d_0(T_0 x_{0n+1}, T_0 z_0)\})$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(z_0, x_{0n+1}), d(x_{0n+1}, z_0)\}) + F(\max\{d_0(z_0, T_0 z_0), d_0(T_0 z_0, z_0)\})$$

$$+F(\max\{d_0(x_{0n+1}, T_0 x_{0n+1}), d(T_0 x_{0n+1}, x_{0n+1})\})\}$$

$$\tau + F(s \max\{d_0(T_0 z_0, T_0 x_{0n+2}), d_0(T_0 x_{0n+2}, T_0 z_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(z_0, x_{0n+1}), d_0(x_{0n+1}, z_0)\}) + F(\max\{d_0(z_0, T_0 z_0), d_0(T_0 z_0, z_0)\})$$

$$+F(\max\{d_0(x_{0n+1}, T_0 x_{0n+2}), d_0(T_0 x_{0n+2}, x_{0n+1})\})\}$$

Now the right-hand side $\to -\infty$.

Since $\max\{d_0(z_0, x_{0n+1}), d_0(x_{0n+1}, z_0)\}) \to 0$

Hence $F(s \max\{d_0(T_0 z_0, x_{0n+2}), d_0(x_{0n+2}, T_0 z_0)\}) \to -\infty$

Consequently, $s \max\{d_0(T_0 z_0, x_{0n+2}), d_0(x_{0n+2}, T_0 z_0)\}) \to 0.$

Therefore $d(T_0 z_0, x_{0n+2}) \to 0,$

and $d(x_{0n+2}, T_0 z_0) \to 0.$

Therefore $x_{0n+2} \to T_0 z_0.$

But $x_{0n+2} \to z_0.$

Therefore $T_0 z_0 = z_0.$

Therefore z_0 is a fixed point of $T_0.$

To show the uniqueness of the fixed point, let z_0' be another fixed point of T_0 with $z_0 \neq z_0'$.

Then

$$\tau + F(s\max\{d_0(T_0z_0, T_0z_0'), d_0(T_0z_0', T_0z_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\}) + F(\max\{d_0(z_0, T_0z_0'), d_0(T_0z_0', z_0)\})$$

$$+F(\max\{d_0(z_0', T_0z_0), d_0(T_0z_0, z_0')\})\})$$

$$\tau + F(s\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\}) + F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$+F(\max\{d_0(z_0', z_0), d_0(z_0, z_0')\})\}$$

$$\tau + F(s\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\}) \leqslant F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\tau + F(d_0(z_0, z_0')) \leqslant F(d_0(z_0', z_0))$$

Hence $\tau \leqslant 0$

which is a contradiction. This proves the result. □

Example 3.1. Consider the dislocated quasi b-metric space (Y_0, d_0, s) where $Y_0 = [0, 1]$ and

$$d_0(x_0, y_0) = \begin{cases} (x_0^2 + y_0)^2 & \text{if } x_0 \neq y_0 \\ 0 & \text{if } x_0 = y_0, \end{cases}$$

Y_0 is complete with $s = 2$. Take $F(x_0) = \log x_0$ for $x_0 > 0$. Let the mapping $T_0 : Y_0 \to Y_0$ be defined by

$$T_0x_0 = \begin{cases} \frac{1}{2} & \text{if } 0 \leqslant x_0 < 1 \\ 0 & \text{if } x_0 = 1. \end{cases}$$

Then T_0 is F-contractive type mapping and hence has unique fixed point. In fact, T_0 has a unique fixed point $x_0 = \frac{1}{2}$.

Proof. We show that T_0 is F-contractive type mapping with $F(x_0) = \log x_0$ for $x_0 > 0$ and $\tau = \frac{2}{3}\log 2$.

Consider the inequality

$$\tau + F(s\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}) + F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\})$$

$$+F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

whenever $1 \times 2 \times 3 \times 4 \neq 0$, where $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $3 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $4 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$.

Now for $x_0, y_0 \in [0, 1)$ and $x_0 \neq y_0$ inequality (3.6) does not hold.

And if $x_0, y_0 = 1$ the inequality (3.6) does not hold.

Again for $x_0 \in [0, 1)$ and $y_0 = 1$,

$$\tau + F(s\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}) + F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\})$$

$$+F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

$$\Rightarrow \tau + \log(\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{3}\{\log(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}) + \log(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\})$$

$$+\log(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

$$\Rightarrow \tau + \log\left(\max\left\{d_0\left(\frac{1}{2}, 0\right), d_0\left(0, \frac{1}{2}\right)\right\}\right)$$

$$\leqslant \frac{1}{3}\{\log(\max\{d_0(1, x_0), d_0(x_0, 1)\}) + \log\left(\max\left\{d_0\left(x_0, \frac{1}{2}\right), d_0\left(\frac{1}{2}, x_0\right)\right\}\right)$$

$$+\log(\max\{d_0(1, 0), d_0(0, 1)\})\}$$

$$\Rightarrow \tau + \log\left(\max\left\{\frac{1}{16}, \frac{1}{4}\right\}\right)$$

$$\leqslant \frac{1}{3}\{\log(\max\{d_0(x_0^2+1)^2, d_0(1+x_0)^2\})$$

$$+\log\left(\max\left\{d_0\left(x_0^2+\frac{1}{2}\right)^2, d_0\left(\frac{1}{4}+x_0\right)^2\right\}\right)$$

$$+\log(\max\{1, 1\})\}$$

$$\Rightarrow \tau + \log\frac{1}{4} \leqslant \frac{1}{3}\log\left((1+x_0)^2 \times \left(\frac{1}{4}\right)^2 \times 1\right)$$

When $y_0 = 0$, we get

$$\tau + \log\frac{1}{4} \leqslant \frac{1}{3}\log\frac{1}{16}$$

$$\tau + \log 1 - \log 4 \leqslant \frac{1}{3}\log 1 - \frac{1}{3}\log 16$$

$$\tau - \log 4 \leqslant -\frac{1}{3}\log 16$$

$$\tau \leqslant \log 4 - \frac{1}{3}\log 16$$

$$\tau \leqslant \log 4 - \frac{4}{3}\log 2$$

$$\tau \leqslant 2\log 2 - \frac{4}{3}\log 2$$

$$\tau \leqslant \log 2\left(2 - \frac{4}{3}\right)$$

$$\tau = \frac{2}{3}\log 2.$$

Similarly for $y_0 = 1$ and $y_0 \in [0, 1)$, we have $\tau = \frac{2}{3}\log 2.$

Thus T_0 is an F-contractive type mapping with $\tau = \frac{2}{3}\log 2$, and T_0 has unique fixed point. We observe that $y_0 = \frac{1}{2}$ is the unique fixed point of T_0.

Definition 3.2. For a dislocated quasi b-metric space (Y_0, d_0, s), a mapping $T_0 : Y_0 \to Y_0$ is said to be an F-contraction type mapping with index $\lambda_0 \in [0, 1)$, if there exists $\tau > 0$ and $F \in \mathcal{F}_s$ such that $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$ implies

$$\tau + F(s\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \lambda\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$$

$$+ F(\frac{\max\{d_0(x_0, T_0x_0), d_0(Tx_0, x_0)\} + \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}}{2})\}$$

whenever $1 \times 2 \times 3 \times 4 \neq 0$, where $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $3 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $4 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, and $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ implies

$$\tau + F(s\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \lambda\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$$

$$+ F(\frac{\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\} + \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}}{2})\}$$

whenever $1 \times 2 \times 5 \times 6 \neq 0$, where $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $5 : \max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}$, $6 : \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}$.

Theorem 3.2. *Suppose* $s > 1$. *Let* (Y_0, d_0, s) *be a complete dislocated quasi b-metric space and let* $T_0 : Y_0 \to Y_0$ *be an F-contractive type mapping with index* λ_0 *i.e., for some* $\tau > 0, 0 \leqslant \lambda < \frac{1}{2}$ *and* $F \in \mathcal{F}_s$ *such that* $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$ *implies*

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \lambda_0\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$$

$$+ F(\frac{\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\} + \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}}{2})\}$$

whenever $1 \times 2 \times 3 \times 4 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $3 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $4 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, *and* $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ *implies*

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \lambda_0\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$$

$$+ F(\frac{\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\} + \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}}{2})\}$$

whenever $1 \times 2 \times 5 \times 6 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $5 : \max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}$, $6 : \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}$. *Then* T_0 *has one and only one fixed point.*

Proof. Let $x_0 \in Y_0$ and consider the sequence $\{x_{0n}\}$ where $x_{0n+1} = T_0x_{0n}$, $n = 0, 1, 2, \ldots$.

Denote $d(x_{0n}, x_{0n+1})$ by u_n and $d(x_{0n+1}, x_{0n})$ by v_n and we may suppose that either $u_n > 0$ or $v_n > 0$ $\forall\, n \in \mathbb{N}$.

Hence we may suppose that $u_n > 0$ for infinitely many n.

Then since T_0 is an F-contractive type mapping when $T_0x_{0n} \neq x_{0n}$, we have

$$\tau + F(s \max\{d_0(T_0x_0, T_0x_{0n+1}), d(T_0x_{0n+1}, T_0x_{0n})\})$$

$$\leqslant \lambda_0\{F(\max\{d(x_{0n}, x_{0n+1}), d(x_{0n+1}, x_{0n})\})$$

$$+ F\left(\frac{\max\{d(x_{0n}, T_0x_{0n}), d(T_0x_{0n}, x_{0n})\} + \max\{d(x_{0n+1}, T_0x_{0n+1}), d(T_0x_{0n+1}, x_{0n+1})\}}{2}\right)\}$$

$$\tau + F(s\max\{d(x_{0n+1}, x_{0n+2}), d(x_{0n+2}, x_{0n+1})\})$$

$$\leqslant \lambda_0\{F(\max\{d(x_{0n}, x_{0n+1}), d(x_{n+1}, x_{0n})\})$$

$$+ F\left(\frac{\max\{d(x_{0n}, x_{0n+1}), d(x_{0n+1}, x_{0n})\} + \max\{d(x_{0n+1}, x_{0n+2}), d(x_{0n+2}, x_{0n+1})\}}{2}\right)\}$$

$$\tau + F(s\max\{u_{n+1}, v_{n+1}\}) \leqslant \lambda_0\{F(\max\{u_n, v_n\})$$

$$+ F\left(\frac{\max\{u_n, v_n\} + \max\{u_{n+1}, v_{n+1}\}}{2}\right)\}$$

Suppose $w_{n+1} = \max\{u_{n+1}, v_{n+1}\}$

$$\tau + F(sw_{n+1}) \leqslant \lambda_0\left\{F(w_n) + F\left(\frac{w_n + w_{n+1}}{2}\right)\right\}$$

Suppose $w_{n+1} \geqslant w_n$. Then

$$\tau + F(sw_{n+1}) \leqslant \lambda_0 F(w_n) + \lambda_0 F(w_{n+1})$$

$$\tau + F(sw_{n+1}) - \lambda_0 F(w_{n+1}) \leqslant \lambda_0 F(w_n)$$

$$\tau + (1-\lambda_0)F(sw_{n+1}) < \tau + F(sw_{n+1}) - \lambda_0 F(w_{n+1}) \leqslant \lambda_0 F(w_n)$$

Hence $\frac{\tau}{1-\lambda_0} + F(sw_{n+1}) \leqslant \frac{\lambda_0}{1-\lambda_0} F(w_n)$

$$\frac{\tau}{1-\lambda_0} + F(s^{n+1}w_{n+1}) \leqslant \frac{\lambda_0}{1-\lambda_0} F(s^n w_n) \text{ for } n = 0, 1, 2, \ldots \text{ (from } F_4)$$

Suppose $0 \leqslant \lambda < \frac{1}{2}$. Then

$$F(s^{n+1}w_{n+1}) \leqslant \frac{\lambda_0}{1-\lambda_0} F(s^n w_n) < F(s^n w_n).$$

Thus $s^{n+1}w_{n+1} < s^n w_n$.

Therefore $s^{n+1}w_{n+1} < s^n w_n$.

By induction

$$F(s^{n+1}w_{n+1}) < F(s^n w_n) - \tau < F(s^{n-1}w_{n-1}) - 2\tau < \ldots < F(w_0) - n\tau$$

$$F(s^{n+1}w_{n+1}) < F(w_0) - n\tau \tag{3.7}$$

We get for limit $n \to \infty$, as

$\lim_{n\to\infty} F(s^{n+1}w_{n+1}) = -\infty$,

so that $\lim_{n\to\infty} s^{n+1}w_{n+1} = 0$. (from F_2)

From condition (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty}(s^{n+1}w_{n+1})^k F(s^{n+1}w_{n+1}) = 0.$$

Multiplying (3.7) with $(s^{n+1}w_{n+1})^k$ yields

$$(s^{n+1}w_{n+1})^k F(s^{n+1}w_{n+1})^k + n(s^{n+1}w_{n+1})^k \tau \leqslant (s^{n+1}w_{n+1})^k F(w_0). \tag{3.8}$$

Taking the limit as $n \to \infty$ to (3.8), we get

$$\lim_{n\to\infty} n(s^{n+1}w_{n+1})^k = 0.$$

This inequality implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^{n+1}w_{n+1})^k \leqslant 1, \forall n \geqslant n_1$. Thus

$$s^{n+1}w_{n+1} \leqslant \frac{1}{n^{\frac{1}{k}}} \tag{3.9}$$

for all $n \geqslant n_1$.

Now we prove that $\{x_{0_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

For $n = 0, 1, 2, \ldots$ and $p = 1, 2, 3, \ldots$, the following chain of inequalities holds:

$$d(x_{0n}, x_{0n+p}) \leqslant s\{d_0(x_{0n}, x_{0n+1}) + d(x_{0n+1}, x_{0n+p})\}$$

$$= su_n + sd_0(x_{0n+1}, x_{0n+p})$$

$$\leqslant su_n + s\{sd(x_{0n+1}, x_{0n+2}) + sd(x_{0n+2}, x_{0n+p})\}$$

$$= su_n + s^2 u_{n+1} + s^2 d(x_{0n+2}, x_{0n+p})$$

$$\leqslant su_n + s^2 u_{n+1} + s^3 u_{n+2} + s^3 d(d_{n+3}, x_{0_{n+p}})$$

$$\vdots$$

$$\leqslant su_n + s^2 u_{n+1} + s^3 u_{n+2} + \ldots + s^{p-1} u_{n+p-2} + s^{p-1} u_{n+p-1}$$

$$\leqslant su_n + s^2 u_{n+1} + s^3 u_{n+2} + \ldots + s^{p-1} u_{n+p-2} + s^{p} u_{n+p-1}$$

$$= \frac{1}{s^{n-1}} \sum_{i=1}^{n+p-1} s^i u_i.$$

Hence $\forall\, n \geqslant n_1$ and $p \geqslant 1$ inequality 3.9 implies

$$d(x_{0_n}, x_{0_{n+p}}) \leqslant \frac{1}{s^{n-1}} \sum_{i=n}^{\infty} s^i u_i \leqslant \frac{1}{s^{n-1}} \sum_{i=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \to 0. \tag{3.10}$$

Similarly

$$d(x_{0n+p}, x_{0n}) \leqslant \frac{1}{s^{n-1}} \sum_{i=n}^{\infty} s^i v_i \leqslant \frac{1}{s^{n-1}} \sum_{i=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \to 0. \tag{3.11}$$

From (3.10) and (3.11), $\{x_{0_n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Since (Y_0, d_0, s) is complete, there exists $z_0 \in Y_0$ such that $\lim_{n \to \infty} x_{0n} = z_0$. Now,

$$\tau + F(s \max\{d_0(T_0 z_0, T_0 x_{0n+1}), d(T_0 x_{0n+1}, T_0 z_0)\})$$

$$\leqslant \lambda_0 \{F(\max\{d(z_0, x_{0n+1}), d(x_{0n+1}, z_0)\})$$

$$+ F\left(\frac{\max\{d_0(z_0, T_0 z_0), d_0(T_0 z_0, z_0)\} + \max\{d_0(x_{0n+1}, T_0 x_{0n+1}), d_0(T_0 x_{0n+1}, x_{0n+1})\}}{2}\right)\}$$

$$\tau + F(s \max\{d_0(T_0z_0, x_{0n+2}), d_0(x_{0n+2}, T_0z_0)\})$$

$$\leqslant \lambda_0\{F(\max\{d_0(z_0, x_{0n+1}), d_0(x_{0n+1}, z_0)\})$$

$$+ F(\frac{\max\{d_0(z_0, T_0z_0), d_0(T_0z_0, z_0)\} + \max\{d_0(x_{0n+1}, x_{0n+2}), d_0(x_{0n+2}, x_{0n+1})\}}{2})\}$$

Now the right-hand side $\to -\infty$.

Since $\max\{d_0(z_0, x_{0n+1}), d_0(x_{0n+1}, z_0)\} \to 0$

Hence $F(s \max\{d_0(T_0z_0, x_{0n+2}), d_0(x_{0n+2}, T_0z_0)\}) \to -\infty$

Consequently, $s \max\{d_0(T_0z_0, x_{0n+2}), d_0(x_{0n+2}, T_0z_0)\} \to 0$.

$$F(s \max\{d_0(T_0z_0, x_{0n+2}), d(x_{0n+2}, T_0z_0)\}) \to -\infty$$

Therefore $d_0(T_0z_0, x_{0n+2}) \to 0$,

and $d_0(x_{0n+2}, T_0z_0) \to 0$.

Therefore $x_{0n+2} \to T_0z_0$.

But $x_{0n+2} \to z_0$.

Therefore $T_0z_0 = z_0$.

Therefore z_0 is a fixed point of T_0.

To show the uniqueness of the fixed point, let z_0' be another fixed point of T_0 with $z_0 \neq z_0'$. Then

$$\tau + F(s \max\{d_0(T_0z_0, T_0z_0'), d_0(T_0z_0', T_0z_0)\})$$

$$\leqslant \lambda_0\{F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$+ F(\frac{\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\} + \max\{d_0(z_0', z_0), d_0(z_0, z_0')\}}{2})\}$$

$$\tau + F(s \max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\leqslant \lambda_0\{F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$+ F(\frac{\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\} + \max\{d_0(z_0', z_0), d_0(z_0, z_0')\}}{2})\}$$

$$\tau + F(s \max\{d_0(z_0, z_0'), d_0(z_0', z_0)\}) \leqslant 2\lambda_0 F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\tau + F(d_0(z_0, z_0')) \leqslant 2\lambda_0 F(d_0(z_0, z_0'))$$

$$\tau + F(d_0(z_0, z_0')) - 2\lambda_0 F(d_0(z_0, z_0')) \leqslant 0$$

which is a contradiction.

This proves the result. □

Definition 3.3. Let (Y_0, d_0, s) be a complete dislocated quasi b-metric space. A mapping $T_0 : Y_0 \to Y_0$ is said to be a Kannan F-contractive type mapping if there exists $\tau > 0$ and $F \in \mathcal{F}_s$ such that $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$ implies

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}) + F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

whenever $1 \times 2 \times 3 \neq 0$, where $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $3 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, and $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ implies

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}) + F(\max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\})\}$$

whenever $1 \times 4 \times 5 \neq 0$, where $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $4 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $5 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$.

Theorem 3.3. *Let (Y_0, d_0, s) be a complete dislocated quasi b-metric space and $T_0 : Y_0 \to Y_0$ be a Kannan F-contractive type mapping i.e., for some $\tau > 0$ and $F \in \mathcal{F}_s$ such that $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$ implies*

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\})$$

$$+ F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

whenever $1 \times 2 \times 3 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $3 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, *and* $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ *implies*

$$\tau + F(s\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\})$$

$$+F(\max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\})\}$$

whenever $1 \times 2 \times 3 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $3 : \max\{d_0(y_0, T_0y_0), d_0(T_0x_0, y_0)\}$.

Then T_0 *has one and only one fixed point.*

Proof. Let $x_0 \in Y_0$ and consider the sequence $\{x_{0n}\}$ where $x_{0n+1} = T_0x_{0n}$, $n = 0, 1, 2, \ldots$.

Denote $d(x_{0n}, x_{0n+1})$ by v_n and $d(x_{0n+1}, x_{0n})$ by v_n and we may suppose that either $u_n > 0$ or $v_n > 0$ $\forall\, n \in \mathbb{N}$.

Hence we may suppose that $u_n > 0$ for infinitely many n.

Then since T_0 is a Kannan F-contractive type mapping when $T_0x_{0n} \neq x_{0n}$, we have

$$\tau + F(s\max\{d_0(T_0x_{0n}, T_0x_{0n+1}), d(T_0x_{0n+1}, T_0x_{0n})\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(x_{0n}, T_0x_{0n}), d_0(T_0x_{0n}, x_{0n})\})$$

$$+F(\max\{d_0(x_{0n+1}, T_0x_{0n+1}), d_0(T_0x_{0n+1}, x_{0n+1})\})\}$$

$$\tau + F(s\max\{d_0(x_{0n+1}, x_{0n+2}), d_0(x_{0n+2}, x_{0n+1})\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(x_{0n}, x_{0n+1}), d_0(x_{0n+1}, x_{0n})\})$$

$$+F(\max\{d_0(x_{0n+1}, x_{0n+2}), d_0(x_{0n+2}, x_{0n+1})\})\}$$

$$\tau + F(s\max\{u_{n+1}, v_{n+1}\}) \leqslant \frac{1}{2}\{F(\max\{u_n, v_n\}) + F(\max\{u_{n+1}, v_{n+1}\})\}$$

$$\tau + F(s\max\{u_{n+1}, v_{n+1}\}) - \frac{1}{2}F(\max\{u_{n+1}, v_{n+1}\}) \leqslant \frac{1}{2}F(\max\{u_n, v_n\})$$

$$\tau + \frac{1}{2}F(s\max\{u_{n+1}, v_{n+1}\}) < \tau + F(s\max\{u_{n+1}, v_{n+1}\})$$

$$-\frac{1}{2}F(\max\{u_{n+1}, v_{n+1}\}) \leqslant \frac{1}{2}F(\max\{u_n, v_n\})$$

$$\tau + \frac{1}{2}F(s\max\{u_{n+1}, v_{n+1}\}) \leqslant \frac{1}{2}F(\max\{u_n, v_n\})$$

Multiplying with 2 on both sides, we get

$$2\tau + F(s\max\{u_{n+1}, v_{n+1}\}) \leqslant F(\max\{u_n, v_n\})$$

write $w_n = \max\{u_n, v_n\}$

$$2\tau + F(sw_{n+1}) \leqslant F(w_n)$$

$$2\tau + F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) \text{ for } n = 0, 1, 2, \ldots \text{ (from } F_4)$$

$$F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) - 2\tau$$

By induction, from F_4 we have

$$F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) - 2\tau$$

and hence by induction

$$F(s^{n+1}w_{n+1}) \leqslant F(s^n w_n) - 2\tau \leqslant \ldots \leqslant F(w_0) - 2n\tau \tag{3.12}$$

In the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} F(s^{n+1}w_{n+1}) = -\infty,$$

so that $\lim_{n\to\infty} s^{n+1}w_{n+1} = 0$. (from F_2)

From condition F_3 there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty}(s^{n+1}w_{n+1})^k F(s^{n+1}w_{n+1}) = 0.$$

Multiplying (3.12) with $(s^{n+1}w_{n+1})^k$ yields

$$(s^{n+1}w_{n+1})^k F(s^{n+1}w_{n+1}) + 2n(s^{n+1}w_{n+1})^k \tau \leqslant (s^{n+1}w_{n+1})^k F(w_0). \tag{3.13}$$

Taking the limit as $n \to \infty$ to 3.13, we get

$$\lim_{n\to\infty} n(s^{n+1}w_{n+1})^k = 0.$$

This inequality implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^{n+1}w_{n+1})^k \leqslant 1, \forall\, n \geqslant n_1$.

Thus

$$s^{n+1}w_{n+1} \leqslant \frac{1}{n^{\frac{1}{k}}} \tag{3.14}$$

for all $n \geqslant n_1$.

Now we prove that $\{x_{0n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

For $n = 0, 1, 2, \ldots$ and $p = 1, 2, 3, \ldots,$ the following chain of inequalities holds:

$$\begin{aligned} d_0(x_{0n}, x_{0n+p}) &\leqslant s\{d_0(x_{0n}, x_{0n+1}) + d_0(x_{0n+1}, x_{0n+p})\} \\ &= su_n + sd_0(x_{0n+1}, x_{0n+p}) \\ &\leqslant su_n + s\{sd_0(x_{0n+1}, x_{0n+2}) + sd_0(x_{0n+2}, x_{0n+p})\} \\ &= su_n + s^2u_{n+1} + s^2d_0(x_{0n+2}, x_{0n+p}) \\ &\leqslant su_n + s^2u_{n+1} + s^3u_{n+2} + s^3d_0(d_{n+3}, x_{0n+p}) \\ &\;\;\vdots \end{aligned}$$

$$\leqslant su_n + s^2 u_{n+1} + s^3 u_{n+2} + \ldots + s^{p-1} u_{n+p-2} + s^{p-1} u_{n+p-1}$$

$$\leqslant su_n + s^2 u_{n+1} + s^3 u_{n+2} + \ldots + s^{p-1} u_{n+p-2} + s^{p} u_{n+p-1}$$

$$= \frac{1}{s^{n-1}} \sum_{i=1}^{n+p-1} s^i u_i.$$

Hence $\forall\, n \geqslant n_1$ and $p \geqslant 1$ inequality 3.3 implies

$$d(x_{0n}, x_{0n+p}) \leqslant \frac{1}{s^{n-1}} \sum_{i=n}^{\infty} s^i u_i \leqslant \frac{1}{s^{n-1}} \sum_{i=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \to 0. \tag{3.15}$$

Similarly

$$d(x_{0n+p}, x_{0n}) \leqslant \frac{1}{s^{n-1}} \sum_{i=n}^{\infty} s^i v_i \leqslant \frac{1}{s^{n-1}} \sum_{i=1}^{\infty} \frac{1}{n^{\frac{1}{k}}} \to 0. \tag{3.16}$$

From (3.15) and (3.16), $\{x_{0n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since (Y_0, d_0, s) is complete, there exists $z_0 \in Y_0$ such that $\lim_{n\to\infty} x_{0n} = z_0$. Now,

$$\tau + F(s \max\{d_0(T_0 z_0, T_0 x_{0n+1}), d_0(T_0 x_{0n+1}, T_0 z_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(z_0, T_0 z_0), d_0(T_0 z_0, z_0)\})$$

$$+ F(\max\{d_0(x_{0n+1}, T_0 x_{0n+1}), d_0(T x_{0n+1}, x_{0n+1})\})\}$$

$$\tau + F(s \max\{d_0(T_0 z_0, x_{0n+2}), d_0(x_{0n+2}, T_0 z_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(z_0, T_0 z_0), d_0(T_0 z_0, z_0)\})$$

$$+ F(\max\{d_0(x_{0n+1}, x_{0n+2}), d_0(x_{0n+2}, x_{0n+1})\})\}$$

Now the right-hand side $\to -\infty$.

Since $\max\{d_0(z_0, x_{0n+1}), d_0(x_{0n+1}, z_0)\} \to 0$

Hence $F(s \max\{d_0(T_0 z_0, x_{0n+2}), d_0(x_{0n+2}, T_0 z_0)\} \to -\infty$

Consequently, $s \max\{d_0(T_0z_0, x_{0n+2}), d_0(x_{0n+2}, T_0z_0)\} \to 0.$

$$F(s \max\{d_0(T_0z_0, x_{0n+2}), d_0(x_{0n+2}, T_0z_0)\}) \to -\infty.$$

Therefore $d_0(T_0z_0, x_{0n+2}) \to 0,$

and $d_0(x_{0n+2}, T_0z_0) \to 0.$

Therefore $x_{0n+2} \to T_0z_0.$

But $x_{0n+2} \to z_0.$

Therefore $T_0z_0 = z_0.$

Therefore z_0 is a fixed point of T_0.

To show the uniqueness of the fixed point, let z_0' be another fixed point of T_0 with $z_0 \neq z_0'$. Then

$$\tau + F(s \max\{d_0(T_0z_0, T_0z_0'), d_0(T_0z_0', T_0z_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(z_0, T_0z_0'), d_0(T_0z_0', z_0)\})$$

$$+ F(\max\{d_0(z_0', T_0z_0), d_0(T_0z_0, z_0')\})\}$$

$$\tau + F(s \max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$+ F(\max\{d_0(z_0', z_0), d_0(z_0, z_0')\})\}$$

$$\tau + F(s \max\{d_0(z_0, z_0'), d_0(z_0', z_0)\}) \leqslant F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\tau + F(d_0(z_0, z_0')) \leqslant F(d_0(z_0, z_0'))$$

which is a contradiction.

This proves the result. □

Now we introduce the notion of a boundedly compact dislocated quasi-b-metric space Y_0 and obtain a fixed point result for a self-map on Y_0.

This is an extension of the notion as boundedly compact metric space [17], to dislocated quasi-b-metric spaces.

Definition 3.4. A dislocated quasi-b-metric space (Y_0, d_0, s) is said to be boundedly compact if every boundedly sequence in Y_0 has a convergent subsequence.

Theorem 3.4. *Let* (Y_0, d_0, s) *be a boundedly compact dislocated quasi* b*-metric space. Suppose* $T_0 : Y_0 \to Y_0$ *be a Kannan F-contractive type mapping. Then* T_0 *has one and only one fixed point.*

Proof. Since (Y_0, d_0, s) is boundedly compact, every Cauchy sequence is bounded and hence contains a convergent subsequence, consequently the sequence itself is convergent (being Cauchy). Thus Y_0 is complete, now the result follows from Theorem 3.3. □

Now we extend the definition of asymptotically regular maps on dislocated quasi b-metric spaces, which is an extension of the notation available in metric spaces [9].

Definition 3.5. For a dislocated quasi b-metric space (Y_0, d_0, s), a mapping $T_0 : Y_0 \to Y_0$ is called asymptotically regular if $\lim_{n\to\infty}(T_0^n x, T_0^{n+1}x_0) = 0$ and $\lim_{n\to\infty}(T_0^{n+1}x_0, T_0^n x_0) = 0\ \forall\, x_0 \in X_0$.

Theorem 3.5. *Let* (Y_0, d_0, s) *be a complete dislocated quasi* b*-metric space. Suppose* $T_0 : Y_0 \to Y_0$ *is an asymptotically regular mapping such that, for some* $\tau > 0$ *and* $F \in \mathcal{F}_s$, *if* $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0 \neq 0$ *implies*

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, Tx_0)\}) + F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})$$

whenever $1 \times 2 \times 3 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $2 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $3 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, *and* $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ *implies*

$$\tau + F(s \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$\leqslant F(\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}) + F(\max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\})$$

whenever $1 \times 4 \times 5 \neq 0$, *where* $1 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $4 : \max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}$, $5 : \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}$.

Then T_0 *has a fixed point* $z_0 \in Y_0$.

Proof. Let $x_0 \in Y_0$ and consider the sequence $\{x_{0n}\}$ where $x_{0n} = T_0^n x_0$, for every $n \in \mathbb{N}$.

Denote $d(x_{0n}, x_{0n+1})$ by u_n and $d(x_{0n+1}, x_{0n})$ by v_n and we may suppose that either $u_n > 0$ or $v_n > 0$ for all $n \in \mathbb{N}$.

Since T_0 is asymptotically regular, we have $\lim_{n\to\infty} u_n = 0$ and $\lim_{n\to\infty} v_n = 0$.

Now, for $n < m$ and $u_n > 0, u_m > 0$ we have

$$\tau + F(s \max\{d_0(x_{0n+1}, x_{0m+1}), d_0(x_{0m+1}, x_{0n+1})\})$$

$$\leqslant F(\max\{d_0(T_0^n x_0, t^{n+1}x_0), d_0(t^{n+1}x_0, T_0^n x_0)\})$$

$$+ F(\max\{d_0(T_0^m x_0, T_0^{m+1}x_0), d_0(T_0^{m+1}x_0, T_0^m x_0)\})$$

$$= F(\max\{u_n, v_n\}) + F(\max\{u_m, v_m\}).$$

on letting $n \to \infty$, we get

$$\lim_{n\to\infty} F(s \max\{d_0(x_{0n+1}, x_{0m+1}), d_0(x_{0m+1}, x_{0n+1})\}) = -\infty$$

(or) $\lim_{n\to\infty} s \max\{d_0(x_{0n+1}, x_{0m+1}), d_0(x_{0m+1}, x_{0n+1})\} = 0$,

showing that $\{x_{0n}\}$ is a Cauchy sequence.

Since (Y_0, d_0, s) is complete, there exists $z_0 \in Y_0$ such that $\lim_{n\to\infty} x_{0n} = z_0$.

By hypothesis, we have $\forall\, n \in Y_0$.

$$\tau + F(s \max\{d_0(T_0z_0, T_0x_{0n}), d_0(T_0x_{0n}, T_0z_0)\})$$

$$\leqslant F(\max\{d_0(z_0, T_0z_0), d_0(T_0z_0, z_0)\})$$

$$+ F(\max\{d_0(x_{0n}, T_0x_{0n}), d_0(T_0x_{0n}, x_{0n})\})$$

Hence on letting $n \to \infty$, we get (since $\lim_{n\to\infty} d_0(x_{0n}, T_0x_{0n}) = 0$).

$$\tau + \lim_{n\to\infty} F(s \max\{d_0(T_0z_0\ T_0x_{0n}), d_0(T_0x_{0n}, T_0z_0)\}) \leqslant -\infty,$$

that is, $\lim_{n\to\infty} s \max\{d_0(T_0z_0\ T_0x_{0n}), d_0(T_0x_{0n}, T_0z_0)\}) = 0$.

Hence $\{T_0x_{0n}\}$ is a fixed point of Y_0.

Since the convergent sequence $\{x_{0n}\}$ converges to both z_0 and T_0z_0 we conclude that $T_0z_0 = z_0$. Thus z_0 is a fixed point of T_0.

To show the uniqueness of the fixed point, let z_0' be another fixed point of T_0 with $z_0 \neq z_0'$. Then

$$\tau + F(s \max\{d_0(T_0z_0, T_0z_0'), d_0(T_0z_0', T_0z_0)\})$$

$$\leqslant F(\max\{d_0(z_0, T_0z_0'), d_0(T_0z_0', z_0)\})$$

$$+F(\max\{d_0(z_0', T_0z_0), d_0(T_0z_0, z_0')\})$$

$$\tau + F(s \max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\leqslant F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$+F(\max\{d_0(z_0', z_0), d_0(z_0, z_0')\}\}$$

$$\tau + F(s \max\{d_0(z_0, z_0'), d_0(z_0', z_0)\}) \leqslant F(\max\{d_0(z_0, z_0'), d_0(z_0', z_0)\})$$

$$\tau + F(d_0(z_0, z_0')) \leqslant F(d_0(z_0, z_0'))$$

which is a contradiction.

This proves the result. □

Now we introduce the notion of F-expanding mapping on a dislocated quasi-b-metric space, this extends the similar notion available in b-metric sequence [20].

Definition 3.6. Let (Y_0, d_0, s) be a dislocated quasi b-metric space and $F \in \mathcal{F}_s$. A mapping $T_0 : Y_0 \to Y_0$ is said to be an F-expanding type mapping if there exists $\tau > 0$ such that $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$ implies

$$\tau + F(s \max\{d_0(x_0, y_0), d_0(y_0, x_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$+F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\})$$

$$+F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

whenever $1 \times 2 \times 3 \times 4 \neq 0$, where $1 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $2 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $3 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $4 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, and $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) = 0$ implies

$$\tau + F(s \max\{d_0(x_0, y_0), d_0(y_0, x_0)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\})$$

$$+F(\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\})$$

$$+F(\max\{d(y_0, T_0x_0), d(T_0x_0, y_0)\})\}$$

whenever $1 \times 2 \times 5 \times 6 \neq 0$, where $1 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $2 : \max\{d_0(T_0x_0, T_0y_0), d_0(T_0y_0, T_0x_0)\}$, $5 : \max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}$, $6 : \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}$.

The following lemma can be easily established.

Lemma 3.2. *Let $Y_0 \neq \phi$. Suppose $T_0 : Y_0 \to Y_0$ is surjective. Then there exists a mapping $T_0^* : Y_0 \to Y_0$ such that $T_0 \circ T_0^*$ is the identity map on Y_0.*

Theorem 2.6. *Let (Y_0, d_0, s) be a complete dislocated quasi b-metric space. Suppose $T_0 : Y_0 \to Y_0$ is surjective is an F-expanding type mapping. Then T_0 has a unique fixed point $z_0 \in Y_0$.*

Proof. By Lemma 3.2, there exists a mapping $T_0^* : Y_0 \to Y_0$ such that $T_0 \circ T_0^*$ is the identity map on Y_0.

Let x_0 and $y_0 \in Y_0$ and $x_0 \neq y_0$. Let $u = T_0^* x_0$ and $v = T_0^* y_0$. Clearly $u \neq v$.

$$\tau + F(s\max\{d_0(u, v), d_0(v, u)\}) \leqslant \frac{1}{3}\{F(\max\{d_0(T_0u, T_0v), d_0(T_0v, T_0u)\})$$

$$+F(\max\{d_0(u, T_0u), d_0(T_0u, u)\}) + F(\max\{d_0(v, T_0v), d_0(T_0v, v)\})\}$$

Since $T_0u = T_0(T_0^*x_0) = x_0$ and $T_0v = T_0(T_0^*y_0) = y_0$, we get

$$\tau + F(s\max\{d(T_0^*x_0, T_0^*y_0), d(T_0^*y, T_0^*x)\})$$

$$\leqslant \frac{1}{3}\{F(\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}) + F(\max\{d_0(x, T_0^*x), d(T_0^*x_0, x_0)\})$$

$$+ F(\max\{d(y, T_0^*y), d(T_0^*y, y)\})\}$$

showing that T_0^* is an F-contractive type mapping.

By theorem 3.1, T_0^* has a unique fixed point $z_0 \in Y_0$ and for every $x_0 \in Y_0$ the sequence $\{T_0^{*n}x_0\}$ converges to z_0.

In particular, z_0 is also a fixed point of T_0 since $T_0^*z_0 = z_0$ implies that $T_0z_0 = T_0(T_0^*z_0) = z_0$.

Finally, if $w = T_0w$ is another fixed point, then

$$\tau + F(s\max\{d_0(z_0, w), d_0(w, z_0)\}) \leqslant \frac{1}{3}\{F(\max\{d_0(T_0z_0, T_0w), d_0(T_0w, T_0z_0)\})$$

$$+F(\max\{d_0(z_0, T_0w), d_0(T_0w, z_0)\}) + F(\max\{d_0(w, T_0z_0), d_0(T_0z_0, w)\})\}$$

Thus $\tau + F(sd_0(z_0, w)) \leqslant F(d_0(z_0, w))$

which is a contradiction.

Hence the fixed point of T_0 is unique.

Now, we can define a Kannan F-expanding type mapping as follows.

$$t + F(s\max\{d_0(x_0, y_0), d_0(y_0, x_0)\})$$

$$\leqslant \frac{1}{2}\{F(\max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\})$$

$$+F(\max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\})\}$$

whenever $1 \times 2 \times 3 \neq 0$, where $1 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $2 : \max\{d_0(x_0, T_0x_0), d_0(T_0x_0, x_0)\}$, $3 : \max\{d_0(y_0, T_0y_0), d_0(T_0y_0, y_0)\}$, if $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$ and

$$\tau + F(s\max\{d_0(x_0, y_0), d_0(y_0, x_0)\}) \leqslant \frac{1}{2}\{F(\max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\})$$
$$+F(\max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\})\}$$

whenever $1 \times 4 \times 5 \neq 0$, where $1 : \max\{d_0(x_0, y_0), d_0(y_0, x_0)\}$, $2 : \max\{d_0(x_0, T_0y_0), d_0(T_0y_0, x_0)\}$, $3 : \max\{d_0(y_0, T_0x_0), d_0(T_0x_0, y_0)\}$, if $d_0(x_0, T_0x_0)d_0(y_0, T_0y_0) \neq 0$.

Theorem 3.7. *Let* (Y_0, d_0, s) *be a complete dislocated quasi-b-metric space. Suppose* $T_0 : Y_0 \to Y_0$ *is surjective and a Kannan F-expanding type mapping. Then* T_0 *has one and only one fixed point* $z_0 \in Y_0$.

References

[1] U. Aksoy, I. M. Erhan, R. Agarwal and E. Karapinar, *F*-contraction mappings on metric-like spaces in connection with integral equations on time scales, RACSAM (2020), 114-147.

[2] B. Alqahtani, A. Fulga, E. Karapinar and P. S. Kumari, Sehgal type contractions on dislocated spaces, Mathematics 7(2) (2019), 153.

[3] H. H. Alsulami, E. Karapinar and H. Piri, Fixed points of generalised *F*-Suzuki type contraction in complete *b*-metric spaces, Discrete Dyn. Nat. Soc. 2015, Article ID 969726 (2015).

[4] H. Aydi, E. Karapinar and H. Yazidi, Modified *F*-Contractions via alpha-Admissible Mappings and Application to Integral Equations, FILOMAT 31(5) (2017), 1141-148.

[5] Badr Alqahtani Andreea Fulga, Fahd Jarad and Erdal Karapinar, Nonlinear *F*-contractions on *b*-metric spaces and differential equations in the frame of fractional derivatives with Mittag-Leffler kernel, Chaos, Solitons and Fractals 128 (2019), 349-354. https://doi.org/10.1016/j.chaos.2019.08.002.

[6] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, Funct. Anal. 30 (1989), 26-37.

[7] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fundam. Math. 3 (1922), 133-181.

[8] M. Boriceanu, Fixed point theory for multi-valued generalized contraction on a set with two *b*-metric, Stud. Univ. Babe s-Bolyai. Math. 3 (2009), 1-14.

[9] F. E. Browder and W. V. Petryshyn, The solution by iteration of non-linear functional equations in Banach spaces, Bull. Am. Math. Soc. 72 (1996), 571-575.

[10] S. Cobzas, Fixed points and completeness in metric and in generalized metric spaces, (2016). arXiv:1508.05173v4 [math.FA]

[11] M. Cosentino, M. Jleli, B. Samet and C. Vetro, Solvability of integrodifferential problems via fixed point theory in b-metric spaces, Fixed Point theory Appl. 70 (2015). https://doi.org/10.1186/s13663-015-0317-2

[12] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.

[13] D. Das and N. Goswami, Some fixed point theorems on the sum and product of operators in tensor product spaces, Int. J. Pure. Appl. Math. 109 (2016), 651-663.

[14] D. Das, N. Goswami and V. N. Mishra, Some results on fixed point theorems in Banach algebras, Int. J. Anal. Appl. 13 (2017), 32-40.

[15] D. Das, N. Goswami and V. N. Mishra, Some results on the projective cone normed tensor product spaces over Banach algebras, Bol. Soc. Parana. Mat. 38(1) (2020), 197-221 (online available).

[16] M. J. Deepmala, A study on fixed point theorems for nonlinear contractions and its applications, Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur, (2014).

[17] H. Garai, T. Senapati and L. K. Dey, A study on Kannan type contractive mapping (2017). arXiv:1707.06383v1[math.FA]

[18] J. Gornicki, Remarks on contractive type mappings, Fixed Point theory Appl. 2017, 8 (2016). https://doi.org/10.1186/s13663-017-0601-4

[19] J. Gornicki, Fixed point theorems for F-expanding mappings, Fixed Point theory Appl. 2017, 9 (2017).

[20] N. Goswami, N. Haokip and V. N. Mishra, F-contractive type mappings in b-metric spaces and some related fixed point results, Fixed Point theory Appl. 2019, 13 (2019). https://doi.org/10.1186/s13663-019-0663-6.

[21] N. Haokip and N. Goswami, Some fixed point theorems for generalized Kannan type mappings in b-metric spaces, Proyecciones J. Math. 38(4) (2019, in press).

[22] P. Hitzler and A. K. Seda, Dislocated topologies, J. Electr. Engin. 51(12/s) (2000), 3-7.

[23] N. Hussain and P. Salimi, Suzuki-Wardowski type fixed point theorems for α-GF-contractions, Taiwan. J. Math. 18(6) (2014), 1879-1895.

[24] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968), 71-76.

[25] E. Karapinar, A short survey on dislocated metric spaces via fixed-point theory, Edited by J. Banas et al. (eds.), Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness, Springer Nature Singapore Pte Ltd. Chapter 13 (2017), 457-483. DOI 10.1007/978-981-10-3722-1

[26] E. Karapinar and P. Salimi, Dislocated Metric Space to Metric Spaces with some fixed point theorems, Fixed Point Theory and Applications (2013), 222 (2013).

[27] E. Karapinar, A. Fulga and R. P. Agarwal, A survey: F-contractions with related fixed point results, Journal of Fixed Point Theory and Applications (2020), 22-69.

[28] X. Liu, M. Zhou, L. N. Mishra, V. N. Mishra and B. Damjanović, Common fixed point theorem of six self-mappings in Menger spaces using (CLRSt) property, Open Math. 16 (2018), 1423-1434.

[29] A. Lukacs and S. Kajanto, Fixed point theorems for various types of F-contractions in complete b-metric spaces, Fixed Point Theory 19(1) (2018), 321-334 https://doi.org/10.24193/fpt-ro.2018.1.25

[30] Meir-Keeler, A note on Meir-Keeler contractions on dislocated quasi-b-metric, Filomat, 31(13) (2017), 4305-4318.

[31] G. Mınak, A. Helvacı and I. Altun, Ciric type generalized F-contractions on complete metric spaces and fixed point results, Filomat 28(6) (2014), 1143-1151.

[32] L. N. Mishra, On existence and behavior of solutions to some nonlinear integral equations with applications, Ph.D. Thesis, National Institute of Technology, Silchar (2017).

[33] L. N. Mishra, K. Jyoti and A. Rani Vandana, Fixed point theorems with digital contractions image processing, Nonlinear Sci. Lett. A 9(2) (2018), 104-115.

[34] L. N. Mishra, S. K. Tiwari and V. N. Mishra, Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, J. Appl. Anal. Comput. 5(4) (2015), 600-612.

[35] L. N. Mishra, S. K. Tiwari, V. N. Mishra, I. A. Khan, Unique fixed point theorems for generalized contractive mappings in partial metric spaces, J. Funct. Spaces 2015, Article ID 960827 (2015).

[36] H. K. Pathak, M. J. Deepmala, Common fixed point theorems for PD-operator pairs under relaxed conditions with applications, J. Comput. Appl. Math. 239 (2013), 103-113.

[37] B. Patir, N. Goswami and V. N. Mishra, Some results on fixed point theory for a class of generalized nonexpansive mappings, Fixed Point Theory Appl. 2018, 19 (2018).

[38] H. Piri and P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, Fixed Point Theory Appl. 2014, 210 (2014).

[39] N. A. Secelean, Weak F-contractions and some fixed point results. Bull. Iran. Math. Soc. 42(3) (2016), 779-798.

[40] P. V. Subrahmanyam, Completeness and fixed points, Monatshefte Math. 80 (1975), 325-330.

[41] X. Udo-turn, On inclusion of F-contractions in (δ, k)-weak contractions, Fixed Point Theory Appl. 2014, 65 (2014).

[42] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012, 94 (2012).

[43] D. Wardowski and N. Van Dung, Fixed points of F-weak contractions on complete metric spaces, Demonstr. Math. 47(1) (2014), 146-155.

[44] Mudasir Younis, Nicola Fabiano, Mirjana Pantović and Stojan Radenović, Some critical remarks of recent results on F-contractions in b-metric spaces, Mathematical Analysis and its Contemporary Applications 04(2) (2022), 1-10.

[45] F. M. Zeyada, G. H. Hassan and M. A. Ahmed, A Generalization of a Fixed point theorem Due to Hitzler and Seda in Dislocated Quasi-Metric Spaces, Arabian J. Sci. Engg. 31 (2005), 111-114.

[46] A. M. Zidan and Rwaily, Asma al, on new type of F-contractive mapping for quasipartial-metric spaces and some results of fixed-point theorem and application, Journal of Mathematics (2020), 1-8.

[47] Kastriot Zoto, Hassen Aydi and Habes Alsamir, Generalizations of some contractions in b-metric-like spaces and applications to boundary value problems, Advances in difference equations, Springer 262 (2021), 1-18.