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GEOSATURATION POLYNOMIAL OF A GRAPH

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Abstract

Let G be a simple graph of order p. The geosaturation number of a graph $G = (V, E)$ is the least positive integer m such that every vertex of G lies in a geodetic set of cardinality m and is denoted by $gs(G)$. The geosaturation polynomial of a graph G of order p is the polynomial $\mathcal{G}(G, x) = \sum_{i=gs(G)}^{|V(G)|} g(G, i)x^i$, where $g(G, i)$ is the number of geodetic sets of G of size i and $gs(G)$ is the geosaturation number of G. If a, b and c are integers such that $2 \leq b \leq a-1$ and $b+1 \leq c \leq a$, then there exists a connected graph G of order a, diameter b and $gs(G) = c$. Moreover, the geosaturation polynomial is $\mathcal{G}(G, x) = \sum_{i=c}^{a} (a-c+1) C_{i-(c-1)} x^i$. In this paper, we obtain several results connecting $g(G)$, $gs(G)$ and other graph theoretic parameters.

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1. Introduction

Throughout this paper, G denotes a graph with order p. By a graph we mean a finite undirected graph without loops or multiple edges. For graph theoretic terms we refer Harary [3]. In particular, for terminology related to domination theory we refer Hayanes [4] and for terminology related to geodetic theory we refer [1].

2. Geodetic Polynomial of a Graph

Definition 2.1. The geodetic polynomial of a graph G of order p is the polynomial $\mathcal{G}(G, x) = \sum_{i=g(G)}^{|V(G)|} g(G, i) x^i$, where $g(G, i)$ is the number of geodetic sets of G of size i and $g(G)$ is the geodetic number of G.

Definition 2.2. A root of $\mathcal{G}(G, x)$ is called a geodetic root of G and is denoted by $Z(\mathcal{G}(G, x))$.

Theorem 2.3. *Let T be any tree,* $\mathcal{G}(T, x) = \sum_{i=n}^{m+n} mC_{i-n} x^i$.

Proof of Theorem 2.3. Let T be a tree with $m+n$ vertices, where m is the cardinality of non-pendent vertices and n is the cardinality of pendent vertices. Let $X = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$. Since x_i's are pendent vertices, x_i's belongs to the geodetic set. Therefore, $\{x_1, x_2, \ldots, x_n\}$ is the geodetic set. If we remove a pendent vertex x_i and add a non-pendent vertex y_j, $\{y_j\} \cup \{x_1, x_2, \ldots, x_n\} - \{x_i\}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, 3, \ldots, m$ is not a geodetic set, since x_i does not lie any geodesic path. Therefore, $\{x_1, x_2, \ldots, x_n\}$ is the one and only minimal geodetic set. Hence $g(T) = n$. Now, the number of geodetic set with cardinality n is 1. The number of geodetic set with cardinality $n+1$ is mC_1. The number of geodetic set with cardinality $n+2$ is mC_2. Proceeding like this, the number of geodetic set with cardinality $n+m$ is mC_m. Therefore,

$$\mathcal{G}(T, x) = 1 \cdot x^n + mC_1 x^{n+1} + \ldots + mC_m x^{n+m}$$

$$= \sum_{i=n}^{m+n} mC_{i-n}x^{i}.$$

2.1 Graph with two Geodetic Roots

Theorem 2.4. *Let T be a tree of order p. Then* $Z(\mathcal{G}(T, x)) = \{0, -1\}$.

Proof of Theorem 2.4. Let T be a tree with $p = a + b$ vertices, where a is the cardinality of non-pendent vertices and b is the cardinality of pendent vertices. Since zero is the geodetic root with multiplicity b, for every tree T, $\mathcal{G}(T, x)$ has two distinct roots, we have $\mathcal{G}(T, x) = x^{b}(x + c)^{p-b}$, for some $c > 0$, where $p = |V(G)|$. Therefore, the coefficient of x^{p-1} is $(p - b)c$ and so $(p - b)c \in N \cup \{0\}$. This means that c is a rational number. Since every rational algebraic integer is an integer, we have $c \in N$. Now, we have to prove that $c = 1$. Since T is a tree, the coefficient of x^{p-1} in $\mathcal{G}(T, x)$ is $p - b$. Then $c = 1$. Therefore-1 is a root of multiplicity $p - b$. Hence $Z(\mathcal{G}(T, x)) = \{0, -1\}$. □

Problem 2.5. Characterize Graphs with three geodetic roots.

Problem 2.6. Characterize geodetic roots of all connected graphs.

3. Geosaturation number and Polynomial of a graph

3.1. Geosaturation number of a Graph

Definition 3.1. The geosaturation number of a graph $G = (V, E)$ is the least positive integer m such that every vertex of G lies in a geodetic set of cardinality m and is denoted by $gs(G)$.

Definition 3.2. A graph G is said to be a class 1 or class 2 according as $gs(G) = g(G)$ or $gs(G) = g(G) + 1$.

Any complete graph K_n is of class 1 and tree T is of class 2.

Observation 3.3. For any graph G, G has a cut-vertex, then G is of class 2.

Theorem 3.4. *Let G be a connected graph of order* $p \geq 2$. *Then*

$g(G) = gs(G) = g_c(G) = p$ if and only if G is the complete graph with p vertices.

Proof of Theorem 3.4. We know that the result holds for $p = 2$. We now consider the case where $p \geq 3$. Assume that $g(G) = gs(G) = g_c(G) = p$. Suppose to the contrary that there are two non-adjacent vertices a, b in G. Let P be an $a - b$ geodesic and let x be a vertex on P which is adjacent to a. Then $V(G)/\{x\}$ is a geodetic set of G, which is a contradiction to our assumption. Hence G is a complete graph. Conversely, if $G = Kp$, then obviously $gs(G) = p$, by theorem in [2], $g(G) = n$ and by theorem in [1], $g_c(G) = p$. Therefore $g(G) = gs(G) = g_c(G) = p$. □

Theorem 3.5. *For any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with $gs(G) = a$ and $|V(G)| = b$.*

Proof of Theorem 3.5. Clearly, the result is true for $2 \leq a \leq b$. Since if $b = 2$, then $G = P_2$, while if $b = 3$, then $G \in \{P_3, K_3\}$. Let us consider the case that $b \geq 4$. If $a = b$, let $G = K_b$ and if $a = b - 1$, let $G = K_1, b - 1$. For $a \leq b - 2$, let G be a graph obtained from the star $K_{1, b-2}$ with support x leaves $x_1, x_2, \ldots, x_{b-2}$ by adding a new vertex y and joining y to the vertices $x_i (a - 1 \leq i \leq b - 2)$. Then $\{x_1, x_2, \ldots, x_{a-2}, y\}$ is the geodetic set. Therefore $g(G) = a - 1$. But the vertices $\{x_{a-1}, x_a, \ldots, x_{b-2}, x\}$ does not belong to any geodetic set of cardinality $a - 1$. Therefore $gs(G) = a$. □

Theorem 3.6. *If G is a connected graph with $\gamma(G) = 1$, then $gs(G) = g_c(G)$.*

Proof of Theorem 3.6. If $G = K_p$, then $\gamma(G) = 1$ and $gs(G) = g_c(G) = p$, so we only have to consider the case $G \neq K_p$. Since $\gamma(G) = 1$, $\Delta(G) = p - 1$ and $diam G \leq 2$. Since $G \neq K_p$, there exists at least two non-adjacent vertices in G. Therefore, $diam G = 2$. Let S be the minimum cardinality geodetic set of G and let $x \notin S$ (such a vertex in G). Since S is a geodetic set, there exists a vertices $x, y \in S$ such that a belongs to a $x - y$ geodesic. Since $diam G = 2$, it follows thatthe $x - y$ geodesic containing a must be the path xay. Also, a does not belong to any geodetic set, $gs(G) = g(G) + 1$. Also, by theorem in [1], a must be in the connected geodetic set. Therefore $gs(G) = g_c(G)$. □

Theorem 3.7. *For every non-trivial tree T of order* n, $gs(T) = p - m + 2$ *if and only if T is a caterpillar*.

Proof of Theorem 3.7. Let T be any non-trivial tree of order p. Let $m = d(u, v)$ and let $P : u = v_0, v_1, \ldots, v_{m-1}v_m = v$ be a diameteral path. Let a be the number of end vertices of T and b be the number of internal vertices of T other than $v_1, v_2, \ldots, v_{m-1}$. Then $m - 1 + b + a = p$. This implies that $a = p - m - b + 1$ therefore $g(T) = a$ and so $g(T) = p - m - b + 1$. T is a caterpillar if and only if all the internal vertices of T lie on the diametrical path P if and only if $b = 0$ if and only if $g(T) = p - m + 1$. But the vertices $v_1, v_2, \ldots, v_{m-1}$ are does not lie on any geodetic set. Then $gs(T) = p - m + 2$. Hence T is a caterpillar if and only if $gs(T) = n - d + 2$. □

Corollary 3.9. *For a wounded sider T of order* n, $gs(T) = p - m + 2$ *if and only if T is obtained from* $K_{1,n}(n \geq 1)$ *by subdividing at most two of its edges*.

Proof of Corollary 3.9. It is clear that an wounded spider T is a caterpillar if and only if T is obtained from $K_{1,n}(n \geq 1)$ by subdividing at most two of its edges. Now, the corollary follows from the above theorem. □

3.2 Geosaturation Polynomial of a graph

Definition 3.10. The geosaturation polynomial of a graph G of order p is the polynomial $\mathcal{G}s(G, x) = \sum_{i=gs(G)}^{|V(G)|} g(G, i)x^i$, where $g(G, i)$ is the number of geodetic sets of G of size i and $gs(G)$ is the geosaturation number of G.

Theorem 3.11. *For any tree* T, $\mathcal{G}s(T, x) = \sum_{i=n+1}^{m+n} mC_{i-n}x^i$.

Proof of Theorem 3.11. Let T be a tree with $m + n$ vertices, where m is the cardinality of non-pendent vertices and n is the cardinality of pendent vertices. Let $X = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_m\}$. Since v_i's are simplicial vertices, v_i's belongs to the geodetic set. Therefore, $\{v_1, v_2, \ldots, v_n\}$ is the geodetic set. Hence $g(T) = n$. But the non-pendent vertices v_i's does not belongs to the geodetic set. Therefore $gs(T) = n+1$. Now, the number of geodetic sets with cardinality $n+1$ is mC_1. The number of geodetic sets with cardinality $n+2$ is mC_2. The number of geodetic sets with cardinality $n+3$ is mC_2. Proceeding like this, the number of geodetic sets with cardinality $n+m$ is mC_m. Therefore, T, $\mathcal{G}s(T, x) = \sum_{i=n+1}^{m+n} mC_{i-n}x^i$. □

Theorem 3.12. *For a positive integers* a, b *and* $c \geq b+1$ *with* $a < b \leq 2a$, *there exists a connected graph* G *with* $radG = a$, $diamG = b$ *and* $gs(G) = c$. *Moreover, the geosaturating polynomial is* $\mathcal{G}s(G, x) = \sum_{i=c}^{a+b+c-3} (a+b-2)C_{i-(c-1)}x^i$.

Proof of Theorem 3.12. Let $a = 1$, then $b = 1$ or 2. If $b = 1$, let $G = K_c$. Then $g(G) = c$. This implies that $gs(G) = c$. If $b = 2$, let $G = K_{1,\, c-1}$. Then $g(G) = c-1$. But the support vertex does not lie on the geodetic set. Therefore, $gs(G) = c$. Now, Let $a < b \leq 2a$. Let C_{2a} be the even cycle of order $2a$ with the vertices $v_1, v_2, \ldots, v_{2a}$ and let P_{b-a+1} be a path of order $b-a+1$ with the vertices $u_0, u_1, \ldots, u_{b-a}$. Let $K_{1,\, c-3}$ be a star graph with vertices $w_0, w_1, \ldots, w_{c-3}$. Let G be a graph obtained from C_{2a} and P_{b-a+1} by identifying v_1 in C_{2a} and u_0 in P_{b-a+1} and also identifying the vertex u_{b-a-1} with w_0 in $G = K_{1,\, c-3}$. The Graph G is shown below:

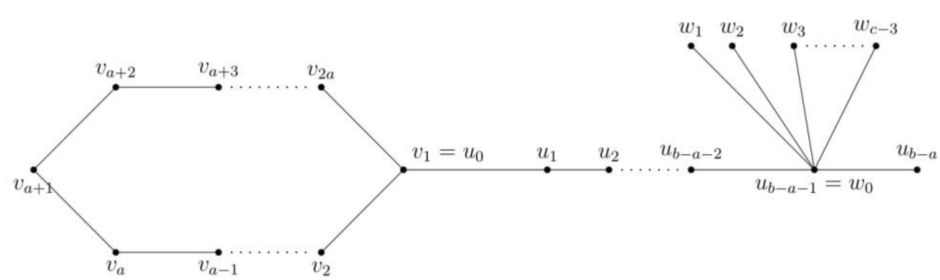

Figure 1. G.

Then $radG = a$ and $diamG = b$. Let $M = \{u_{b-a}, w_1, w_2, \ldots, w_{c-3}\}$ be the pendent vertices of the graph G with $|M| = c-2$. Clearly, all the pendent vertices belongs to the geodetic set. Let $N = M \cup \{v_{a+1}\}$. Clearly N is a geodetic set with $|N| = c-1$. Therefore $g(G) = c-1$. But $\{v_1, v_2, \ldots, v_{a-1}, v_{a+1}, \ldots, u_1, u_2, \ldots, u_{b-a-1}\}$ does not lie on the geodetic set. Therefore $gs(G) = c$. Now, we form a geosaturation polynomial. Let G be a graph with $a+b+c-3$ vertices. Since $g(G) = c-1$, $gs(G) = c$ and this can be done in $(a+b-2)C_1$ ways. Therefore, the number of geodetic set with cardinality c is $(a+b-2)C_1$. Now, the number of geodetic set with cardinality $c+1$ is $(a+b-2)C_1$. Also, the number of geodetic set with cardinality $c+2$ is $(a+b-2)C_3$. Proceeding like this, The number of geodetic set with cardinality $a+b+c-3$ is $(a+b-2)C_{a+b-2}$. Therefore, the geosaturation polynomial is

$$\mathcal{G}s(G, x)$$

$$= (a+b-2)C_1 x^c + (a+b-2)C_2 x^{c+1} + \ldots + (a+b-2)C_{(a+b-2)} x^{a+b+c-2}$$

$$\mathcal{G}s(G, x) = \sum_{i=c}^{a+b+c-3} (a+b-2)C_{i-(c-1)} x^i$$

v_{a+1}.

Problem 3.13. For any three positive integers a, b and $c \geq b+1$ such that $a = b \leq 2a$, does there exist a connected graph G with $radG = a$, $diamG = b$ and $gs(G) = c$.

Theorem 3.14. *If a, b and c are integers such that $2 \leq b \leq a-1$ and $b+1 \leq c \leq a$, then there exists a connected graph G of order a, diameter b and $gs(G) = c$. Moreover, the geosaturation polynomial is* $\mathcal{G}s(G, x) = \sum_{i=c}^{a} (a-b+1)C_{i-(c-1)} x^i$.

Proof of Theorem 3.14. We prove this theorem by considering three cases.

Case (i). Let $b = 2$. If $c = b + 1$, then $c = 3$. Let $P_3 : u_1, u_2, u_3$ be a path of order 3. Now, we choose $a - 3$ new vertices $w_1, w_2, \ldots, w_{a-3}$ and joining each $w_i (1 \leq i \leq a - 3)$ to u_1 and u_3. The graph G in Figure 2 is the resultant graph.

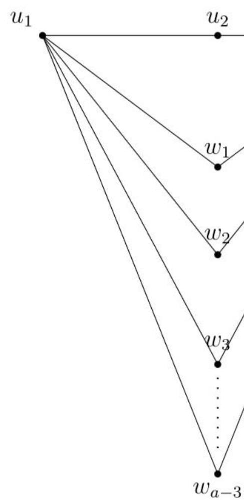

Figure 2. G.

Then G has order a and diameter 2. Clearly, $M = \{u_1, u_3\}$ is the minimum cardinality geodetic set of G. Therefore $g(G) = 2 = c - 1$. But $\{u_2, w_1, w_2, \ldots, w_{a-3}\}$ does not lie on the geodetic set. Thus $gs(G) = c$. Now, let $b + 1 \leq c \leq a$. Consider a complete graph K_{a-1}, $\{w_1, w_2, \ldots, w_{a-c+1}, v_1, v_2, \ldots, v_{c-2}\}$ as its vertex set. Now, add a new vertex x to K_{a-1}. Then a graph G by joining x with $w_i (1 \leq i \leq a - c + 1)$. The graph G in Figure 3 is the resultant graph.

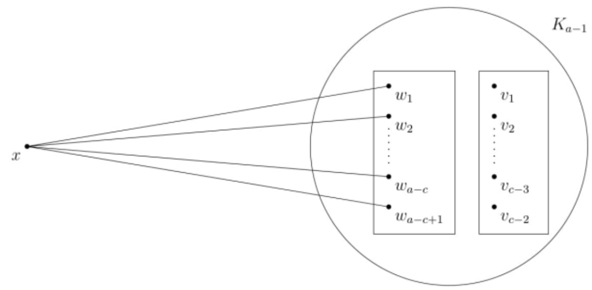

Figure 3. G.

Then G has order a and diameter $b = 2$. Let $M = \{v_1, v_2, \ldots, v_{c-2}, x\}$. Clearly M is the minimum cardinality geodetic set of G. Therefore, $g(G) = c - 1$. But $\{w_1, w_2, \ldots, w_{a-c+1}\}$ does not lie on the geodetic set. Thus $gs(G) = c$.

Case (ii). Let $3 \leq b \leq a - 2$. Consider a path $P_{b+1}, u_1, u_2, \ldots, u_{b+1}$ as its vertex set of length b. Now, choose $a - b - c - 2$ new vertices $w_1, w_2, \ldots, w_{a-b-c+2}$ and joining $w_i (1 \leq i \leq a - b - c + 2)$ to u_1 and u_3. Also, we can choose $c - 3$ new vertices $v_1, v_2, \ldots, v_{c-3}$ and joining $v_i (1 \leq i \leq c - 3)$ to u_b. Then graph G in Figure 4 is the resultant graph.

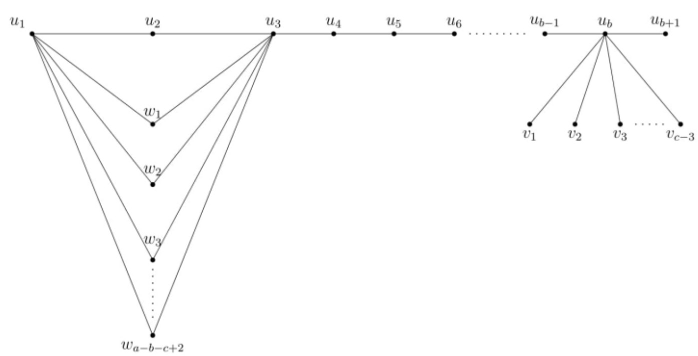

Figure 4. G.

Then G has order a and diameter b. Let $M = \{v_1, v_2, \ldots, v_{c-2}, u_{b+1}\}$ be the set of all pendent vertices. Clearly, all the pendent vertices belongs to geodetic set. Now, $M \cup \{u_1\}$ is a geodetic set. Therefore $g(G) = c - 1$. But

$\{w_1, w_2, \ldots, w_{a-b-c+2}, u_2, u_3, \ldots, u_b\}$ does not lie on the geodetic set. Thus $gs(G) = c$.

Case (iii). Let $b = a - 1$. Then $c = a$. Let G be the complete graph of order c. Then $g(G) = c$. This implies that $gs(G) = c$. Now, we form a geosaturation polynomial. Let G be a graph with c vertices. Since $g(G) = c - 1$, $gs(G) = c$ and this can be done in $(a - c + 1)C_1$ ways. Therefore, the number of geodetic set with cardinality c is $(a - c + 1)C_1$. Now, the number of geodetic set with cardinality $c + 1$ is $(a - c + 1)C_2$. Also, the number of geodetic set with cardinality $c + 2$ is $(a - c + 1)C_3$. Proceeding like this, the number of geodetic set with cardinality a is $(a - c + 1)C_{a-c+1}$. Therefore,

$\mathcal{G}s(G, x)$

$$= (a - c + 1)C_1 x^c + (a - c + 1)C_2 x^2 + \ldots + (a - c + 1)C_{(a-c+1)} x^{a-c+1}$$

$$\mathcal{G}s(G, x) = \sum_{i=c}^{a} (a + c - 2)C_{i-(c-1)} x^i.$$

4. The Geosaturation Polynomial of $G \circ K_1$.

In this section, we study the geosaturation number and geosaturation polynomial of $G \circ K_1$.

Lemma 4.1. *For a connected graph G of order* $p - 1$, $gs(G \circ K_1) = p$.

Proof of Lemma 4.1. Let $\{v_1, v_2, \ldots, v_{p-1}\}$ be the vertices of a connected graph G. Add $p - 1$ new vertices $\{u_1, u_2, \ldots, u_{p-1}\}$ to G. Now, connect u_i to v_i for $1 \le i \le p - 1$. If T is a geodetic set of G, then for every $i, 1 \le i \le p - 1, u_i \in T$. This implies that $|T| = p - 1$. But $\{v_1, v_2, \ldots, v_{p-1}\}$ does not lie any geodetic set. Therefore, $gs(G \circ K_1) = p$.

Remark 4.2. By lemma 4.1, $g(G \circ K_{1,k}) = 0$, for every $k, k < p$. So we shall compute $g(G \circ K_{1,k})$ for each $k, p < k \le 2p$.

Theorem 4.3. *For any graph G of order p and* $p < k \leq 2p$, *we have* $g(G \circ K_{1,k}) = \binom{p}{k-p}$. *Hence* $\mathcal{G}s(G, x) = x^p[(x+1)^p - 1]$.

Proof of Theorem 4.3. Let G be any graph with vertex set $\{v_1, v_2, \ldots, v_p\}$. Add p new vertices $\{u_1, u_2, \ldots, u_p\}$ and join u_i to v_i for $1 \leq i \leq p$. By previous lemma 4.1, $gs(G \circ K_1) = p + 1$. Suppose that T is a geodetic set of $G \circ K_1$ of size k. There are $\binom{p}{k-p}$ possibilities to choose the remaining vertices. Therefore, $g(G \circ K_{1,k}) = \binom{p}{k-p}$.

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