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OPTIMALITY CONDITIONS FOR FUZZY NON-LINEAR EQUALITY CONSTRAINED MINIMIZATION PROBLEMS

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Abstract

In this paper, an optimality conditions for fuzzy non-linear equality constrained minimization problems are discussed. Here the cost coefficients and constrained coefficients are represented by a triangular fuzzy number. Some numerical illustrations are discussed by using these optimality conditions.

1. Introduction

Many authors considered different types of the fuzzy non-linear programming problems and proposed several approaches by solving these problems. R. E. Bellman and L. A. Zadeh [5] have introduced the decision making in a fuzzy environment. Hsien-Chung Wu [10] has presented an (α, β)- optimal solution concept in fuzzy optimization problems and also he [11] discussed the optimality conditions for optimization problems with fuzzy-valued objective functions. V. D. Pathak and U. M. Pirzada [19] have introduced the necessary and sufficient optimality conditions for nonlinear fuzzy optimization problem. R. Saranya and Palanivel Kaliyaperumal [20] have presented fuzzy nonlinear programming problem for inequality

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constraints with alpha optimal solution in terms of trapezoidal membership functions.

Here, either an objective function or constraints or both of them are non-linear and all the variables are considered as fuzzy variables. Here, the necessary and sufficient optimality conditions are based on the concept of partial differentiability of the Lagrangian function are discussed. Numerical examples based on these optimality conditions are given.

2. Preliminaries

2.1. Fuzzy non-linear programming problem:

It refers to an optimization problem in which the variables are continuous variables and the problem is of the following general form:

Minimize $\theta(\tilde{x})$

Subject to $h_i(\tilde{x}) = 0,\ i = 1, 2, \ldots, m.$

$$g_p(\tilde{x}) \geq 0,\ p = 1, 2, \ldots, t.$$

where $\theta(\tilde{x}), h_i(\tilde{x}), g_p(\tilde{x})$ are all real valued continuous functions of $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbf{R}^n$.

2.2. Fuzzy non-linear equality constrained minimization problem:

If there are no inequality constraints on the variables in fuzzy non-linear programming problem.

2.3. Fuzzy local minimum:

Consider a fuzzy non-linear programming problem in which a function $\theta(\tilde{x})$ is required to be optimized subject to some constraints on the variables $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$. Let $\tilde{K}$ denote the set of fuzzy feasible solutions for this problem. For this problem a fuzzy feasible solution $\tilde{x} \in \tilde{\mathbf{K}}$ is said to be a local minimum, if there exists an $\in < 0$ such that $\theta(\tilde{x}) \geq \theta(\tilde{x})$ for all $\tilde{x} \in \tilde{\mathbf{K}} \cap \{\tilde{x} : \| \tilde{x} - \tilde{x} < \in\}$.

2.4. Fuzzy set

A fuzzy set $\tilde{P}$ is defined by $\tilde{P} = \{(s, \mu_P(s)) : s \in P, \mu_P(s) \in [0, 1]\}$. In the pair $(s, \mu_P(s))$, the first element s belong to the classical set P, the second element $\mu_P(s)$ belong to the interval [0, 1], called membership function.

2.5. Fuzzy number

The notion of fuzzy numbers was introduced by Dubois D and Prade H [26]. A fuzzy subset $\tilde{P}$ of the real line R with membership function $\mu_{\tilde{P}} : R \to [0, 1]$ is called a fuzzy number if

(i) A fuzzy set $\tilde{P}$ is normal.

(ii) $\tilde{P}$ is fuzzy convex, that is

$$\mu_{\tilde{P}}[\lambda_{s_1} + (1 - \lambda)s_2] \geq \mu_{\tilde{P}}(s_1) \wedge \mu_{\tilde{P}}(s_2),\ s_1, s_2 \in R, \forall \lambda \in [0, 1].$$

(iii) $\mu_{\tilde{P}}$ is upper continuous and

Supp $\tilde{P}$ is bounded, where supp $\tilde{P} = \{s \in R : \mu_{\tilde{P}}(s) > 0\}$.

2.6. Triangular Fuzzy Number

The triangular fuzzy number can be denoted as $\tilde{P} = (p_1, p_2, p_3)$, where p_2 is the central value, $\mu_{\tilde{P}}(p_2) = 1$, such that $p_1 < p_2 < p_3$ are defined in R.

The α-cut of a triangular fuzzy number is,

$$\tilde{P}_\alpha = [(p_2 - p_1)\alpha + p_1, -(p_3 - p_2)\alpha + p_3].$$

The membership function $\mu_{\tilde{P}}(s)$ is given by,

$$\mu_{\tilde{P}}(s) = \begin{cases} 0 & \text{for } s < p_1 \\ \dfrac{s - p_1}{p_2 - p_1} & \text{for } p_1 \leq s \leq p_2 \\ \dfrac{p_3 - s}{p_3 - p_2} & \text{for } p_2 \leq s \leq p_3 \\ 0 & \text{for } s > p_3. \end{cases}$$

2.7. Operations of Triangular Fuzzy Number using Function Principle

Let $\tilde{P} = (p_1, p_2, p_3)$ and $\tilde{Q} = (q_1, q_2, q_3)$ be two triangular fuzzy numbers.

Then

(i) The addition of $\tilde{P}$ and $\tilde{Q}$ is

$$\tilde{P} + \tilde{Q} = (p_1 + q_1,\ p_2 + q_2,\ p_3 + q_3)$$

where $p_1, p_2, p_3, q_1, q_2, q_3$ are real numbers.

(ii) The product of $\tilde{P}$ and $\tilde{Q}$ is

$$\tilde{P} \times \tilde{Q} = (c_1, c_2, c_3), \text{ where } T = \{p_1 q_1,\ p_2 q_2,\ p_3 q_3\}$$

where $c_1 = \min\ \{T\}$, $c_2 = p_2 q_2$, $c_3 = \max\ \{T\}$.

If $p_1, p_2, p_3, q_1, q_2, q_3$ are all non-zero positive real numbers, then

$$\tilde{P} \times \tilde{Q} = (p_1 q_1,\ p_2 q_2,\ p_3 q_3).$$

(iii) $-\tilde{Q} = (-q_3, -q_2, -q_1)$.

Then the subtraction of $\tilde{Q}$ from $\tilde{P}$ is

$$\tilde{P} - \tilde{Q} = (p_1 - q_3,\ p_2 - q_2,\ p_3 - q_1),$$

where $p_1, p_2, p_3, q_1, q_2, q_3$ are real numbers.

(iv) $\frac{1}{\tilde{Q}} = \tilde{Q}^{-1} = \left(\frac{1}{q_3}, \frac{1}{q_2}, \frac{1}{q_1}\right)$

$\frac{\tilde{P}}{\tilde{Q}} = (c_1, c_2, c_3)$, where $T = \left(\frac{p_1}{q_3}, \frac{p_2}{q_2}, \frac{p_3}{q_1}\right)$ where

$$c_1 = \min\ \{T\},\ c_2 = \frac{p_2}{q_2},\ c_3 = \max\ \{T\}.$$

If $p_1, p_2, p_3, q_1, q_2, q_3$ are all non-zero positive real numbers, then

$$\frac{\tilde{P}}{\tilde{Q}} = \left(\frac{p_1}{q_3}, \frac{p_2}{q_2}, \frac{p_3}{q_1}\right).$$

(v) Let $\alpha \in R$, then $\alpha\tilde{P} = (\alpha p_1, \alpha p_2, \alpha p_3)$, if $\alpha \geq 0$

$= (\alpha p_3, \alpha p_2, \alpha p_1)$ if $\alpha < 0$.

2.8. Triangular Fuzzy Matrix

A triangular fuzzy matrix of order $m \times n$ is defined as $P = (\tilde{p}_{ij})_{m \times n}$, where $\tilde{p}_{ij} = (p_{ij1}, p_{ij2}, p_{ij3})$ is the ij^{th} element of P.

2.9. Operations on Triangular Fuzzy Matrices

Let $S = (\tilde{s}_{ij})$ and $T = (\tilde{t}_{ij})$ be two triangular fuzzy matrices of same order. Then

(i) $S + T = (\tilde{s}_{ij} + \tilde{t}_{ij})$

(ii) $S - T = (\tilde{s}_{ij} - \tilde{t}_{ij})$

(iii) For $S = (\tilde{s}_{ij})_{m \times n}$ and $T = (\tilde{t}_{ij})_{n \times k}$ then $ST = (\tilde{c}_{ij})_{m \times k}$

where $\tilde{c}_{ij} = \sum_{p=1}^{n} \tilde{s}_{ip} \cdot \tilde{t}_{pj}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, k$.

(iv) $S^T = (\tilde{s}_{ji})$

(v) $KS = (K\tilde{s}_{ij})$ where K is scalar.

2.10. Positive semi-definite fuzzy matrix

A fuzzy square matrix $\tilde{A} = (\tilde{a}_{ij})$ of order *n*, whether it is symmetric or not, is said to be a positive semi definite fuzzy matrix if $\tilde{x}^T \tilde{A} \tilde{x} \geq 0$ for all $\tilde{x} \in \mathbf{R}^n$.

2.11. Positive definite fuzzy matrix

A fuzzy square matrix $\tilde{A} = (\tilde{a}_{ij})$ of order *n*, whether it is symmetric or not, is said to be a positive definite fuzzy matrix if $\tilde{x}^T \tilde{A} \tilde{x} > 0$ for all $\tilde{x} \neq 0$.

3. Optimality Conditions for Fuzzy non-linear Equality Constrained Minimization Problems

Consider the fuzzy non-linear programming problem,

minimize $\theta(\tilde{x})$

Subject to $h_i(\tilde{x}) = 0,\ i = 1$ to m (3.1)

where $\theta(\tilde{x}),\ h_i(\tilde{x})$ are all real valued continuously differentiable functions defined on R^n. Let $h(\tilde{x}) = (h_1(\tilde{x}), \ldots, h_m(\tilde{x}))^T$. The set of fuzzy feasible solutions is a surface in R^n, and it is smooth if each $h_i(\tilde{x})$ is a smooth function (i.e., continuously differentiable). If $\bar{\tilde{x}}$, is a fuzzy feasible point, when some of the $h_i(\tilde{x})$ are nonlinear, there may be no fuzzy feasible direction at $\tilde{x}$. In order to retain fuzzy feasibility while moving from $\bar{\tilde{x}}$, one has to follow a nonlinear curve through $\bar{\tilde{x}}$ which lies on the fuzzy feasible surface.

A curve in R^n is the locus of a point $\tilde{x}(\lambda) = (\tilde{x}_j(\lambda))$ where each $\tilde{x}_j(\lambda)$ is a real valued function of the real parameter λ, as the parameter varies over some interval of the real line.

The curve $\tilde{x}(\lambda) = (\tilde{x}_j(\lambda))$ is said to be differentiable at λ if $\frac{d\tilde{x}_j(\lambda)}{d\lambda}$ exists for all *j*, and twice differentiable if $\frac{d^2\tilde{x}_j(\lambda)}{d\lambda^2}$ exists for all *j*. The curve $\tilde{x}(\lambda)$ is said to pass through the point $\bar{\tilde{x}}$ if $\bar{\tilde{x}} = \tilde{x}(\bar{\lambda})$ for some $\bar{\lambda}$.

If the curve $\tilde{x}(\lambda)$ defined over $a < \lambda < b$ is differentiable at $\bar{\lambda},\ a < \bar{\lambda} < b$, then the line $\{\tilde{x} = \tilde{x}(\bar{\lambda}) + \delta \frac{d\tilde{x}}{d\lambda}(\bar{\lambda}) : \delta$ real$\}$ is the tangent line to the curve at the point $\tilde{x}(\bar{\lambda})$ on it.

The tangent plane at a fuzzy feasible point $\bar{\tilde{x}}$ to (3.1) is defined to be the set of all directions $\left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)_{\lambda=0}$, where $\tilde{x}(\lambda)$ is a differential curve in the fuzzy feasible region with $\tilde{x}(0) = \bar{\tilde{x}}$.

Theorem 3.1. *If* $\bar{\tilde{x}}$ *is a fuzzy regular point for* (3.1), *the tangent plane for* (3.1) *at* $\bar{\tilde{x}}$ *is* $\{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$.

Proof. Let $\tilde{x}(\alpha)$ be a differentiable curve lying in the fuzzy feasible region for α lying in an interval around zero, with $\tilde{x}(0) = \bar{\tilde{x}}$ and $\frac{d\tilde{x}(0)}{d\alpha} = \tilde{y}$. So $h(\tilde{x}(\alpha)) = 0$ for all values of α lying in an interval around zero, and hence $\left(\frac{dh(\tilde{x}(\alpha))}{d\alpha}\right)_{\alpha=0} = 0$, that is $(\nabla h(\bar{\tilde{x}}))\tilde{y} = 0$. This implies that the tangent plane is a subset of $\{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$.

Suppose $\tilde{y} \in \{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$ and $\tilde{y} \neq 0$.

Define new variables $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)^T$. Consider the following system of m equations in $m + 1$ variables $\tilde{u}_1, \ldots, \tilde{u}_m, \alpha$.

$$\tilde{g}_i(\tilde{u}, \alpha) = \tilde{h}_i(\bar{\tilde{x}} + \alpha\tilde{y} + (\nabla h(\bar{\tilde{x}}))^T \tilde{u}) = 0, \; i = 1 \text{ to } m. \tag{3.2}$$

$\tilde{g}(0, 0) = 0$ and the Jacobian matrix of $\tilde{g}(\tilde{u}, \alpha)$ with respect to $\tilde{u}$ is non-singular at $\tilde{u} = 0$, $\alpha = 0$ (since $\bar{\tilde{x}}$ is a regular point of (3.1)). So by applying the implicit function theorem on (3.2), we can express $\tilde{u}$ as a differentiable function of α, say $\tilde{u}(\alpha)$, in an interval around $\alpha = 0$, and that (3.2) holds as an identity in this interval when $\tilde{u}$ in (3.2) is replaced by $\tilde{u}(\alpha)$, and that $\tilde{u}(0) = 0$, and $\frac{d\tilde{u}(0)}{d\alpha}$ is obtained by solving

$\left(\frac{d}{d\alpha} h(\bar{\tilde{x}} + \alpha\tilde{y} + (\nabla h(\bar{\tilde{x}}))^T \tilde{u}(\alpha)\right)_{\alpha=0} = 0$ which leads to $\frac{d}{d\alpha}\tilde{u}(0) = 0$ since $\nabla h(\bar{\tilde{x}})$ has rank m.

So if we define $\tilde{x}(\alpha) = \bar{\tilde{x}} + \alpha\tilde{y} + (\nabla h(\bar{\tilde{x}}))^T \tilde{u}(\alpha)$.

This defines a differentiable curve lying in the feasible region for (3.1) for values of α in an interval around $\alpha = 0$, and that $\frac{d\tilde{x}}{d\alpha}\tilde{u} = \tilde{y}$, which implies that $\tilde{y}$ is in the tangent plane for (3.1) at $\bar{\tilde{x}}$.

We will now derive optimality conditions for (3.1) by using theorem (3.1).

If $\bar{\tilde{x}}$ is a fuzzy feasible regular point for (3.1), and it is a local minimum, clearly along every differentiable curve $\tilde{x}(\alpha)$ lying in the fuzzy feasible region for (3.1) for values of α in an interval around $\alpha = 0$, satisfying $\bar{\tilde{x}}(0) = \bar{\tilde{x}}$; $\alpha = 0$ must be a local minimum for $\theta(\bar{\tilde{x}})$ on this curve. That is, for the problem of minimizing $\theta(\bar{\tilde{x}}(\alpha))$ over this interval for α, $\alpha = 0$ must be a local minimum. Since $\alpha = 0$ is an interior point of this interval this implies that $\frac{d\theta}{d\alpha}(\tilde{x}(0))$ must be zero. Applying this to all such curves and using theorem 3.1 we conclude that $(\nabla\theta(\tilde{x}))\tilde{y} = 0$ for all $\tilde{y}$ satisfying $(\nabla h(\bar{\tilde{x}}))\tilde{y} = 0$.

There must exist $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)$ such that $(\nabla\theta(\bar{\tilde{x}})) - \sum_{i=1}^{m} \bar{\mu}_i \nabla h_i(\bar{\tilde{x}}) = 0$ and by feasibility $h_i(\bar{\tilde{x}}) = 0$ (3.3) the conditions (3.3) are the first order necessary optimality conditions for (3.1), the vector $\tilde{\mu}$ is the vector of Lagrange multipliers. (3.1) is a system of $(n + m)$ equations in $(n + m)$ unknowns (including $\bar{\tilde{x}}$ and $\bar{\tilde{\mu}}$) and it may be possible to solve (3.3) using algorithms for solving nonlinear equations. If we define the Lagrangian for (3.1) to be $L(\tilde{x}, \mu) = \theta(\tilde{x}) - \mu h(\tilde{x})$ where $\mu = (\mu_1, \mu_m)$, $h(\tilde{x}) = (h(\tilde{x}), h_m(\tilde{x}))^T$, (3.3) becomes: $(\bar{\tilde{x}}, \tilde{\mu})$ satisfies

$$h(\tilde{x}) = 0$$

$$\nabla_{\tilde{x}} L(\tilde{x}, \mu) = 0. \tag{3.4}$$

We will now derive the second order necessary optimality conditions for (3.1). Suppose the functions $\theta(\tilde{x}), h_i(\tilde{x})$ are all twice continuously differentiable. Let $\bar{\tilde{x}}$ be a fuzzy feasible solution for (3.1) which is a regular point. If $\bar{\tilde{x}}$ is a local minimum for (3.1), by the first order necessary optimality conditions (3.4), there must exist a row vector of Lagrange multipliers, $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)$ such that $\nabla_{\tilde{x}} L(\tilde{x}, \bar{\mu}) = 0$, where $L(\tilde{x}, \bar{\mu}) = \theta(\tilde{x}) - \bar{\mu} h(\tilde{x})$ is the Lagrangian. Since $\bar{\tilde{x}}$ is a fuzzy regular point, the tangent plane to (3.1) at $\bar{\tilde{x}}$ is $T = \{\tilde{y} : (\nabla h(\bar{\tilde{x}}))\tilde{y} = 0\}$. Suppose there exists a $\tilde{\bar{y}} \in T$ satisfying $\tilde{\bar{y}}^T H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))\tilde{\bar{y}} < 0$. Since $\tilde{\bar{y}} \in T$, and all the functions are twice continuously differentiable, there exists a twice differentiable curve $\tilde{x}(\lambda)$ through $\bar{\tilde{x}}$ lying in the feasible region (i.e., $\tilde{x}(0) = \bar{\tilde{x}}$, and the curve is defined in an interval of λ with 0 as an interior point, with $h(\tilde{x}(\lambda)) = 0$ for all λ in this interval), such that $\left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)_{\lambda=0} = \bar{\tilde{y}}$.

Now, $\frac{d}{d\lambda} L(\tilde{x}, \lambda, \bar{\mu}) = (\nabla_{\tilde{x}} L(\tilde{x}(\lambda), \tilde{\bar{\mu}}))\left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)$

$$\frac{d^2}{d\lambda^2} L(\tilde{x}(\lambda), \bar{\mu}) = \left(\frac{d\tilde{x}(\lambda)}{d\lambda}\right)^T H_{\tilde{x}}(L(x(\lambda), \bar{\mu}))\frac{d\tilde{x}(\lambda)}{d\lambda} + (\nabla_{\tilde{x}} L(\tilde{x}(\lambda), \bar{\mu}))\left(\frac{d^2\tilde{x}(\lambda)}{d\lambda^2}\right),$$

where $\nabla_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))$, $H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))$ are the fuzzy row vector of partial derivatives with respect to $\tilde{x}$ and the Hessian matrix with respect to $\tilde{x}$ of $L(\bar{\tilde{x}}, \bar{\mu})$ at $\tilde{x} = \bar{\tilde{x}}$ respectively. At $\lambda = 0$, we have $\nabla_{\tilde{x}} L(\tilde{x}(0), \bar{\mu}) = \nabla_{\tilde{x}} L(\bar{\tilde{x}}, \bar{\mu}) = 0$ by the first order necessary optimality conditions.

So, from the above

$$\left(\frac{d}{d\lambda} L(\tilde{x}(\lambda), \bar{\mu})\right)_{\lambda=0} = 0$$

$$\left(\frac{d^2}{d\lambda^2} L(\tilde{x}(\lambda), \bar{\mu})\right)_{\lambda=0} = \tilde{\bar{y}}^T H_{\tilde{x}}(L\bar{\tilde{x}}, \bar{\mu}))\tilde{\bar{y}}.$$

Using these in a Taylor series expansion for $f(\lambda) = L(\tilde{x}(\lambda), \bar{\mu})$ up to second order around $\lambda = 0$ leads to $f(\lambda) = L(\tilde{x}(\lambda), \bar{\mu}) = L(\bar{\tilde{x}}, \bar{\mu}) + \frac{\lambda^2}{2}\tilde{\bar{y}}^T$ $H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))\tilde{\bar{y}} + o(\lambda)$ where $o(\lambda) = 0$, since $\lim_{\lambda \to 0} \frac{o(\lambda)}{\lambda^2} = 0$.

Since $h(\tilde{x}(\lambda)) = 0$ for every point on the curve, we have $f(\lambda) = L(\tilde{x}(\lambda), \bar{\mu}) = \theta(\tilde{x}(\lambda))$ for all λ in the interval of λ on which the curve is defined. So in the neighbourhood of $\lambda = 0$ on the curve we have from the above

$$\frac{2(\theta(\tilde{x}(\lambda))) - \theta(\bar{\tilde{x}})}{\lambda^2} = \frac{2(f(\lambda) - f(0))}{\lambda^2} = \bar{\tilde{y}}^T H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))\bar{\tilde{y}} + \frac{2(o(\lambda))}{\lambda^2}$$

and since $\bar{\tilde{y}}^T H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))\bar{\tilde{y}} < 0$ and $\lim_{\lambda \to 0} \frac{o(\lambda)}{\lambda^2} = 0$, for all λ sufficiently small $\theta(\tilde{x}(\lambda)) - \theta(\bar{\tilde{x}}) < 0$. For all these λ, $\tilde{x}(\lambda)$ is a point on the curve in the fuzzy feasible region in the neighbourhood of $\tilde{x}$, and this is a contradiction to the fact that $\bar{\tilde{x}}$, is a fuzzy local minimum for (3.1).

In fact it can be verified that $\bar{\tilde{y}}^T H_{\tilde{x}}(L(L\bar{\tilde{x}}, \bar{\mu}))\bar{\tilde{y}} = \left(\frac{d^2 f(\lambda)}{d\lambda^2}\right)_{\lambda=0}$ and if this quantity is < 0, $\lambda = 0$ cannot be a local minimum for the one variable minimization problem of minimizing $f(\lambda) = \theta(\tilde{x}(\lambda))$ over λ; or equivalently, that $\bar{\tilde{x}} = \tilde{x}(0)$ is not a local minimum for $\theta(\tilde{x})$ along the curve $\tilde{x}(\lambda)$.

These facts imply that if $\theta(\tilde{x})$, $h_i(\tilde{x})$ are all twice continuously differentiable, and $\bar{\tilde{x}}$ is a fuzzy regular point which is a fuzzy feasible solution and a fuzzy local minimum for (3.1), there must exist a Lagrange multiplier fuzzy vector $\bar{\mu}$ such that the following conditions hold.

$$h(\bar{\tilde{x}}) = 0$$

$$\nabla_{\tilde{x}} L(\bar{\tilde{x}}, \bar{\mu}) = \nabla\theta(\bar{\tilde{x}}) - \bar{\mu}\nabla h(\bar{\tilde{x}}) = 0$$

$$\tilde{y}^T H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu})) \geq 0 \text{ for all } \tilde{y} \in T = \{\tilde{y} : (\nabla h((\bar{\tilde{x}})))\tilde{y} = 0\}, \tag{3.5}$$

i.e., $H_{\tilde{x}}(L((\bar{\tilde{x}}), \bar{\mu}))$ is *PSD* on the subspace *T*.

These are the second order necessary optimality conditions for a fuzzy regular feasible point $\bar{\tilde{x}}$ to be a fuzzy local minimum for (3.1).

Theorem 3.2 (Sufficient optimality condition for (3.1)). *Suppose* $\theta(\tilde{x})$, $h_i(\tilde{x})$, $i = 1$ *to m are all twice continuously differentiable functions, and* $\bar{\tilde{x}}$ *is a fuzzy feasible point such that there exists a Lagrange multiplier vector* $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m)$ *which together satisfy*

$$h(\bar{\tilde{x}}) = 0$$

$$\nabla \theta(\bar{\tilde{x}}) - \bar{\mu} \nabla h(\bar{\tilde{x}}) = 0$$

$$\tilde{y}^T H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))\tilde{y} > 0 \text{ for all } \tilde{y} \in \{\tilde{y} : \nabla h((\bar{\tilde{x}})) \tilde{y} = 0\}, \tilde{y} \neq 0 \tag{3.6}$$

where $L(\tilde{x}, \bar{\mu}) = \theta(\tilde{x}) - \bar{\mu} h(\tilde{x})$ *is the Lagrangian for* (3.1).

Then $\bar{\tilde{x}}$ is a local minimum for (3.1).

Proof. Suppose $\bar{\tilde{x}}$ is not a local minimum for (3.1). There must exist a sequence of distinct fuzzy feasible points $\{\tilde{x}^r : r = 1, 2, \ldots\}$ converging to $\bar{\tilde{x}}$ such that $\theta(\tilde{x}^r) < \theta(\bar{\tilde{x}})$ for all *r*.

Let $\delta_r = | \bar{\tilde{x}} - \tilde{x}^r |, \tilde{y}^r = (\bar{\tilde{x}} - \tilde{x}^r)/\delta_r$.

Then $\| \tilde{y}^r \| = 1$ for all *r* and $\tilde{x}^r = \bar{\tilde{x}} + \delta_r \tilde{y}^r$.

Thus $\delta_r \to 0^+$ as $r \to \infty$.

Since the sequence of points $\{\tilde{y}^r : r = 1, 2, \ldots, \}$ all lie on the surface of the unit sphere in R^n, a compact set, the sequence has atleast one limit point.

Let $\tilde{\tilde{y}}$ be a limit point of $\{\tilde{y}^r : r = 1, 2, \ldots, \}$.

There must exist a subsequence of $\{\tilde{y}^r : r = 1, 2, \ldots, \}$ which converges to $\tilde{\tilde{y}}$, eliminate all points other than those in this subsequence, and for simplicity call the remaining sequence by the same notation $\{\tilde{y}^r : r = 1, 2, \ldots, \}$.

So now we have a sequence of points $\tilde{x}^r = \bar{\tilde{x}} + \delta_r \tilde{y}^r$ all of them fuzzy feasible, such that $\| \tilde{y}^r \| = 1$ for all r, $\tilde{y}^r \to \tilde{\tilde{y}}$ and $\delta_r \to 0$ as $r \to \infty$.

By feasibility $h(\bar{\tilde{x}} + \delta_r \tilde{y}^r) = 0$ for all *r*, and by the differentiability of $h(\tilde{x})$ we have

$$0 = h(\bar{\tilde{x}} + \delta_r \tilde{y}^r) = h(\bar{\tilde{x}}) + \delta_r \nabla h(\bar{\tilde{x}}) \tilde{y}^r + o(\delta_r)$$

$$= \delta_r \nabla h(\bar{\tilde{x}}) \tilde{y}^r + o(\delta_r).$$

Dividing by $\delta_r > 0$, and taking the limit as $r \to \infty$ we see that $\nabla h(\bar{\tilde{x}}) \bar{\tilde{y}} = 0$.

Since $L(\tilde{x}, \bar{\mu})$ is a twice continuously differentiable function in $\tilde{x}$, applying Taylor's theorem to it, we conclude that for each *r*, there exists a $0 \le \alpha_r \le \delta_r$ such that $L(\bar{\tilde{x}} + \delta_r \tilde{y}^r, \bar{\mu}) = L(\bar{\tilde{x}}\ \bar{\mu}) + \delta_r \nabla_{\tilde{x}} L(\bar{\tilde{x}}, \bar{\mu}) \tilde{y}^r$ $+ (1/2)\delta_r^2 (\tilde{y}^r)^T H_{\tilde{x}} (\bar{\tilde{x}} + \alpha_r \tilde{y}^r, \bar{\mu}))\tilde{y}^r$.

From the fact that $\bar{\tilde{x}} + \delta_r \tilde{y}^r = \tilde{x}^r$ and $\bar{\tilde{x}}$ are feasible, we have $L(\tilde{x}^r, \bar{\mu}) = \theta(\tilde{x}^r)$ and $L(\bar{\tilde{x}}, \bar{\mu}) = \theta(\bar{\tilde{x}})$. Also, from (3.6), $\nabla_{\tilde{x}} L(\bar{\tilde{x}}, \bar{\mu}) = 0$. So, from the above equation, we have

$$\theta(\tilde{x}^r) - \theta(\bar{\tilde{x}}) = (1/2)\delta_r^2 (\tilde{y}^r)^T H_{\tilde{x}} (L(\bar{\tilde{x}} + \alpha_r \tilde{y}^r, \bar{\mu}))\tilde{y}^r. \tag{3.7}$$

Since $0 \le \alpha_r \le \delta_r$ and $\delta_r \to 0$ as $r \to \infty$, and by continuity, $H_{\tilde{x}}(L(\bar{\tilde{x}} + \alpha_r \bar{\tilde{y}}_r, \bar{\mu}))$ converges to $H_{\tilde{x}}(L(\bar{\tilde{x}}, \bar{\mu}))$ as $r \to \infty$. Since $\tilde{y}^r \to \tilde{y}$ as $r \to \infty$, and $\nabla h(\bar{\tilde{x}}) \bar{\tilde{y}} = 0$, from the last condition in (3.6) and contiuity we conclude that when *r* is sufficiently large, the right-hand side of (3.7) is ≥ 0, while the left-hand side is < 0, a contradiction. So, $\bar{\tilde{x}}$ must be a fuzzy local minimum for (3.1).

Thus, (3.7) provides a sufficient condition for a fuzzy feasible point $\bar{\tilde{x}}$ to be a fuzzy local minimum for (3.1).

4. Numerical Example

Example 4.1. Consider the problem minimize $(-1.25, -1, -0.75)\tilde{s}_1 + (-1.25, -1, -0.75)\tilde{s}_2$ subject to $(0.75, 1, 1.25)\tilde{s}_1^2 + (0.75, 1, 1.25)\tilde{s}_2^2 + (-8.25, -8, -7.75) = 0$.

Solution

Given constraint is

$$(0.75, 1, 1.25)\tilde{s}_1^2 + (0.75, 1, 1.25)\tilde{s}_2^2 + (-8.25, -8, -7.75) = 0. \tag{4.1}$$

The Lagrangian is

$$L(\tilde{s}, \lambda) = (-1.25, -1, -0.75)\tilde{s}_1 + (-1.25, -1, -0.75)\tilde{s}_2$$

$$- \lambda[(0.75, 1, 1.25)\tilde{s}_1^2 + (0.75, 1, 1.25)\tilde{s}_2^2 + (-8.25, -8, -7.75)].$$

The first order necessary optimality conditions are

$$\frac{\partial L(\tilde{s}, \lambda)}{\partial \tilde{s}} = [(-1.25, -1, 0.75) - 2(0.75, 1, 1.25)\lambda\tilde{s}_1, (-1.25, -1, -0.75)$$

$$-2(0.75, 1, 1.25)\lambda\tilde{s}_2 = 0.$$

$$\Rightarrow s_1 = \frac{1}{\lambda}(-0.5, -0.5, -0.5) \tag{4.2}$$

$$\tilde{s}_2 = \frac{1}{\lambda}(-0.5, -0.5, -0.5) \tag{4.3}$$

Using (4.2) and (4.3) in (4.1), we get

$$\lambda^2 = (0.05, 0.06, 0.08)$$

$$\lambda = \mp(0.22, 0.24, 0.28)$$

$$\lambda = -(0.22, 0.24, 0.28).$$

$$\tilde{s}_1 = (1.78, 2.08, 2.27)$$

$$\tilde{s}_2 = (1.78, 2.08, 2.27)$$

Therefore, $\bar{\tilde{s}} = \begin{bmatrix} (1.78, 2.08, 2.27) \\ (1 - 78, 2.08, 2.27) \end{bmatrix}$

$$\bar{\lambda} = -(0.22, 0.24, 0.28)$$

$$\frac{\partial L(\tilde{s})}{\partial \tilde{s}} = [(-1.25, -1, -0.75) + (0.33, 0.48, 0.7)\tilde{s}_1, (-1.25, -1, -0.75) + (0.33, 0.48, 0.7)\tilde{s}_1].$$

The Hessian of the Lagrangian is

$$H_{\tilde{s}}(L(\bar{\tilde{s}}, \bar{\lambda})) = \begin{bmatrix} (0.33, 0.48, 0.7) & (0, 0, 0) \\ (0, 0, 0) & (0.33, 0.48, 0.7) \end{bmatrix}$$

$$\tilde{t}^T H_{\tilde{s}}(L(\bar{\tilde{s}}, \bar{\lambda}))\tilde{t} = [\tilde{t}_1, \tilde{t}_2]\begin{bmatrix}(0.33, 0.48, 0.7) & (0, 0, 0) \\ (0, 0, 0) & (0.33, 0.48, 0.7)\end{bmatrix}\begin{bmatrix}\tilde{t}_1 \\ \tilde{t}_2\end{bmatrix}$$

$$= (0.33, 0.48, 0.7)\tilde{t}_1^2 + (0.33, 0.48, 0.7)\tilde{t}_2^2$$

$$= (0.33, 0.48, 0.7)[\tilde{t}_1^2 + \tilde{t}_2^2]$$

$$> \tilde{0}$$

$H_{\tilde{s}}(L(\bar{\tilde{s}}, \bar{\lambda}))$ is *PD*.

Hence $\bar{\tilde{s}}$ satisfies the sufficient condition for being a fuzzy local minimum in this problem.

5. Conclusion

In this paper, the fuzzy nonlinear equality constrained minimization problem is defined and the optimality conditions for this problem are stated. Some examples are discussed based on these optimality conditions.

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