Advances and Applications in Mathematical Sciences
Volume 21, Issue 11, September 2022, Pages 6235-6250

BEST PROXIMITY POINTS FOR GENERALIZED CONTRACTION MAPPINGS ON TOPOLOGICAL SPACES

S. SAHAYARAJJOSEPH NIRMALKUMAR[1] and S. SUJITH[2]

[1]Research Scholar (Reg. No. 20221282091006)
[2]Assistant Professor
Department of Mathematics
St. Xavier's College (Autonomous), Palayamkottai
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli-627012, Tamilnadu, India
E-mail: lssnirmal@gmail.com
sujithsr123@gmail.com

Abstract

In this manuscript, we proved the existence and uniqueness of best proximity points for a map f defined on a union of subsets A, B of a topological space satisfying the conditions (i) $f(A) \subset B, f(B) \subset A$ and (ii) with respect to a continuous function $g : X \times X \to \mathbb{R}, |g(fx, fy)| \leq \frac{\ell}{3}[|g(x, y)| + |g(fx, x)| + |g(fy, y)|] + (1-\ell)D_g(A, B)$ for $\ell \in (0, 1)$ and $\forall x \in A, \forall y \in B$, where $D_g(A, B) = \inf\{|g(x, y)| : x \in A, y \in B\}$. Also, the existence and uniqueness of best proximity points for cyclic topologically Ciric type contraction mapping is proved.

1. Introduction and Preliminaries

Suppose that A and B are non-empty closed subsets of a complete metric space (X, d). A map $f : A \cup B \to A \cup B$ is called cyclic if $f(A) \subset B$, $f(B) \subset A$. Kirk et al. [5] introduced the concept of cyclical contractive mappings in 2003, and extended Banach fixed point result [2] to the category of cyclic mappings. They demonstrated that, for a cyclic contraction map $f : A \cup B \to A \cup B$ (i.e., there exists $\kappa \in [0, 1)$ such that for all $x \in A$ and

2020 Mathematics Subject Classification: 47H09, 47H10.

Keywords: Fixed point, best proximity point, cyclic contraction, cyclic Ciric type contraction, cyclic topologically contraction, topological P-property, orbit sequence.

Received January 12, 2022; Accepted March 5, 2022

$y \in B$, f satisfies $d(fx, fy) \le \kappa d(x, y)$) unique fixed point exists in $A \cap B$. In [4], Eldred and Veeramani are concerned with the case when $A \cap B = \emptyset$, and they weren't looking for a fixed point of f in this case, instead, they were looking for the best proximity point. For instance, they have shown that, for a cyclic contraction map $f : A \cup B \to A \cup B$ (i.e., there exists $\kappa \in [0, 1)$ such that for all $x \in A$ and $y \in B$, f satisfies $d(fx, fy) \le \kappa d(x, y) + (1 - \kappa) dist(A, B)$, where $dist(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$) unique best proximity point exists in $A \cup B$. Several extensions of cyclic contraction map and existence of best proximity points in various generalized spaces have been published in the literature, for example see [8, 12, 13] and the references therein.

In [9], Liepins introduced the concept of fixed point results from standard metric spaces to any arbitrary topological spaces. In [10], Sankar raj et al. expanded the best proximity point results from standard metric spaces to topological spaces. He also proved the best proximity point for r-contractive map is unique. For proving uniqueness he introduced and used the P-property concept. We are interested in the existence and uniqueness of best proximity points for generalized cyclic topologically contraction map and cyclic topologically Ciric type contraction map on topological spaces in this paper.

The following concepts are used throughout the manuscript:

Consider a topological space X and a mapping $f : X \to X$. Fix $x_0 \in X$. For any $n \in \mathbb{N}$, the orbit of f beginning at x_0 is defined as

$$0(x_0, f) := \{f^n(x_0) : n \in \mathbb{N} \cup \{0\}\},$$

where $f^0(x_0) = x_0$ and $f^n(x_0) = f(f^{n-1}(x_0))$. Assume that $g : X \to X$ is a function. Then $g(0(x_0, f)) = \{g(f^n(x_0)) : n \in \mathbb{N} \cup \{0\}\}$ is the image of $0(x_0, f)$ under g. The limit point set $Lim\{x_n\}$ of a sequence $\{x_n\}$ in a topological space X is defined as

$$Lim_{\{x_n\}} := \bigcap_{n=1}^{\infty} \overline{\{x_m : m \ge n\}},$$

Where $\overline{A}$ represents the closure of A in X. If $g : X \to X$ is a continuous mapping and $x \in Lim\,0(x_o, f)$, then $g(x) \in Lim\,g(0(x_0, f))$ is easy to prove. Throughout this paper A and B are nonempty subsets of a topological space X.

Definition 1. If $g : X \times X \to \mathbb{R}$ is a continuous function then $D_g(A, B) = \inf\{|g(x, y)| : x \in A, y \in B\}$ and $\delta_g(A, B) = \sup\{|g(x, y)| : x \in A, y \in B\}$.

Note that, if (X, d) is a metric space and $g = d$ is used in the definition, $D_g(A, B)$ is simply the distance between the sets A and B, which we normally refer to as $dist(A, B)$. Let's look at how cyclic contraction mapping is defined in topological space.

Definition 2. [11] A cyclic map $f : A \cup B \to A \cup B$ is a cyclic topologically contraction map with respect to a continuous function $g : X \times X \to \mathbb{R}$ if for some $\ell \in (0, 1)$ and $\forall x \in A, \forall y \in B, |g(fx, fy)| \leq \ell|g(x, y)| + (1 - \ell)D_g(A, B)$.

In metric space, the concept of cyclic topologically contraction mapping is reduced to Eldred's cyclic contraction mapping [4]. A point $x \in A \cup B$ is said to be a best proximity point of f with respect to g if $|g(x, fx)| = D_g(A, B)$.

2. Main Results

In this section we define generalized cyclic contraction mapping introduced by Karapinar [7] on a topological space and some new results are obtained.

Definition 3. A cyclic mapping $f : A \cup B \to A \cup B$ is called generalized cyclic topologically contraction map with respect to a continuous function $g : X \times X \to \mathbb{R}$ if for some $\ell \in (0, 1)$, the condition

$$|g(fx, fy)| \leq \frac{\ell}{3}[|g(x, y)| + |g(fx, x)| + |g(fy, y)|] + (1 - \ell)D_g(A, B) \qquad (1)$$

holds for all $x \in A, y \in B$.

The following example shows that a cyclic mapping $f : A \cup B \to A \cup B$ is generalized cyclic topologically contraction with respect to a continuous mapping $g : X \times X \to \mathbb{R}$ but not cyclic topologically contraction with respect to the same mapping g.

Example 1. Consider $\mathbb{R}^2$, which has a standard topology. Let $A := \{(0, x) : -1 \leq x \leq 1\}$ and $B := \{(1, y) : -1 \leq y \leq 1\}$ be subsets of $\mathbb{R}^2$. Define a cyclic mapping $f : A \cup B \to A \cup B$ as follows:

$$f(t, x) := \begin{cases} \left(1, \dfrac{-x}{2}\right), & \text{if } t = 0, \\ \left(0, \dfrac{x}{2}\right), & \text{if } t = 1. \end{cases}$$

Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined as $g((x, y), (u, v)) = | y - v |$, for all $(x, y), (u, v) \in \mathbb{R}^2$.

Clearly, g is a continuous function. It is worth noting that $D_g(A, B) = 0$. It is simple to verify that the function f is generalized cyclic topologically contraction with respect to g. But if we take $x = (0, 1)$, $y = (1, 1)$ then $| g(fx, fy) | = 1, | g(x, y) | = 0$. Hence there exists no $\ell \in (0, 1)$ such that $| g(fx, fy) | \leq \ell | g(x, y) | + (1 - \ell) D_g(A, B)$.

Let us now show that the best proximity point on a topological space exists and is unique:

Theorem 4. *Let* $g : X \times X \to \mathbb{R}$ *be a continuous mapping satisfying the following conditions*:

(1) $g(x, y) = g(y, x)$, *for all* $x \in A, y \in B$

(2) $g(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$, *for all* $x_1, x_2 \in A$ *(or* $x_1, x_2 \in B$*)*

Let $f : A \cup B \to A \cup B$ *be a continuous generalized cyclic topologically contraction mapping with respect to g. Suppose that the pair* (A, B) *has P-property with respect to g. Then for any* $u \in A \cup B$, *then set Lim* $O(u, f)$ *is* *either empty or* $Lim O(u, f) \cap A$ *is a singleton set* $\{x\}$, *which satisfy* $|g(x, fx)| = D_g(A, B)$.

Proof. Let $u \in A \cup B$. If Lim $O(u, f)$ is not empty, then for any $y \in Lim O(u, f)$,

$$f(y) \in Lim\, f(O(u, f)) \subseteq Lim O(u, f).$$

Define $h : A \cup B \to \mathbb{R}$ by $h(x) := |g(x, fx)|$. Since f, g are continuous, the function h defined above is also continuous. Hence,

$$h(y) \in Lim\, h(O(u, f)) \tag{2}$$

For some $\ell \in (0, 1)$, we have

$$|g(f^n u, f^{n+1} u)| \leq \frac{\ell}{3}[|g(f^{n-1}u, f^n u)| + |g(f^n u, f^{n-1} u)| + |g(f^{n+1}u, f^n u)|] + (1 - \ell) D_g(A, B)$$

$$\left(1 - \frac{\ell}{3}\right) |g(f^n u, f^{n+1}u)| \leq \frac{2\ell}{3} |g(f^{n-1}u, f^n u)| + (1 - \ell) D_g(A, B)$$

and hence

$$|g(f^n u, f^{n+1}u)| \leq \frac{2\ell}{3 - \ell} |g(f^{n-1}u, f^n u)| + \frac{3(1 - \ell)}{3 - \ell} D_g(A, B)$$

Now, we have

$$|g(f^n u, f^{n+1}u)| \leq \frac{2\ell}{3 - \ell}\left[\frac{2\ell}{3 - \ell} |g(f^{n-2}u, f^{n-1}u)| + \frac{3(1 - \ell)}{3 - \ell} D_g(A, B)\right] + \frac{3(1 - \ell)}{3 - \ell} D_g(A, B)$$

$$= \left(\frac{2\ell}{3 - \ell}\right)^2 |g(f^{n-2}u, f^{n-1}u)| + \left(\frac{2\ell}{3 - \ell}\right) \frac{3(3 - \ell)}{3 - \ell} D_g(A, B) + \frac{3(1 - \ell)}{3 - \ell} D_g(A, B)$$

$$\leq \left(\frac{2\ell}{3 - \ell}\right)^2 + \left[\frac{2\ell}{3 - \ell} |g(f^{n-3}u, f^{n-2}u)| + \frac{3(1 - \ell)}{3 - \ell} D_g(A, B)\right]$$

$$+\left(\frac{2\ell}{3-\ell}\right)\frac{3(1-\ell)}{3-\ell}D_g(A, B)+\frac{3(1-\ell)}{3-\ell}D_g(A, B)$$

$$=\left(\frac{2\ell}{3-\ell}\right)^3|g(f^{n-3}u, f^{n-2}u)|+\left(\frac{2\ell}{3-\ell}\right)^2\frac{3(1-\ell)}{3-\ell}D_g(A, B)$$

$$+\left(\frac{2\ell}{3-\ell}\right)\frac{3(1-\ell)}{3-\ell}D_g(A, B)+\frac{3(1-\ell)}{3-\ell}D_g(A, B)$$

Inductively, we have

$$|g(f^n u, f^{n+1}u)| \le \left(\frac{2\ell}{3-\ell}\right)^n |g(u, fu)|$$

$$+\left[\left(\frac{2\ell}{3-\ell}\right)^{n-1}+\ldots+\left(\frac{2\ell}{3-\ell}\right)+1\right]\frac{3(1-\ell)}{3-\ell}D_g(A, B)$$

As $n\to\infty$, since $\ell\in(0, 1)$, $\left(\frac{(2\ell)}{(3-\ell)}\right)<1$. Therefore $\left(\frac{(2\ell)}{(3-\ell)}\right)^n\to 0$ and $\sum_{i=0}^{n-1}\left(\frac{(2\ell)}{(3-\ell)}\right)^i\to\frac{3-\ell}{3-3\ell}$. Hence the coefficient of $D_g(A, B)$ tends to 1. Thus,

$$\lim_{n\to\infty}|g(f^n u, f^{n+1}u)| \le D_g(A, B) \tag{3}$$

From the definition of $D_g(A, B)$, we have

$$\lim_{n\to\infty}|g(f^n u, f^{n+1}u)| \ge D_g(A, B) \tag{4}$$

From (3) and (4) we have

$$\lim_{n\to\infty}|g(f^n u, f^{n+1}u)| = D_g(A, B)$$

Hence $\{h(f^n u)\}$ converges to $D_g(A, B)$. i.e., $h(O(u, f))$ is a singleton set. From (2), $h(y)=D_g(A, B)$. i.e., $|g(y, fy)|=D_g(A, B)$. i.e., y is a best proximity point of f with respect to g.

Let us now show that the best proximity point is unique. Without loss of generality, assume that y is in A. Suppose there exist two points x_0, y_0 in A such that $x_0, y_0 \in LimO(u, f)$ satisfying

$$|g(x_0, fx_0)| = D_g(A, B) \tag{5}$$

$$|g(y_0, fy_0)| = D_g(A, B) \tag{6}$$

Since f is generalized cyclic topologically contraction with respect to g and using equation (5), for some $\ell \in (0, 1)$,

$$|g(fx_0, f^2x_0)| \leq \frac{\ell}{3}[|g(x_0, fx_0)| + |g(fx_0, x_0)| + |g(f^2x_0, fx_0)|] + (1-\ell)D_g(A, B)$$

$$\left(1 - \frac{\ell}{3}\right)|g(fx_0, f^2x_0)| \leq \frac{2\ell}{3}D_g(A, B) + (1-\ell)D_g(A, B)$$

and hence $|g(fx_0, f^2x_0)| \leq D_g(A, B)$.

Suppose that $|g(fx_0, f^2x_0)| < D_g(A, B)$ it will contradict the definition of $D_g(A, B)$.

Therefore $|g(fx_0, f^2x_0)| = D_g(A, B)$

Hence, $|g(x_0, fx_0)| = |g(f^2x_0, fx_0)| = D_g(A, B)$. By P-property, we have $|g(x_0, f^2x_0)| = |g(fx_0, fx_0)| = 0$. Thus $x_0 = f^2x_0$. Similarly $y_0 = f^2y_0$.

Now $|g(y_0, fx_0)| = |g(f^2y_0, fx_0)|$

$$\leq \frac{\ell}{3}[|g(fy_0, x_0)| + |g(f^2y_0, fy_0)| + |g(fx_0, x_0)|] + (1-\ell)D_g(A, B)$$

$$= \frac{\ell}{3}|g(fy_0, x_0)| + \frac{\ell}{3}D_g(A, B) + \frac{\ell}{3}D_g(A, B) + (1-\ell)D_g(A, B)$$

$$= \frac{\ell}{3}|g(fy_0, x_0)| + \frac{3-\ell}{3}D_g(A, B)$$

$$\leq \frac{\ell}{3} | g(fy_0, x_0) | + \frac{3-\ell}{3} | g(fy_0, x_0) |$$

this is

$$| g(y_0, fx_0) | \leq | g(fy_0, x_0) | \tag{7}$$

Now $| g(x_0, fy_0) | = | g(f^2 x_0, fy_0) |$

$$\leq \frac{\ell}{3} [| g(fx_0, y_0) | + | g(f^2 x_0, fx_0) | + | g(fy_0, y_0) |] + (1-\ell) D_g(A, B)$$

$$= \frac{\ell}{3} | g(fx_0, y_0) | + \frac{\ell}{3} D_g(A, B) + \frac{\ell}{3} D_g(A, B) + (1-\ell) D_g(A, B)$$

$$= \frac{\ell}{3} | g(fx_0, y_0) | + \frac{3-\ell}{3} D_g(A, B)$$

$$\leq \frac{\ell}{3} | g(fx_0, y_0) | + \frac{3-\ell}{3} | g(fx_0, y_0) |$$

that is,

$$| g(x_0, fy_0) | \leq | g(fx_0, y_0) | \tag{8}$$

From (7) and (8), we get that $| g(y_0, fx_0) | = | g(x_0, fy_0) |$. Also,

$$\begin{aligned} | g(y_0, fx_0) | &= | g(f^2 y_0, fx_0) | \\ &\leq \frac{\ell}{3} | g(fy_0, x_0) | + \frac{3-\ell}{3} D_g(A, B) \\ &= \frac{\ell}{3} | g(y_0, fx_0) | + \frac{3-\ell}{3} D_g(A, B) \end{aligned}$$

$$\left(1 - \frac{\ell}{3}\right) | g(y_0, fx_0) | \leq \frac{3-\ell}{3} D_g(A, B)$$

$$| g(y_0, fx_0) | \leq D_g(A, B)$$

From the definition of $D_g(A, B)$, we have $| g(y_0, fx_0) | = D_g(A, B)$

Hence $| g(y_0, fx_0) | = | g(x_0, fy_0) | = D_g(A, B)$.

Using equation (5) and P-property, $|g(x_0, fx_0)| = |g(y_0, fx_0)| = D_g(A, B) \Rightarrow |g(x_0, y_0)| = |g(fx_0, fx_0)| = 0$. Hence $x_0 = y_0$. i.e., the best proximity point of f with respect to g is unique in A.

Example 2. Consider the subsets A and B of $\mathbb{R}^2$ and the functions f and g are given as in Example 1 and we get $D_g(A, B) = 0$.

When $x = 0 \Rightarrow f(t, 0) = \begin{cases} (1, 0), & \text{if } t = 0, \\ (0, 0), & \text{if } t = 1. \end{cases}$

We also have $|g((0, 0), f(0, 0))| = |g((0, 0), (1, 0))| = 0 = D_g(A, B)$ and $|g((1, 0), f(1, 0))| = |g((1, 0), (0, 0))| = 0 = D_g(A, B)$. Hence, for any $u = (x, y) \in A \cup B$, $Lim\, O(u, f) = \{(0, 0), (1, 0)\}$ and Lim $O(u, f) \cap A = (0, 0)$. As a result, the best proximity point of f with respect to g is $(0, 0)$.

Let (X, d) be a metric space. Then $d : X \times X \to \mathbb{R}$ is a continuous function. As a result, if we take $g = d$ then we obtain the Theorem 2.3 by Karapinar [6].

Now let us define the notion of cyclic Ciric type contraction mapping [1, 3] on a topological space.

Definition 5. A cyclic mapping $f : A \cup B \to A \cup B$ is said to be a cyclic topologically Ciric type contraction map with respect to a continuous function $g : X \times X \to \mathbb{R}$ if there exists a number ℓ, $0 \leq \ell < 1$, such that

$$|g(fx, fy)| \leq \ell \max\{|g(x, y)|, |g(x, fx)|, |g(y, fy)|\} + (1 - \ell) D_g(A, B)$$

holds for every $x \in A$, $y \in B$.

Let us now see an example.

Example 3. Consider $\mathbb{R}^2$ with usual topology. Let $A := \{0\} \times [0, 1]$, $B := \{1\} \times [0, 1]$ be subsets of $\mathbb{R}^2$. Define a cyclic mapping $f : A \cup B \to A \cup B$ as

$$f(t, x) := \begin{cases} \left(1, \dfrac{x}{4}\right), & \text{if } t = 0, \\ \left(0, \dfrac{x}{4}\right), & \text{if } t = 1. \end{cases}$$

Let $g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined as $g((x, y), (u, v)) = | y - v |$, for all $(x, y), (u, v) \in \mathbb{R}^2$. Clearly, g is a continuous function. Note that $D_g(A, B) = 0$. It is easy to verify that the function f is cyclic topologically Ciric type contraction with respect to g.

Let us now prove the existence and uniqueness of best proximity pairs for cyclic topologically Ciric type contraction mapping.

Lemma 1. *Let $f : A \cup B \to A \cup B$ be a cyclic topologically Ciric type contraction mapping with respect to a continuous function $g : X \times X \to \mathbb{R}$. Then for each $x \in A \cup B$ and for all $i, j, n \in \mathbb{N}$ with $i, j \leq n$ and $i + j$ is an odd integer, implies $| g(f^i x, f^j x) | \leq \ell \cdot \delta_g[O(x, n)] + (1 - \ell) D_g(A, B)$.*

Proof. Let $x \in A \cup B$ be arbitrary and $n \in \mathbb{N}$. Let i, j be any two positive integers such that $i, j \leq n$ and $i + j$ is an odd integer. Since f is a cyclic topologically Ciric type contraction, we have

$$\begin{aligned} | g(f^i x, f^j x) | &= | g(ff^{i-1} x, ff^{j-1} x) | \\ &\leq \ell \cdot \max \{| g(f^{i-1} x, f^{j-1} x) |, | g(f^{i-1} x, f^i x) |, | g(f^{j-1} x, f^j x) |\} \\ &\quad + (1 - \ell) D_g(A, B) \\ &\leq \ell \cdot \delta_g[O(x, n)] + (1 - \ell) D_g(A, B). \end{aligned}$$

Lemma 2. *If $f : A \cup B \to A \cup B$ is a cyclic topologically Ciric type contraction mapping with respect to a continuous function $g : X \times X \to \mathbb{R}$ and $x \in A \cup B$, then for every positive integer n there exists a positive integer $m \leq n$ such that $(x, f^m x) \in A \times B$ (or $B \times A$) and $| (x, f^m x) | = \delta_g[O(x, n)]$.*

Proof. Let $n \in \mathbb{N}$ and i, j be any two positive integers such that $i, j \leq n$ and $i + j$ is an odd integer. Let $x \in A \cup B$, since f is a cyclic topologically Ciric type contraction, by Lemma 1, we have $| g(f^i x, f^j x) | \leq \ell \cdot \delta_g[O(x, n)] + (1 - \ell) D_g(A, B)$. We know that $D_g(A, B) \leq \delta_g[O(x, n)]$. Suppose $D_g(A, B) < \delta_g[O(x, n)]$, then $| g(f^i x, f^j x) | \leq \ell \cdot \delta_g[O(x, n)] + (1 - \ell)\delta_g[O(x, n)]$, that is $| g(f^i x, f^j x) | < \delta_g[O(x, n)]$. Therefore, there exists a positive integer $m \leq n$ such that $| g(x, f^m x) = \delta_g[O(x, n)]$. On the other hand, if $D_g(A, B) = \delta_g[O(x, n)]$, then by the inequality $D_g(A, B) \leq | g(x, f^m x) | \leq \delta_g[O(x, n)]$, we get $| g(x, f^m x) | = \delta_g[O(x, n)]$, which proves the lemma.

Lemma 3. *Let* $f : A \cup B \to A \cup B$ *be a cyclic topologically Ciric type contraction mapping with respect to a continuous function* $g : X \times X \to \mathbb{R}$ *such that* $| g(x, z) | \leq | g(x, y) | + | g(y, z) |$ *for all* $x, y, z, \in X$, *then*

$$\delta_g[O(x, \infty)] \leq \left(\frac{1}{1 - \ell}\right) | g(x, fx) | + D_g(A, B)$$

holds for all $x \in A \cup B$.

Proof. Let $x \in A \cup B$ be arbitrary. The lemma will follow if we show that $\delta_g[O(x, n)] \leq \left(\frac{1}{1 - \ell}\right) | g(x, fx) | + D_g(A, B)$ for all $n \in \mathbb{N}$.

Let n be any positive integer. By Lemma 2, there exists $f^m x \in O(x, n) (1 \leq m \leq n)$ such that $| g(x, f^m x) | = \delta_g[O(x, n)]$. Applying triangle inequality and Lemma 1, we get

$$| g(x, f^m x) | \leq | g(x, fx) | + | g(fx, f^m x) |$$

$$\leq | g(x, fx) | + \ell \cdot \delta_g[O(x, n)] + (1 - \ell) D_g(A, B)$$

$$= | g(x, fx) | + \ell \cdot | g(x, f^m x) | + (1 - \ell) D_g(A, B).$$

Therefore, $\delta_g[O(x, n)] = | g(x, f^m x) | \leq \left(\frac{1}{1 - \ell}\right) | g(x, fx) | + D_g(A, B)$. Since n was arbitrary the proof is completed.

Theorem 6. *Let* $g: X \times X \to \mathbb{R}$ *be a continuous mapping satisfying the following conditions:*

(1) $g(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$, *for all* $x_1, x_2 \in A$ *(or* $x_1, x_2 \in B$*)*

(2) $g(x, y) = g(y, x)$, *for all* $x \in A, y \in B$

(3) $|g(x, z)| \leq |g(x, y)| + |g(y, z)|$ *for all* $x, y, z \in X$

Let $f: A \cup B \to A \cup B$ *be a continuous cyclic topologically Ciric type contraction mapping with respect to g. Suppose that the pair* (A, B) *has P-property with respect to g. Then for any* $u \in A \cup B$, *the set* $Lim O(u, f)$ *is either empty or* $Lim O(u, f) \cap A$ *is a singleton set* $\{x\}$ *which satisfy* $|g(x, fx)| = D_g(A, B)$.

Proof. Let $u \in A \cup B$. Suppose that $Lim O(u, f)$ is nonempty then for any $y \in Lim O(u, f)$,

$$f(y) \in Lim f(O(u, f)) \subseteq Lim O(u, f)$$

Define $h: A \cup B$. by $h(x) := |g(x, fx)|$. Since f, g are continuous, the function h defined above is also continuous. Hence,

$$h(y) \in Lim h(O(u, f)) \tag{9}$$

For some $\ell \in [0, 1)$, $n \in \mathbb{N}$, and it follows from lemma 1 that

$$|g(f^n u, f^{n+1} u)| = |g(ff^{n-1} u, f^2 f^{n+1} u)|$$

$$\leq \ell \cdot \delta_g[O(f^{n-1} u, 2)] + (1 - \ell) D_g(A, B)$$

From the Lemma 2, there exists a positive integer $m_1 \leq 2$, such that

$$\delta_g[O(f^{n-1} u, 2)] = |g(f^{n-1} u, f^{m_1} f^{n-1} u)|$$

Again, by Lemma 1, we have

$$|g(f^{n-1} u, f^{m_1} f^{n-1} u)| = |g(ff^{n-2} u, f^{m_1 + 1} f^{n-1} u)|$$

$$\leq \ell \cdot \delta_g[O(f^{n-2} u, m_1 + 1)] + (1 - \ell) D_g(A, B)$$

$$\leq \ell \cdot \delta_g[O(f^{n-2}u, 3)] + (1-\ell)D_g(A, B)$$

Therefore, we have the following system of inequalities,

$$\begin{aligned} | g(f^n u, f^{n+1}u) | &\leq \ell \cdot \delta_g[O(f^{n-1}u, 2)] + (1-\ell)D_g(A, B) \\ &\leq \ell \cdot \{\ell \cdot \delta_g[O(f^{n-2}u, 3)] + (1-\ell)D_g(A, B)\} \\ &\quad + (1-\ell)D_g(A, B) \\ &= \ell^2 \cdot \delta_g[O(f^{n-2}u, 3)] + \ell(1-\ell)D_g(A, B) \\ &\quad + (1-\ell)D_g(A, B) \\ &= \ell^2 \cdot \delta_g[O(f^{n-2}u, 3)] + (1-\ell^2)D_g(A, B) \end{aligned}$$

Inductively, we have

$$| g(f^n u, f^{n+1}u) | \leq \ell^n \cdot \delta_g[O(u, n+1)] + (1-\ell^n)D_g(A, B).$$

Since $n \in \mathbb{N}$, it follows from Lemma 3 that

$$\begin{aligned} | g(f^n u, f^{n+1}u) | &\leq \ell^n\left(\left(\frac{1}{1-\ell}\right)| g(u, fu) | + D_g(A, B)\right) + (1-\ell^n)D_g(A, B). \\ &= \left(\frac{\ell^n}{1-\ell}\right)| g(u, fu) | + \ell^n D_g(A, B) + (1-\ell^n)D_g(A, B) \\ &= \left(\frac{\ell^n}{1-\ell}\right)| g(u, fu) | + D_g(A, B) \end{aligned}$$

Since $\ell \in [0, 1)$, $\ell \to 0$ as $n \to \infty$ Therefore,

$$\lim_{n\to\infty} | g(f^n u, f^{n+1}u) | \leq D_g(A, B) \tag{10}$$

From the definition of $D_g(A, B)$, we have

$$\lim_{n\to\infty} | g(f^n u, f^{n+1}u) | \geq D_g(A, B) \tag{11}$$

From (10) and (11) we have

$$\lim_{n\to\infty} | g(f^n u, f^{n+1} u) | = D_g(A, B)$$

Hence $\{h(f^n)\}$ converges to $D_g(A, B)$. i.e., $Lim h(O(u, f))$ is a singleton set. From (9), $h(y) = D_g(A, B)$. i.e., $| g(y, fy) | = D_g(A, B)$. i.e., y is a best proximity point of f with respect to g.

Let us now show that the best proximity point is unique. Without loss of generality, assume that y is in A. Suppose there exist two points x_0, y_0 in A such that $x_0, y_0 \in Lim O(u, f)$ satisfying

$$| g(x_0, fx_0) | = D_g(A, B) \tag{12}$$

$$| g(y_0, fy_0) | = D_g(A, B) \tag{13}$$

Then

$$| g(fx_0, f^2 x_0) | \leq \ell \cdot \max\{| g(x_0, fx_0) |, | g(x_0, fx_0) |, | g(x_0, f^2 x_0) |\}$$
$$+ (1 - \ell) D_g(A, B)$$

which implies

$$| g(fx_0, f^2 x_0) | \leq \ell | g(fx_0, f^2 x_0) | + (1 - \ell) D_g(A, B).$$

That is $| g(fx_0, f^2 x_0) | \leq D_g(A, B)$. By the definition of $D_g(A, B)$ we have

$$| g(fx_0, f^2 x_0) | = D_g(A, B).$$

Hence $| g(x_0, fx_0) | = | g(f^2 x_0, fx_0) | = D_g(A, B)$. By P-property, we have $| g(x_0, f^2 x_0) | = | g(fx_0, fx_0) | = 0$. Thus $x_0 = f^2 x_0$. Similarly $y_0 = f^2 y_0$.

Now $| g(fx_0, y_0) | = | g(fx_0, f^2 y_0) |$

$$\leq \ell \cdot \max\{| g(x_0, fy_0) |, | g(x_0, fx_0) |, | g(fy_0, f^2 y_0) |\}$$
$$+ (1 - \ell) D_g(A, B)$$

that is,

$$| g(fx_0, y_0) | \leq \ell | g(x_0, fy_0) | + (1 - \ell) D_g(A, B)$$

$$= \ell | g(f^2 x_0, fy_0) | + (1 - \ell) D_g(A, B)$$

$$\leq \ell [\ell \cdot \max \{| g(fx_0, y_0) |, | g(fx_0, f^2 fx_0) |, | g(y_0, fy_0) |\} + (1 - \ell) D_g(A, B)]$$

$$+ (1 - \ell) D_g(A, B)$$

hence

$$| g(fx_0, y_0) | \leq \ell^2 | g(fx_0, y_0) | + (1 - \ell^2) D_g(A, B)$$

$$(1 - \ell^2) | g(fx_0, y_0) | \leq (1 - \ell^2) D_g(A, B)$$

$$| g(fx_0, y_0) | \leq D_g(A, B)$$

From the definition of $D_g(A, B)$, we have $| g(fx_0, y_0) | \leq D_g(A, B)$. Using equation (12) and P-property, $| g(fx_0, x_0) | = | g(fx_0, y_0) | = D_g(A, B)$ $\Rightarrow | g(x_0, y_0) | = | g(fx_0, fx_0) | = 0.$ Hence x_0, y_0 i.e., the best proximity point of f with respect to g is unique in A.

Example 4. Consider the subsets A and B of $\mathbb{R}^2$ and the functions f and g as shown in Example 3 and we get $D_g(A, B) = 0$.

When

$$x = 0 \Rightarrow f(t, 0) = \begin{cases} (1, 0), & \text{if } t = 0, \\ (0, 0), & \text{if } t = 1. \end{cases}$$

There's also $| g((0, 0), f(0, 0)) | = | g((0, 0), (1, 0)) | = 0 = D_g(A, B)$ and $| g((1, 0), f(1, 0)) | = | g((1, 0), (0, 0)) | = 0 = D_g(A, B)$. Hence, for any $u = (x, y) \in A \cup B$, $LimO(u, f) = \{(0, 0), (1, 0)\}$ and $LimO(u, f) \cap A = (0, 0)$. That is, $(0, 0)$ is the best proximity point of f with respect to g.

If (X, d) is a metric space then $d : X \times X \to \mathbb{R}$ is continuous function. As a result, if we take $g = d$ then we obtain the Theorem 2.5 by Aydi [1] and the Theorem 2.4 by Karapinar [6].

References

[1] H. Aydi and A. Felhi, Best proximity points for cyclic Kannan-Chatterjea-Ciric type contractions on metric-like spaces, J. Nonlinear Sci. Appl. 9 (2016), 2458-2466.

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrals, Fund. Math. 3(1) (1992), 133-181.

[3] L. J. B. Ciric, A generalization of Banach's Contraction Principle, 45, Proceedings of the American Mathematical Society, (1974).

[4] A. A. Eldred and P. Veeramani, Existence and convergence of the best proximity points, J. Math. Anal. Appl. 323(2) (2006), 1001-1006.

[5] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contrative conditions, Fixed Point Theory 4(1) (2003), 79-89.

[6] E. Karapinar and I. M. Erhan, Best proximity point on different type contractions, Appl. Math. Inf. Sci. 5(3) (2011), 558-569.

[7] E. Karapinar, Best proximity points of cyclic mappings, Applied Mathematics Letter 25(11) (2012), 1761-1766.

[8] M. S. Khan, M. Menaka, Geno Kadwin Jacob and M. Marudai, Proximity points for cyclic 2-convex contraction mappings, Nonlinear Functional Analysis and Applications 25(1) (2020), 2466-0973.

[9] A. Liepins, Edelstein's fixed point theorem in topological spaces, Numer. Funct. Anal. Optim. 2(5) (1980), 387-396.

[10] V. Sankar Raj and T. Piramatchi, Best proximity point theorems in topological spaces, J. Fixed Point Theory Appl. 22 Article, 2 (2020).

[11] S. Sahayarajjoseph Nirmalkumar and S. Sujith, Best proximity points for cyclic contraction mapping on topological spaces, (communicated).

[12] S. Sujith and P. S. Srinivasan, Fixed point theorem for k-quasi contraction map, IJPAM 114(6) (2017), 87-88.

[13] S. Sujith, A study on fixed point theorems and best proximity pair theorems, Ph.D. Thesis, Department of Mathematics, Bharathidasan University, Trichy, India.