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KY FAN TYPE INEQUALITIES

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Abstract

In this paper, Ky Fan inequality of arithmetic, geometric, harmonic, contra harmonic, heron mean and complementary of arithmetic, geometric mean with respect to Heron mean are discussed.

1. Introduction

On the basis of proportion ten Greek means were defined, the familiar ones are

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$$A = \frac{n_1 + n_2}{2},\ G = \sqrt{(n_1 n_2)},\ C = \frac{n_1^2 n_2^2}{n_1 + n_2},\ H = \frac{2n_1 n_2}{n_1 + n_2},$$

$$\text{and } H_e = \frac{1}{3}(n_1 + \sqrt{n_1 n_2} + n_2).$$

Definition [2]**.** A mean V is called complementary to U with respect to P (or P-complementary to U), if it verifies $P(U, V) = P$, it is denoted by $U^P = V$.

$$A^{He}(n_1, n_2) = \frac{1}{4}[3n_1 + 3n_2 + 4\sqrt{n_1 n_2} - \sqrt{(n_1 + n_2)(5n_1 + 5n_2 + 8\sqrt{n_1 n_2})}]$$

$$G^{He}(n_1, n_2) = \frac{1}{2}[2n_1 + 2n_2 + \sqrt{n_1 n_2} - \sqrt{4(n_1 + n_2)\sqrt{n_1 n_2} + n_1 n_2}],$$

$$C^{He}(n_1, n_2) = n_1 + n_2 + \sqrt{n_1 n_2}$$

$$- \frac{n_1^2 n_2^2 + \sqrt{(n_1^2 n_2^2)^2 + 8n_1 n_2(n_1^2 n_2^2) + 4\sqrt{n_1 n_2}(n_1^2 n_2^2)(n_1 + n_2)}}{2(n_1 + n_2)},$$

and

$$H^{He}(n_1, n_2) = n_1 + n_2 - \frac{n_1 n_2}{n_1 + n_2} + \sqrt{n_1 n_2}$$

$$- \frac{\sqrt{n_1 n_2(2(n_1 + n_2)^2 - 3n_1 n_2 + 2\sqrt{n_1 n_2}(n_1 + n_2))}}{n_1 + n_2}$$

are respectively complementary of arithmetic mean, geometric mean, Contra harmonic mean, Harmonic mean with respect to Heron mean. Results on Ky Fan Inequality are found in [1, 2, 9], similar type mean inequalities in ([3]-[10], [15], [16]) and their strengthening them in ([11]-[17]).

2. Results

For any arbitrary non negative real numbers $h \in \left(0, \frac{1}{2}\right]$ and $(1 - h) \in \left[\frac{1}{2}, 1\right)$ is represented as a function given below,

$$f(h) = \begin{cases} h & 0 < h \le \frac{1}{2} \\ 1 - h & \frac{1}{2} \le h < 1. \end{cases}$$

For two variables $n_1, n_2 \in \left(0, \frac{1}{2}\right]$,

Let $f_1(n_1, n_2) = \dfrac{A^{He}(n_1, n_2)}{A^{He'}(n_1, n_2)}$

$$= \frac{\left[3n_1 + 3n_2 + 4\sqrt{n_1 n_2} - \sqrt{(n_1 + n_2)(5n_1 + 5n_2 + 8\sqrt{n_1 n_2})}\right]}{\left[3(1 - n_1) + 3(1 - n_2) + 4\sqrt{(1 - n_1)(1 - n_2)} - \sqrt{\begin{array}{l}(2 - n_1 - n_2)(10 - n_1 - n_2) \\ + 8\sqrt{(1 - n_1)(1 - n_2)}\end{array}}\right]}$$

$$f_2(n_1, n_2) = \frac{H(n_1, n_2)}{H'(n_1, n_2)} = \frac{n_1 n_2}{n_1 + n_2} \frac{(1 - n_2 - n_1 + n_1 n_2)}{2 - (n_1 + n_2)}$$

$$f_3(n_1, n_2) = \frac{G(n_1, n_2)}{G'(n_1, n_2)} = \frac{\sqrt{n_1 n_2}}{\sqrt{1 - n_1 - n_2 + n_1 n_2}}$$

$$f_4(n_1, n_2) = \frac{H_e(n_1, n_2)}{H'_e(n_1, n_2)} = \frac{n_1 + \sqrt{n_1 n_2} + n_2}{2 - n_1 - n_2 + \sqrt{1 - n_1 - n_2 + n_1 n_2}}$$

$$f_5(n_1, n_2) = \frac{A(n_1, n_2)}{A'(n_1, n_2)} = \frac{n_1 + n_2}{2 - (n_1 + n_2)},$$

$$f_6(n_1, n_2) = \frac{G^{He}(n_1, n_2)}{G^{He'}(n_1, n_2)}$$

$$= \frac{2n_1 + 2n_2 + \sqrt{n_1 n_2} - \sqrt{4(n_1 + n_2)\sqrt{n_1 n_2} + n_1 n_2}}{2(1 - n_1) + 2(1 - n_2) + \sqrt{(1 - n_1)(1 - n_2)} - \sqrt{\begin{array}{l}4(2 - n_1 - n_2)\sqrt{(1 - n_1)(1 - n_2)} \\ + (1 - n_1)(1 - n_2)\end{array}}}$$

$$f_7(n_1, n_2) = \frac{C(n_1, n_2)}{C'(n_1, n_2)} = \frac{n_1^2 n_2^2}{n_1 + n_2} \frac{2 - (n_1 + n_2)}{2 + n_1^2 n_2^2 - 2(n_1 + n_2)}$$

Put $n_2 = \frac{1}{2}$ and $n_1 = t \in (0, 1/2]$, then by using Taylor's series expansion

$$f_1\left(t, \frac{1}{2}\right) = 1 + 2\left(t - \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)^2 + 3.3333\left(t - \frac{1}{2}\right)^3 + 4.66667\left(t - \frac{1}{2}\right)^4 + \dots$$

$$f_2\left(t, \frac{1}{2}\right) = 1 + 2\left(t - \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)^2 + 6\left(t - \frac{1}{2}\right)^3 + 10\left(t - \frac{1}{2}\right)^4 + \dots$$

$$f_3\left(t, \frac{1}{2}\right) = 1 + 2\left(t - \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)^2 + 4\left(t - \frac{1}{2}\right)^3 + 6\left(t - \frac{1}{2}\right)^4 + \dots$$

$$f_4\left(t, \frac{1}{2}\right) = 1 + 2\left(t + \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)^2 + \frac{8}{3}\left(t - \frac{1}{2}\right)^3 + \frac{10}{3}\left(t - \frac{1}{2}\right)^4 + \dots$$

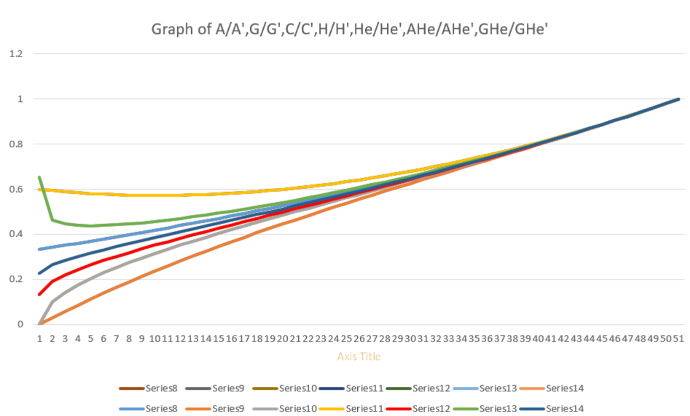
$$f_5\left(t, \frac{1}{2}\right) = 1 + 2\left(t - \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)^2 + 2\left(t - \frac{1}{2}\right)^3 + 2\left(t - \frac{1}{2}\right)^4 + \dots$$

$$f_6\left(t, \frac{1}{2}\right) = 1 + 2\left(t - \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)^2 + 1.3333\left(t - \frac{1}{2}\right)^3 + 0.66667\left(t - \frac{1}{2}\right)^4 + \dots$$

$$f_7\left(t, \frac{1}{2}\right) = 1 + 2\left(t - \frac{1}{2}\right) + 2\left(t - \frac{1}{2}\right)^2 - 2\left(t - \frac{1}{2}\right)^3 - 6\left(t - \frac{1}{2}\right)^4 + \dots.$$

It was observed that the series expansions are same up to 2nd degree term and illustration by graph. By considering 3rd and 4th degree terms, the following interpolating Ky Fan type inequality chain holds.

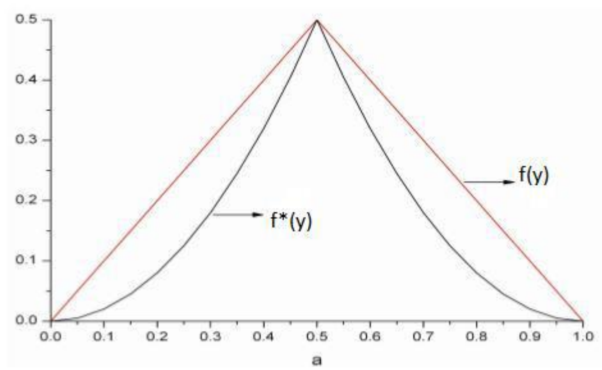
$$\frac{H}{H^1} < \frac{G}{G^1} < \frac{A^{He}}{A^{He'}} < \frac{H_e}{H_e^1} < \frac{A}{A'} < \frac{G^{He}}{G^{He'}} < \frac{C}{C'}.$$



The motivation of the work carried out by the eminent researchers and discussion with experts, results in study a function which is parabolic in nature given below;

$$f^*(h) = \begin{cases} 2h^2 & 0 < h \le \frac{1}{2} \\ 2(1-h)^2 & \frac{1}{2} \le h < 1. \end{cases}$$

The functions $f(h)$ and $f^*(h)$ are graphically represented as shown below.



Let $f(h) = 2h^2$ for $h \in (0, 1/2]$ and $2(1-h)^2$ for $h \in [1/2, 1)$ then,

$A^*(n_1, n_2) = n_1^2 + n_2^2,\ G^*(n_1, n_2) = 2n_1n_2$

$C^*(n_1, n_2) = \dfrac{2n_1^4 + 2n_2^4}{n_1^2 + n_2^2},\ H^*(n_1, n_2) = \dfrac{4n_1^2n_2^2}{n_1^2 + n_2^2}$, and

$H_e^*(n_1, n_2) = \dfrac{2n_1^2 + 2n_2^2 + 2n_1n_2}{3}$

$A^{(He)^*}(n_1, n_2) = \dfrac{1}{2}\left(3n_1^2 + 3n_2^2 + 4n_1n_2 - \sqrt{(n_1^2n_2^2)(5n_1^2 + 5n_2^2 + 8n_1n_2)}\right),$

$G^{(He)^*}(n_1, n_2) = 2n_1^2 + 2n_2^2 + n_1n_2 - \sqrt{4(n_1^2n_2^2)n_1n_2 + n_1^2n_2^2}$

For $n_1' = 1 - n_1,\ n_2' = 1 - n_2 \in [1/2, 1)$, the above said means are given by

$$A^{*'}(n_1, n_2) = (1-n_1)^2 + (1-n_2)^2, \; G^{*'}(n_1, n_2) = 2(1-n_1)(1-n_2),$$

$$C^{*'}(n_1, n_2) = \frac{2((1-n_1)^4 + (1-n_2)^4)}{(1-n_1)^2 + (1-n_2)^2}, \; H^{*'}(n_1, n_2) = \frac{4((1-n_1)^4 + (1-n_2)^4)}{(1-n_1)^2 + (1-n_2)^2},$$

$$H_e^{*'}(n_1, n_2) = \frac{2}{3}((1-n_1)^2 + (1-n_2)^2 + (1-n_1)(1-n_2))$$

$$A^{(He)^{*'}}(n_1, n_2) = \frac{1}{2}[3(1-n_1)^2 + 3(1-n_2)^2 + 4(1-n_1)(1-n_2)$$

$$-\frac{1}{2}\sqrt{(1-n_1)^2 + (1-n_2)^2(5(1-n_1)^2 + 5(1-n_2)^2 + 8(1-n_1)(1-n_2)}$$

$$G^{(He)^{*'}}(n_1, n_2) = 2(1-n_1)^2 + 2(1-n_2)^2 + (1-n_1)(1-n_2)$$

$$-\sqrt{4((1-n_1)^2(1-n_2)^2(1-n_1)(1-n_2) + (1-n_1)^2(1-n_2)^2}.$$

Let $f_8(n_1, n_2) = \dfrac{A^{(He)^{*}}(n_1, n_2)}{A^{(He)^{*'}}(n_1, n_2)}$

$$= \frac{3n_1^2 + 3n_2^2 + 4n_1n_2 - \sqrt{(n_1^2n_2^2)(5n_1^2 + 5n_2^2 + 8n_1n_2)}}{\begin{array}{l}3(1-n_1)^2 + 3(1-n_2)^2 + 4(1-n_1)(1-n_2)\\ -\sqrt{(1-n_1)^2 + (1-n_2)^2(5(1-n_1)^2 + 5(1-n_2)^2 + 8(1-n_1)(1-n_2)}\end{array}}$$

$$f_9(n_1, n_2) = \frac{H^{*}(n_1, n_2)}{H^{*'}(n_1, n_2)} = \frac{n_1^2n_2^2}{n_1^2 + n_2^2}\,\frac{(1-n_1)^2 + (1-n_2)^2}{(1-n_1)^2(1-n_2)^2}$$

$$f_{10}(n_1, n_2) = \frac{G^{*}(n_1, n_2)}{G^{*'}(n_1, n_2)} = \frac{n_1n_2}{(1-n_1)(1-n_2)}$$

$$f_{11}(n_1, n_2) = \frac{H_e^{*}(n_1, n_2)}{H_e^{*'}(n_1, n_2)} = \frac{n_1^2 + n_2^2 + n_1n_2}{(1-n_1)^2 + (1-n_2)^2 + (1-n_1)(1-n_2)}$$

$$f_{12}(n_1, n_2) = \frac{A^{*}(n_1, n_2)}{A^{*'}(n_1, n_2)} = \frac{n_1^2 + n_2^2}{(1-n_1)^2 + (1-n_2)^2},$$

$$f_{13}(n_1, n_2) = \frac{G^{(He)^*}(n_1, n_2)}{G^{(He)^{*'}}(n_1, n_2)}$$

$$= \frac{2n_1^2 + 2n_2^2 + n_1 n_2 - \sqrt{4(n_1^2 n_2^2) n_1 n_2 + n_1^2 n_2^2}}{2(1-n_1)^2 + 2(1-n_2)^2 + (1-n_1)(1-n_2) - \sqrt{\begin{array}{l}4((1-n_1)^2(1-n_2)^2(1-n_1) \\ (1-n_2) + (1-n_1)^2(1-n_2)^2\end{array}}}$$

$$f_{14}(n_1, n_2) = \frac{C^*(n_1, n_2)}{C^{*'}(n_1, n_2)} = \frac{n_1^2 + n_2^2}{n_1^2 + n_2^2} \frac{(1-n_1)^2 + (1-n_2)^2}{(1-n_1)^4 + (1-n_2)^4}.$$

Put $n_1 = 1/2$ and $n_2 = t \in (0, 1/2]$ then by using Taylor's series expansion

$$f_8\left(t, \frac{1}{2}\right) = 1 + 4\left(t - \frac{1}{2}\right) + 8\left(t - \frac{1}{2}\right)^2 + 13.3333\left(t - \frac{1}{2}\right)^3 + 21.3333\left(t - \frac{1}{2}\right)^4 + \ldots$$

$$f_9\left(t, \frac{1}{2}\right) = 1 + 4\left(t - \frac{1}{2}\right) + 8\left(t - \frac{1}{2}\right)^2 + 24\left(t - \frac{1}{2}\right)^3 + 64\left(t - \frac{1}{2}\right)^4 + \ldots$$

$$f_{10}\left(t, \frac{1}{2}\right) = 1 + 4\left(t - \frac{1}{2}\right) + 8\left(t - \frac{1}{2}\right)^2 + 16\left(t - \frac{1}{2}\right)^3 + 32\left(t - \frac{1}{2}\right)^4 + \ldots$$

$$f_{11}\left(t, \frac{1}{2}\right) = 1 + 4\left(t - \frac{1}{2}\right) + 8\left(t - \frac{1}{2}\right)^2 + 10.66\left(t - \frac{1}{2}\right)^3 + 20.66\left(t - \frac{1}{2}\right)^4 + \ldots$$

$$f_{12}\left(t, \frac{1}{2}\right) = 1 + 4\left(t - \frac{1}{2}\right) + 8\left(t - \frac{1}{2}\right)^2 + 8\left(t - \frac{1}{2}\right)^3 - 16\left(t - \frac{1}{2}\right)^4 + \ldots$$

$$f_{13}\left(t, \frac{1}{2}\right) = 1 + 4\left(t - \frac{1}{2}\right) + 8\left(t - \frac{1}{2}\right)^2 + 5.3333\left(t - \frac{1}{2}\right)^3 - 10.666\left(t - \frac{1}{2}\right)^4 + \ldots$$

$$f_{14}\left(t, \frac{1}{2}\right) = 1 + 4\left(t - \frac{1}{2}\right) + 8\left(t - \frac{1}{2}\right)^2 - 8\left(t - \frac{1}{2}\right)^3 - 64\left(t - \frac{1}{2}\right)^4 + \ldots.$$

It was observed that the series expansions are same up to 2nd degree term. By considering 3rd and 4th degree terms, the following interpolating Ky Fan type inequality chain holds.

$$\frac{H^*}{H^{*'}} < \frac{G^*}{G^{*'}} < \frac{A^{(He)^*}}{A^{(He)^{*'}}} < \frac{H_e^*}{H_e^{*'}} < \frac{A^*}{A^{*'}} < \frac{G^{He^*}}{G^{He^{*'}}} < \frac{C^*}{C^{*'}}.$$

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