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FIXED POINT RESULTS FOR RATIONAL TYPE CONTRACTION IN A-METRIC SPACES

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Abstract

The goal of this paper is to define rational contraction in the context of A-metric spaces and develop various fixed point theorems in order to elaborate, generalize, and synthesize a number of previously published results. Finally, to illustrate the new theorem, an example is given.

1. Introduction

Fixed point theory is crucial in science and mathematics. This topic has drawn a lot of interest from academics in the last two decades due to its wide range of applications in disciplines such as nonlinear analysis, topology, and engineering difficulties. The Banach contraction principle [2] is the starting

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point for most generalizations of metric fixed point theorems. It's difficult to enumerate all of this principle's generalizations. The Banach fixed-point theorem [2] ensures the existence and uniqueness of fixed points of particular self-maps of metric spaces, as well as a constructive approach for discovering them. The S-metric space was introduced by Sedghi et al. [5]. It's a three-dimensional space called the S-metric space. The concept of A-metric space was established by Abbas et al. [1], which is a generalization of S-metric space. Jaggi [4], Das and Gupta [3] discovered the fixed point theorem for rational contractive type conditions in metric space. The goal of this paper is to define rational contraction in the setting of A-metric spaces, as well as to create various fixed point theorems to elaborate, generalize, and synthesize several previously published results. Finally, an example is given to demonstrate the new theorem.

2. Preliminaries

Some valuable information and ideas will be presented in this section. Abbas et al. [1] established the concept of A-metric space in 2015.

Definition 2.1 (see [1]). Let $\mathfrak{D}$ be a nonempty set. A mapping $A : \mathfrak{D}^n \to [0, +\infty)$ is called an A-metric on $\mathfrak{D}$ if and only if for all $\eta_i, a \in \mathfrak{D}, i = 1, 2, 3, \ldots, n:$ the following conditions hold:

(A1). $A(\eta_1, \eta_2, \eta_3, \ldots, \eta_{n-1}, \eta_n) \geq 0,$

(A2). $A(\eta_1, \eta_2, \eta_3, \ldots, \eta_{n-1}, \eta_n) = 0$ if and only if $\eta_1 = \eta_2 = \ldots = \eta_{n-1} = \eta_n,$

(A3).
$$\begin{aligned} A(\eta_1, \eta_2, \eta_3, \ldots, \eta_{n-1}, \eta_n) \leq\ & A(\eta_1, \eta_1, \eta_1, \ldots, (\eta_1)_{n-1}, a) \\ &+ A(\eta_2, \eta_2, \eta_2, \ldots, (\eta_2)_{n-1}, a) \\ &+ A(\eta_3, \eta_3, \eta_3, \ldots, (\eta_3)_{n-1}, a) + \ldots \\ &+ A(\eta_{n-1}, \eta_{n-1}, \eta_{n-1}, \ldots, (\eta_{n-1})_{n-1}, a) \\ &+ A(\eta_n, \eta_n, \eta_n, \ldots, (\eta_n)_{n-1}, a). \end{aligned}$$

The pair $(\mathfrak{D}, A)$ is called an A-metric space.

The following is the intuitive geometric example for A-metric spaces.

Example 2.2 (see [1]). Let $\mathfrak{D} = [1, +\infty)$. Define $A : \mathfrak{D}^n \to [0, +\infty)$ by

$$A(\eta_1, \eta_2, \eta_3, \ldots, \eta_{n-1}, \eta_n) = \sum_{i=1}^{n} \sum_{i<j} | \eta_i - \eta_j |$$

for all $\eta_i \in X$, $i = 1, 2, \ldots, n$.

Example 2.3 (see [1]). Let $\mathfrak{D} = \mathbb{R}$. Define $A : \mathfrak{D}^n \to [0, +\infty)$ by

$$\begin{aligned} A(\eta_1, \eta_2, \eta_3, \ldots, \eta_{n-1}, \eta_n) = & \left| \sum_{i=n}^{2} \eta_i - (n-1)\eta_1 \right| \\ & + \left| \sum_{i=n}^{3} \eta_i - (n-2)\eta_2 \right| \\ & + \left| \sum_{i=n}^{n-3} \eta_i - 3\eta_{n-3} \right| \\ & + \left| \sum_{i=n}^{n-2} \eta_i - 2\eta_{n-2} \right| \\ & + | \eta_n - \eta_{n-1} | \end{aligned}$$

for all $\eta_i \in \mathfrak{D}$, $i = 1, 2, \ldots, n$.

Lemma 2.4 (see [1]). *Let $(\mathfrak{D}, A)$ be an A-metric space. Then for all $\eta, \mu \in \mathfrak{D}$,*

$$A(\eta, \eta, \eta, \eta \ldots, (\eta)_{n-1}, \mu) = A(\mu, \mu, \mu, \mu \ldots, (\mu)_{n-1}, \eta)$$

Lemma 2.5 (see [1]). *Let $(\mathfrak{D}, A)$ be an A-metric space. Then for all $\eta, \mu, \zeta \in \mathfrak{D}$,*

$$\begin{aligned} A(\eta, \eta, \eta, \eta, \ldots, (\eta)_{n-1}, \zeta) \le & (n-1) A(\eta, \eta, \eta, \eta, \ldots, (\eta)_{n-1}, \mu) \\ & + A_b(\zeta, \zeta, \zeta, \zeta, \ldots, (\zeta)_{n-1}, \mu) \end{aligned}$$

and

$$A(\eta, \eta, \eta, \eta, \ldots, (\eta)_{n-1}, \zeta) \leq (n-1)A(\eta, \eta, \eta, \eta, \ldots, (\eta)_{n-1}, \mu) + A(\mu, \mu, \mu, \mu, \ldots, (\mu)_{n-1}, \zeta)$$

Lemma 2.6 (see [1]). *Let* $(\mathfrak{D}, A)$ *be an A-metric space. Then* $(\mathfrak{D} \times \mathfrak{D}, D_A)$ *is an A-metric space on* $\mathfrak{D} \times \mathfrak{D}$, *where* D_A *is given by for all* $\eta_i, \mu_j \in \mathfrak{D}$, $i, j = 1, 2, \ldots, n$:

$$D_A((\eta_1, \mu_1), (\eta_2, \mu_2), (\eta_3, \mu_3), \ldots, (\eta_n, \mu_n)) = A(\eta_1, \eta_2, \eta_3, \ldots, \eta_n) + A(\mu_1, \mu_2, \mu_3, \ldots, \mu_n).$$

Definition 2.7 (see [1]). Let $(\mathfrak{D}, A)$ be an A-metric space. Then

1. A sequence $\{\eta_k\}$ is called convergent to η in $(\mathfrak{D}, A)$ if

$$\lim_{k \to +\infty} A(\eta_k, \eta_k, \eta_k, \eta_k, \ldots (\eta_k)_{n-1}, \eta) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have

$$A(\eta_k, \eta_k, \eta_k, \eta_k, \ldots (\eta_k)_{n-1}, \eta) \leq \epsilon$$

and we write $\lim_{k \to +\infty} \eta_k = \eta$.

2. A sequence $\{\eta_k\}$ is called Cauchy in $(\mathfrak{D}, A)$ if

$$\lim_{k, m \to +\infty} A(\eta_k, \eta_k, \eta_k, \eta_k, \ldots (\eta_k)_{n-1}, \eta_m) = 0.$$

That is, for each $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have

$$A(\eta_k, \eta_k, \eta_k, \eta_k, \ldots (\eta_k)_{n-1}, \eta_m) \leq \epsilon.$$

3. $(\mathfrak{D}, A)$ is said to be complete if every Cauchy sequence in $(\mathfrak{D}, A)$ is a convergent.

Lemma 2.8 (see [1]). *Let* $(\mathfrak{D}, A)$ *be an A-metric space. If the sequence* $\{\eta_k\}$ *in* $\mathfrak{D}$ *converges to* η, *then* η *is unique.*

Lemma 2.9 (see [1]). *Every convergent sequence in A-metric space* $(\mathfrak{D}, A)$ *is a Cauchy sequence.*

Definition 2.10 (see [6]). Let $(\mathfrak{D}, \preccurlyeq)$ be a partially ordered set and let $\Gamma : \mathfrak{D} \to \mathfrak{D}$ be a mapping. Then

1. Elements $\eta, \mu \in \mathfrak{D}$ are comparable, if $\eta \preceq \mu$ or $\mu \preceq \eta$ holds;

2. A non empty set $\mathfrak{D}$ is called well ordered set, if every two elements of it are comparable;

3. Γ is said to be monotone non-decreasing w.r.t. $\preceq$, if for all $\eta, \mu \in \mathfrak{D}$, $\eta \preceq \mu$ implies $\Gamma\eta \preceq \Gamma\mu$;

4. Γ is said to be monotone non-increasing w.r.t. $\preceq$, if for all $\eta, \mu \in \mathfrak{D}$, $\eta \preceq \mu$ implies $\Gamma\eta \succeq \Gamma\mu$.

3. Main Results

First, we introduce following definitions.

Definition 3.1. The triple $(\mathfrak{D}, A, \preccurlyeq)$ is called partially ordered A-metric spaces if $(\mathfrak{D}, \preccurlyeq)$ could be a partial ordered set and $(\mathfrak{D}, A)$ be an A-metric space.

Definition 3.2. If A is complete A-metric, then $(\mathfrak{D}, A, \preccurlyeq)$ is called complete partially ordered metric space.

Definition 3.3. A partially ordered A-metric space $(\mathfrak{D}, A, \preccurlyeq)$ is called an ordered complete (OC), if for each convergent sequence $\{\eta_k\} \subset \mathfrak{D}$, the subsequent condition holds: either

- if $\{\eta_k\} \subset \mathfrak{D}$ is a non-increasing sequence such that $\eta_k \to \eta \in \mathfrak{D}$, then $\eta_k \preccurlyeq \eta$, for all $k \in \mathbb{N}$, that is, $\eta = \inf\{\eta_k\}$, or

- if $\{\eta_k\} \subset \mathfrak{D}$ is a non-decreasing sequence such that $\eta_k \to \eta$ implies that $\eta_k \preccurlyeq \eta$, for all $k \in \mathbb{N}$, that is, $\eta = \sup\{\eta_k\}$.

The following is our first main outcome.

Theorem 3.1. *Let* $(\mathfrak{D}, A, \preccurlyeq)$ *be a complete partially ordered* A*-metric* *space. Suppose a self map* Γ *on* $\mathfrak{D}$ *is continuous, non-decreasing and satisfies the contraction condition*

$$A(\Gamma\eta, \Gamma\eta, \ldots, \Gamma\eta, \Gamma\mu) \le a\frac{A(\eta, \eta, \ldots, \eta, \Gamma\eta)A(\mu, \mu, \ldots, \mu, \Gamma\mu)}{A(\eta, \eta, \ldots, \eta, \mu)}$$

$$+ b[A(\eta, \eta, \ldots, \eta, \Gamma\eta) + A(\mu, \mu, \ldots, \mu, \Gamma\mu)]$$

$$+ cA(\eta, \eta, \ldots, \eta, \mu) + L\min\{A(\eta, \eta, \ldots, \eta, \Gamma\mu), A(\mu, \mu, \ldots, \mu, \Gamma\eta)\} \tag{3.1}$$

for any $\eta \neq \mu \in \mathfrak{D}$ *with* $\eta \preccurlyeq \mu$, *where* $L \ge 0$, *and* $a, b, c \in [0, 1)$ *with* $0 \le a + 2b + c < 1$. *If* $\eta_0 \preccurlyeq \Gamma\eta_0$ *for certain* $\eta_0 \in \mathfrak{D}$, *then* Γ *has a fixed point.*

Proof. Define a sequence, $\eta_{k+1} = \Gamma\eta_k$ for $\eta_0 \in \mathfrak{D}$. If $\eta_{k_0+1} = \eta_{k_0}$ for certain $\eta_0 \in \mathbb{N}$, then η_{k_0} is a fixed point Γ. Assume that $\eta_{k+1} \neq \eta_k$ for each k. But $\eta_0 \preccurlyeq \Gamma\eta_0$ and Γ is non-decreasing as by induction we obtain that

$$\eta_0 \preccurlyeq \eta_1 \preccurlyeq \eta_2 \preccurlyeq \ldots \preccurlyeq \eta_k \preccurlyeq \eta_{k+1} \le \ldots \tag{3.2}$$

By (3.1), we have

$$A(\eta_{k+1}, \eta_{k+1}, \ldots, \eta_{k+1}, \eta_k) = A(\Gamma\eta_k, \Gamma\eta_k, \ldots, \Gamma\eta_k, \Gamma\eta_{k-1})$$

$$\le a\frac{A(\eta_k, \eta_k, \eta_k, \Gamma\eta_k)A(\eta_{k-1}, \eta_{k-1}, \ldots, \eta_{k-1}, \Gamma\eta_{k-1})}{A(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k-1})}$$

$$+ b[A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma\eta_k) + A(\eta_{k-1}, \eta_{k-1}, \ldots, \eta_{k-1}, \Gamma\eta_{k-1})]$$

$$+ cA(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k-1})$$

$$+ L\min\{A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma\eta_{k-1}), A(\eta_{k-1}, \eta_{k-1}, \ldots, \eta_{k-1}, \Gamma\eta_k)\}$$

$$= a\frac{A(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k+1})A(\eta_{k-1}, \eta_{k-1}, \ldots, \eta_{k-1}, \eta_k)}{A(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k-1})}$$

$$+ b[A(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k+1}) + A(\eta_{k-1}, \eta_{k-1}, \ldots, \eta_{k-1}, \eta_k)]$$

$$+ cA(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k-1})$$

$$+ L\min\{A(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k-1}), A(\eta_{k-1}, \eta_{k-1}, \ldots, \eta_{k-1}, \eta_{k+1})\}$$

$$= a\frac{A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_k)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k-1})}{A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k-1})}$$

$$+ b[A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_k) + A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k-1})]$$

$$+ cA(\eta_k, \eta_k, \dots, \eta_k, \eta_{k-1})$$

$$= (a+b)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_k)$$

$$+ (b+c)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k-1})$$

which infer that

$$A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_k) \le \left(\frac{b+c}{1-a-b}\right)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k-1})$$

$$\le \left(\frac{b+c}{1-a-b}\right)^k A(\eta_1, \eta_1, \dots, \eta_1, \eta_0) \le \dots \tag{3.3}$$

For $m, k \in \mathbb{N}$ with $m > k$, by repeated use of (A3), we have

$$A(\eta_k, \eta_k, \dots, \eta_k, \eta_m) \le (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1})$$

$$+A(\eta_m, \eta_m, \dots, \eta_m, \eta_{k+1})$$

$$\le (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_m)$$

$$\le (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2})$$

$$+ A(\eta_m, \eta_m, \dots, \eta_m, \eta_{k+2})$$

$$\le (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2})$$

$$+ A(\eta_{k+2}, \eta_{k+2}, \dots, \eta_{k+2}, S\eta_m)$$

$$\le (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2})$$

$$+ (n-1)A(\eta_{k+2}, \eta_{k+2}, \dots, \eta_{k+2}, \eta_{k+3}) + A(\eta_m, \eta_m, \dots, \eta_m, \eta_{k+3})$$

$$\le (n-1)A(\eta_k, \eta_k, \dots, \eta_k, \eta_{k+1}) + (n-1)A(\eta_{k+1}, \eta_{k+1}, \dots, \eta_{k+1}, \eta_{k+2})$$

$$+ (n-1)A(\eta_{k+2}, \eta_{k+2}, \dots, \eta_{k+2}, \eta_{k+3})$$

$$+ (n-1)A(\eta_{k+3}, \eta_{k+3}, \dots, \eta_{k+3}, \eta_{k+4}) + \dots$$

$$+ (n-1)A(\eta_{m-2}, \eta_{m-2}, \ldots, \eta_{m-2}, \eta_{m-1})$$

$$+ A(\eta_{m-1}, \eta_{m-1}, \ldots, \eta_{m-1}, \eta_m)$$

$$\leq (n-1)[\lambda^k + \lambda^{k+1} + \ldots + \lambda^{m-2}]A(\eta_0, \eta_0, \ldots, \eta_0, \eta_1)$$

$$+ \lambda^{m-1}A(\eta_0, \eta_0, \ldots, \eta_0, \eta_1)$$

$$= (n-1)\lambda^k[1 + \lambda + \lambda^2 + \ldots + \lambda^{m-k-2}]A(\eta_0, \eta_0, \ldots, \eta_0, \eta_1)$$

$$+ \lambda^{m-k-1}A(\eta_0, \eta_0, \ldots, \eta_0, \eta_1)$$

$$\leq (n-1)\lambda^k[1 + \lambda + \lambda^2 + \lambda^3 + \ldots]A(\eta_0, \eta_0, \ldots, \eta_0, \eta_1)$$

$$\leq (n-1)\frac{\lambda^k}{1-\lambda} A(\eta_0, \eta_0, \ldots, \eta_0, \eta_1)$$

where $\lambda = \frac{b+c}{1-a-b}$. As $k, m \to \infty$ in inequality (3.6), we obtain $\lim_{k, m\to\infty} A(\eta_k, \eta_k, \ldots, \eta_k, \eta_m) = 0$. This shows that $\{\eta_k\} \subset \mathfrak{D}$ is a Cauchy sequence and then $\eta_k \to \zeta \in \mathfrak{D}$ by its completeness. Besides, the continuity of Γ implies that

$$\Gamma\zeta = \Gamma(\lim_{k\to\infty} \eta_k) = \lim_{k\to\infty} \Gamma\eta_k = \lim_{k\to\infty} \eta_{k+1} = \zeta$$

Therefore, ζ is a fixed point of Γ in $\mathfrak{D}$.

Extracting the continuity of a map Γ in Theorem 3.1, we have the below result.

Theorem 3.2. *If $\mathfrak{D}$ has an ordered complete (OC) property in Theorem* 3.1, *then a non-decreasing mapping Γ has a fixed point in $\mathfrak{D}$.*

Proof. We only claim that $\Gamma\zeta = \zeta$. By an ordered complete metrical property of $\mathfrak{D}$, we have $\zeta = \sup\{\eta_k\}$, for $k \in \mathbb{N}$ as $\eta_k \to \zeta \in \mathfrak{D}$ is a non-decreasing sequence. The non-decreasing property of a map Γ implies that $\Gamma\eta_k \preccurlyeq \Gamma\zeta$ or, equivalently, $\eta_{k+1} \preccurlyeq \Gamma\zeta$, for $k \geq 0$. Since, $\eta_k \prec \eta_1 \preccurlyeq \Gamma\zeta$ and $\zeta = \sup\{\eta_k\}$ as a result, we get $\zeta \preccurlyeq \Gamma\zeta$. Assume $\zeta \prec \Gamma\zeta$. From Theorem 3.1, there is a non-decreasing sequence $\Gamma^k\zeta \in \mathfrak{D}$ with $\lim_{k\to\infty}\Gamma^k\zeta = \varepsilon \in \mathfrak{D}$. Again by an ordered complete (OC) property of $\mathfrak{D}$, we obtain that $\varepsilon = \sup\{\Gamma^k\zeta\}$. Furthermore, $\eta_k = \Gamma^k\eta_0 \preccurlyeq \Gamma^k\zeta$, for $k \geq 1$ as a result, $\eta_k \prec \Gamma^k\zeta$, for $k \geq 1$, since $\eta_k \preccurlyeq \zeta \prec \Gamma\zeta \preccurlyeq \Gamma^k\zeta$, for $k \geq 1$ whereas η_k and $\Gamma^k\zeta$, for $k \geq 1$ are distinct and comparable.

Now we have the discussion below in the subsequent cases.

Case 1. If $A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^k\zeta) \neq 0$, then (3.1) becomes,

$$A(\eta_{k+1}, \eta_{k+1}, \ldots, \eta_{k+1}, \Gamma^{k+1}\zeta) = A(\Gamma\eta_k, \Gamma\eta_k, \ldots, \Gamma\eta_k, \Gamma(\Gamma^k\zeta))$$

$$\leq a\,\frac{A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma\eta_k)A(\Gamma^k\zeta, \Gamma^k\zeta, \ldots, \Gamma^k\zeta, \Gamma^{k+1}\zeta)}{A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^k\zeta)}$$

$$+ b[A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma\eta_k) + A(\Gamma^k\zeta, \Gamma^k\zeta, \ldots, \Gamma^k\zeta, \Gamma^{k+1}\zeta)]$$

$$+ cA(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^k\zeta)$$

$$+ L\min\{A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^{k+1}\zeta), A(\Gamma^k\zeta, \Gamma^k\zeta, \ldots, \Gamma^k\zeta, \Gamma\eta_k)\}$$

$$= a\,\frac{A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma\eta_{k+1})A(\Gamma^k\zeta, \Gamma^k\zeta, \ldots, \Gamma^k\zeta, \Gamma^{k+1}\zeta)}{A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^k\zeta)}$$

$$+ b[A(\eta_k, \eta_k, \ldots, \eta_k, \eta_{k+1}) + A(\Gamma^k\zeta, \Gamma^k\zeta, \ldots, \Gamma^k\zeta, \Gamma^{k+1}\zeta)]$$

$$+ cA(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^k\zeta)$$

$$+ L\min\{A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^{k+1}\zeta), A(\Gamma^k\zeta, \Gamma^k\zeta, \ldots, \Gamma^k\zeta, \eta_{k+1})\} \tag{3.5}$$

As $k \to \infty$ in (3.5), we get

$$A(\zeta, \zeta, \ldots, \zeta, \varepsilon) \leq cA(\zeta, \zeta, \ldots, \zeta, \varepsilon) + L\min\{A(\zeta, \zeta, \ldots, \zeta, \varepsilon), A(\varepsilon, \varepsilon, \ldots, \varepsilon, \zeta)\}$$

$$\leq (c + L)A(\zeta, \zeta, \ldots, \zeta, \varepsilon)$$

as a result we have, $A(\zeta, \zeta, \ldots, \zeta, \varepsilon) = 0$. Hence $\zeta = \varepsilon$. In particular,

$\zeta = \varepsilon = \sup\{\Gamma^k \zeta\}$ in consequence, we get $\Gamma\zeta \preccurlyeq \zeta$, a contradiction. Therefore, $\Gamma\zeta = \zeta$.

Case 2. If $A(\eta_k, \eta_k, \ldots, \eta_k, \Gamma^k\zeta) = 0$, then, $A(\zeta, \zeta, \ldots, \zeta, \varepsilon) = 0$ as $k \to \infty$. By following the similar argument in Case 2, we get $\Gamma\zeta = \zeta$.

Corollary 3.1. *Let* $(\mathfrak{D}, A, \preccurlyeq)$ *be a complete partially ordered A-metric space. Suppose a self map* Γ *on* $\mathfrak{D}$ *is continuous, non-decreasing and satisfies the contraction condition*

$$A(\Gamma\eta, \Gamma\eta, \ldots, \Gamma\eta, \Gamma\mu) \leq a \frac{A(\eta, \eta, \ldots, \eta, \Gamma\eta) A(\mu, \mu, \ldots, \mu, \Gamma\mu)}{A(\eta, \eta, \ldots, \eta, \mu)}$$

$$+ b[A(\eta, \eta, \ldots, \eta, \Gamma\eta) + A(\mu, \mu, \ldots, \mu, \Gamma\mu)] + cA(\eta, \eta, \ldots, \eta, \mu) \tag{3.6}$$

for any $\eta \neq \mu \in \mathfrak{D}$ *with* $\eta \preccurlyeq \mu$, *where* $a, b, c \in [0, 1)$ *with* $0 \leq a + 2b + c < 1$. *If* $\eta_0 \preccurlyeq \Gamma\eta_0$ *for certain* $\eta_0 \in \mathfrak{D}$, *then* Γ *has a fixed point.*

Proof. It follows by $L = 0$ in Theorem 3.1.

Corollary 3.2. *Let* $(\mathfrak{D}, A, \preccurlyeq)$ *be a complete partially ordered A-metric space. Suppose a self map* Γ *on* $\mathfrak{D}$ *is continuous, non-decreasing and satisfies the contraction condition*

$$A(\Gamma\eta, \Gamma\eta, \ldots, \Gamma\eta, \Gamma\mu) \leq a \frac{A(\eta, \eta, \ldots, \eta, \Gamma\eta) A(\mu, \mu, \ldots, \mu, \Gamma\mu)}{A(\eta, \eta, \ldots, \eta, \mu)}$$

$$+ cA(\eta, \eta, \ldots, \eta, \mu) \tag{3.7}$$

for any $\eta \neq \mu \in \mathfrak{D}$ *with* $\eta \preccurlyeq \mu$, *where* $L \geq 0$, *and* $a, b, c \in [0, 1)$ *with* $0 \leq a + 2b + c < 1$. *If* $\eta_0 \preccurlyeq \Gamma\eta_0$ *for certain* $\eta_0 \in \mathfrak{D}$, *then* Γ *has a fixed point.*

Proof. Taking $b = 0$, $L = 0$ in Theorem 3.1, we obtain the desired result.

We conclude with an example.

Example 3.1. Let $(\mathbb{R}, A, \preccurlyeq)$ be a totally ordered complete A-metric space with A-metric defined as in Example 2.3. Let $\Gamma : \mathbb{R} \to \mathbb{R}$ be two maps defined by $\Gamma(\eta) = \frac{3\eta + 24n - 3}{24n}$ for all $n \geq 1$. It is evident that Γ is continuous and non-decreasing in $\mathbb{R}$ and $\eta_0 = 0 \in \mathbb{R}$ such that $\eta_0 = 0 \preccurlyeq \Gamma\eta_0$. Taking $a = 0, b = 0, c = \frac{1}{n}, L = 0$. For $\eta \preccurlyeq \mu$, we have

$$A(\Gamma\eta, \Gamma\eta, \ldots, \Gamma\eta, \Gamma\eta) = (n-1)|\Gamma\eta - \Gamma\mu|$$

$$= (n-1)\left|\frac{3\eta + 24n - 3}{24n} - \frac{3\mu + 24n - 3}{24n}\right|$$

$$= (n-1)\left|\frac{3(\eta - \mu)}{24}\right| = \left(\frac{n-1}{n}\right)\left|\frac{\eta - \mu}{8}\right|$$

$$\leq \left(\frac{n-1}{n}\right)|\eta - \mu| = \frac{1}{n}A(\eta, \eta, \ldots, \eta, \mu)$$

$$\leq a\frac{A(\eta, \eta, \ldots, \eta, \Gamma\eta)A(\mu, \mu, \ldots, \mu, \Gamma\mu)}{A(\eta, \eta, \ldots, \eta, \mu)}$$

$$+ b[A(\eta, \eta, \ldots, \eta, \Gamma\eta) + A(\mu, \mu, \ldots, \mu, \Gamma\mu)]$$

$$+ cA(\eta, \eta, \ldots, \eta, \mu) + L\min\{A(\eta, \eta, \ldots, \eta, \Gamma\mu), A(\mu, \mu, \ldots, \mu, \Gamma\eta)\}$$

holds for every $\eta, \mu \in \mathbb{R}$. For $L \geq 0$ and $a, b, c \in [0, 1)$ such that $0 \leq a + 2b + c < 1$, in particular, if we take $a = 0, b = 0, c = \frac{1}{n}, L = 0$, then $0 \leq a + 2b + c < 1$ and $1 \in \mathbb{R}$ is a fixed point of Γ as all the conditions of Theorem 3.1 are satisfied.

Conflict of Interest

No conflict of interest was declared by the authors.

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Author's Contributions

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