Advances and Applications in Mathematical Sciences
Volume 21, Issue 12, October 2022, Pages 6721-6729

DOM-CHROMATIC NUMBER OF CERTAIN PATH RELATED GRAPHS

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Abstract

Let $G(V, E, \psi_G)$ be a graph with χ-coloring. A dominating set of G which is a subset of $V(G)$ is called a dom-coloring set if it contains a minimum of one vertex from each color class of G. The minimum number of such vertices taken over all coloring sets is called the cardinality of that set which yields the dom-chromatic number. It is denoted by $\gamma_{dc}(G)$. In this paper, the basic concepts of domination and coloring have been used to determine the dom-chromatic number of caterpillar, coconut tree, lobster, tadpole, pan and lollipop graphs.

1. Introduction

Graph Theory is a branch of Mathematics which has a wide range of applications in the research field of Mathematics. The basic concepts of domination and coloring in Graph Theory play a vital role in every field of Modern Science and Engineering. In 1962, Ore was the first to use the term "domination" for undirected graphs and he denoted the domination number

2020 Mathematics Subject Classification: 62.
Keywords: Dominating set, Chromatic number, Dom-coloring set, Dom-chromatic number.
Received February 3, 2022; Accepted March 2, 2022

by $\delta(G)$ [5]. In 1977, Cockayne and Hedetniemi introduced the accepted notation $\gamma(G)$ to denote the domination number [3]. Graph coloring originated with the four-color conjecture in 1852. The combination of the two broad concepts give rise to a new problem called the dom-chromatic problem, which deals with the determination of the dom-chromatic number. Chaluvaraju and Appajigowda has introduced $\gamma_{dc}(G)$ in 2016 [2].

2. Preliminaries

Definition 2.1 [1]**.** Let G be an undirected graph. A subset D of vertex set V of G is a dominating set, if every vertex in G is either in D or adjacent to some vertex in D. The set D with minimum number of vertices gives the cardinality of D which is called the domination number $\gamma(G)$ of G.

Definition 2.2 [4, 8]**.** The chromatic number $\chi(G)$ is the minimum number of colors used to color graph G such that any two adjacent vertices receive distinct colors.

Definition 2.3 [2]**.** Let $G(V, E, \psi_G)$ be a graph with χ-coloring. A dominating set of G is called a dom-coloring set, if it contains minimum of one vertex from each color class of G. The minimum number of such vertices taken over all dom-coloring sets is called the cardinality of that set which yields the dom-chromatic number. It is denoted by $\gamma_{dc}(G)$.

Definition 2.4 [6]**.** A path P_n is a tree on n vertices with 2 pendant vertices of degree 1 and remaining $n-1$ internal vertices of degree 2.

Definition 2.5 [6]**.** A coconut tree $CT(m, l)$ is constructed by identifying l copies of P_2 to an end vertex of the path P_m.

Definition 2.6 [6]**.** A caterpillar $CP(m, k)$, with k-pendant edges is an extension of $CT(m, l)$ in which every $m-1$ vertices of $CT(m, l)$, leaving the end vertex is attached with at least one copy of P_2.

Definition 2.7 [6]**.** A lobster $L(m, k)$ is an extension of $CP(m, k)$ in which each k pendent vertices of $CP(m, k)$ is attached with one pendant edge. In other words, removal of leaf nodes leaves a caterpillar.

Definition 2.8 [7]**.** Tadpole graph $T_{m,n}$ is obtained by joining a cycle C_m to a path P_n with a bridge.

Definition 2.9 [7] A pan graph $T_{m,1}$ is obtained from a tadpole graph $T_{m,n}$ by replacing n by 1. In other words, $T_{m,1}$ is obtained by joining a cycle C_m to a singleton graph K_1 with a bridge.

Definition 2.10 [10]**.** A lollipop graph $L_{m,n}$ is obtained by joining a complete graph K_m to a path $P_n : u_1u_2, \ldots u_n$ with a bridge.

3. Main Results

In this section certain graphs like coconut tree, caterpillar, lobster, tadpole, pan graph and lollipop graphs have been considered to determine the dom-chromatic number. The coloring algorithms have been designed for acyclic and connected graphs like coconut tree, caterpillar and lobster graphs.

Remark. In all these graphs, the vertices of the central path P_n are labelled as $\{u_i, 1 \le i \le n\}$ and the pendant vertices adjacent to the central path receive labels $\{u_{ik}, 2 \le i \le n-1, 1 \le k \le \deg(u_i) - 2\} \cup \{u_{ik} : i = 1, n$ and $1 \le k \le \deg(u_i) - 1\}$. In case of a lobster graph, each vertex of the central path P_n will be adjacent to at least one path P_2. Each P_2 path adjacent to $u_i, 1 \le i \le n$ are labelled as $\{u'_{ik}, u''_{ik}, 2 \le i \le n-1, 1 \le k \le \deg(u_i) - 2\}$ $\cup \{u'_{ik}, u''_{ik} : i = 1, n$ and $1 \le k \le \deg(u_i) - 1\}$.

3.1 Coloring Algorithm for Coconut Tree and Caterpillar Graphs.

Input: $CT(m, l)$ or $CP(m, k)$ with $m \ge 3$ vertices.

Step 1. For path P_m in $CT(m, l)$ or $CP(m, k)$, color $u_{2i-1} = 1, u_{2i} = 2$ when $1 \le i \le m-1$.

Step 2. For pendant vertices adjacent to $u_1, 1 \le i \le m$, color $u_{ik} = 1$ if $u_i = 2$ and color $u_{ik} = 2$ if $u_i = 1$.

Output. χ-coloring of $CT(m, l)$ or $CP(m, k)$.

3.2 Coloring Algorithm for Lobster Graphs

Input. $L(m, k)$ with $m \geq 3$ vertices.

Step 1. For path P_m in $L(m, k)$, color $u_{2i-1} = 1, u_{2i} = 2$ when $1 \leq i \leq m-1$.

Step 2. Color each path $P_2 : \{u'_{ik}, u''_{ik}, 2 \leq i \leq n-1, 1 \leq k \deg(u_i) - 2\} \bigcup \{u'_{ik}, u''_{ik}, i = 1, n$ and $1 \leq k \leq \deg(u_i) - 1\}$ adjacent to u_i such that if $u_i = 1$, color $u'_{ik} = 2, u''_{ik} = 1$, else color $u'_{ik} = 1, u''_{ik} = 2$.

Output. χ-coloring of $L(m, k)$.

3.3 Dom-Chromatic Number of Certain Path Related Graphs

The dom-chromatic number for certain path related graphs like coconut tree, caterpillar, lobster, tadpole, pan graph and lollipop graphs have been determined in this section.

Theorem 3.3.1 [2]. *For any graph* G, $\max\{\gamma(G), \chi(G)\} \leq \gamma_{dc}(G) \leq \gamma(G) + \chi(G) - 1$. *The bounds are sharp*.

Theorem 3.3.2. *Let* $CT(m, l)$ *be a coconut tree with* $m, l \geq 2$. *Then*

$$\gamma_{dc}(CT(m, l)) = \begin{cases} \frac{m}{3} + 1, & m \equiv 0 \bmod 3 \\ \left\lceil \frac{m}{3} \right\rceil, & otherwise \end{cases}.$$

Proof. Let $G = CT(m, l)$ be a coconut tree with $\{P_m : u_i, 1 \leq i \leq m\}$ as the central path and l pendant vertices adjacent to u_m are $u_{ml}, l > 1$. By Coloring Algorithm 3.1, $CT(m, l)$ yields a proper coloring.

Case (i). For $m \equiv 0 \bmod 3$.

Let $D = \{u_{3i-1}, 1 \leq i \leq \frac{m}{3}\} \bigcup \{u_m\}$ be a subset of $V(CT(m, l))$ such that every vertex in $V(CT(m, l)) - D$ and some vertex in D are adjacent. Hence D is a dominating set with minimum of one vertex from every color class of $CT(m, l)$. Therefore, D is a dom-chromatic set with cardinality $\gamma_{dc}(G) = \frac{m}{3} + 1$.

Case (ii). For $m \not\equiv 0 \bmod 3$.

Let $D = \left\{u_{3i-1}, 1 \le i \le \left\lfloor \frac{m}{3} \right\rfloor\right\} \cup \{u_m\}$ be a subset of $V(CT(m, l))$ such that every vertex in $V(CT(m, l)) - D$ and some vertex in D are adjacent. Hence D is a dominating set with a minimum of one vertex from every color class of $CT(m, l)$. Therefore, D is a dom-chromatic set with cardinality $\gamma_{dc}(G) = \left\lfloor \frac{m}{3} \right\rfloor + 1 = \left\lceil \frac{m}{3} \right\rceil$.

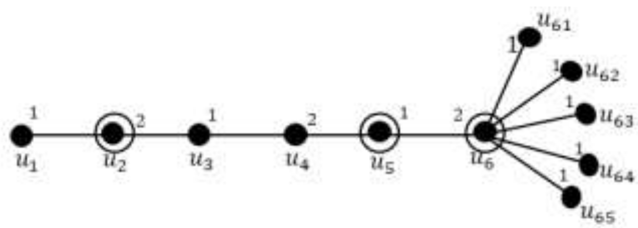

Figure 1. Coconut tree graph $CT(6, 5)$.

Theorem 3.3.3. *Let $CT(m, k)$ be a caterpillar graph with $m, k \ge 2$. Then $\gamma_{dc}(CT(m, k)) = m$.*

Proof. Let $G = CT(m, k)$ be a caterpillar graph with $\{P_m : u_i, 1 \le i \le m\}$ as the central path and the pendant vertices adjacent to the central path are $\{u_{ik}, 2 \le i \le n-1, 1 \le k \le \deg(u_i) - 2$ and $u_{ik}, i = 1, m$ and $1 \le k \le \deg(u_i) - 1\}$. By Coloring Algorithm 3.1, $CT(m, k)$ yields a proper coloring. Let $D = \{u_i, 1 \le i \le m\}$ be a subset of $V(CT(m, k)) - D$ such that every vertex in $V(CT(m, k)) - D$ and some vertex in D are adjacent. Hence D is a dominating set with minimum of one vertex from every color class of $CT(m, k)$. The dominating set D is minimum because one vertex less than m does not satisfy the domination property. Hence D is a dom-chromatic set with cardinality $\gamma_{dc}(G) = m$.

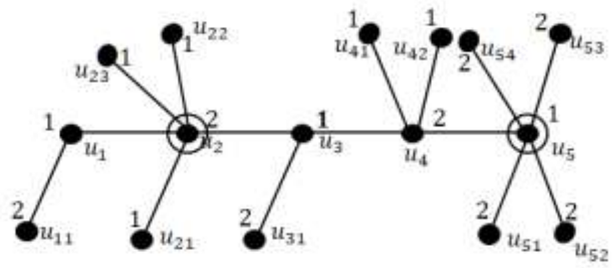

Figure 2. Caterpillar graph $CP(5, 11)$.

Theorem 3.3.4. *Let* $L(m, k)$ *be a lobster graph with* $m, k \geq 2$. *Then* $\gamma_{dc}(L(m, k)) = \sum_{i=1}^{m} d(u_i) - 2(m-1)$.

Proof. Consider $G = L(m, k)$ to be a lobster with $\{P_m : u_i, 1 \leq i \leq m\}$ as the central path and each vertex of the central path P_m will be adjacent to at least one path of length 2. Each path P_2 adjacent to $u_i, 1 \leq i \leq m$ are labelled as $\{P_2 : u'_{ik}u'_{ik}, 2 \leq i \leq m-1, 1 \leq k \leq \deg(u_i) - 2$ and $u_{ik}, i = 1, n$ and $1 \leq k \leq \deg(u_i) - 1\}$. The Coloring Algorithm 3.2, yields a proper coloring of $L(m, k)$. Let $D = \{u'_{ik}, 2 \leq i \leq m-1, 1 \leq \deg(u_i) - 2$ and $u_{ik}, i = 1, m$ and $1 \leq k \leq \deg(u_i) - 1\}$ be a subset of $V(L(m, k))$ such that every vertex in $V(L(m, k)) - D$ and some vertex in D are adjacent. Hence D is a dominating set with minimum one vertex from every color class of $L(m, k)$. Therefore D is a dom-chromatic set with cardinality $\gamma_{dc}(G) = \sum_{i=1}^{m} d(u_i) - 2(m-2) - 2$ $= \sum_{i=1}^{m} d(u_i) - 2m + 4 - 2m + 4 - 2 = \sum_{i=1}^{m} d(u_i) - 2m + 2 = \sum_{i=1}^{m} d(u_i) - (m-1)$.

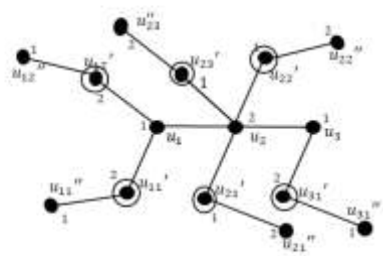

Figure 3. Lobster $L(3, 6)$.

Theorem 3.3.5 [2]. *For any path P_p with $p \geq 4$ vertices,* $\gamma_{dc}(P_p) = \gamma(P_p) = \left\lceil \frac{p}{3} \right\rceil$.

Theorem 3.3.6 [9]. *Let G be a cycle C_n, of length $n > 5$. Then* $\gamma_{dc}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem 3.3.7 [2]. *Let G be an arbitrary graph. Then $\gamma_{dc}(G) = p$ if and only $G \cong K_p$* or $\overline{K_p}$.

Theorem 3.3.8. *Let $T_{m,n}$ be a tadpole graph $m \geq 5, n \geq 2$. Then* $\gamma_{dc}(C_m) = \left\lceil \frac{m}{3} \right\rceil$ and $\gamma_{dc}(P_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. Consider a tadpole $T_{m,n}$ which has a cycle $C_m : v_1 v_2 \ldots v_m v_1$ and path $P_n : u_1 u_2 \ldots u_n$ joined by a bridge, $m \geq 5, n \geq 2$. Choose v_1 as the vertex which joins C_m with the path P_n such that v_1 is adjacent with u_1. By the theorems 3.3.5 and 3.3.6, $\gamma_{dc}(C_m) = \left\lceil \frac{m}{3} \right\rceil$ and $\gamma_{dc}(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Since the vertex v_1 of C_m dominates u_1 of P_1, the dom-chromatic number is obtained for $P_{n-1} : u_2 u_3 \ldots u_n$ starting from vertex v_2. Hence $\gamma_{dc}(P_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil$. Therefore we get $\gamma_{dc}(T_{m,n}) = \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil$.

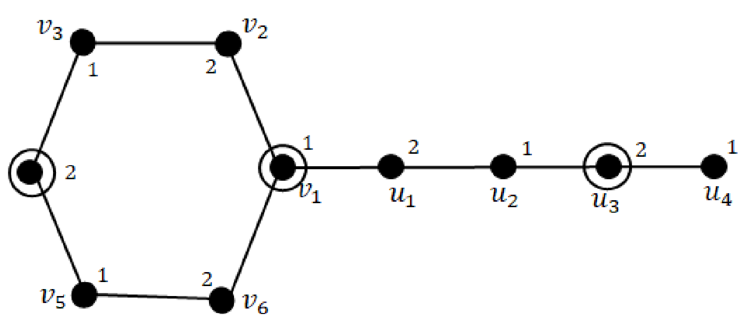

Figure 4. Tadpole graph $T_{6,4}$.

Theorem 3.3.9. *Let* $T_{m,1}$ *be a pan graph with* $m \geq 5$. *Then* $\gamma_{dc}(T_{m,1}) = \gamma_{dc}(C_m) = \left\lceil \frac{m}{3} \right\rceil$.

Proof. Consider a pan graph $T_{m,1}$ which has a cycle $C_m : v_1 v_2 \ldots v_m v_1$, $m \geq 5$ and K_1 joined by a bridge. Choose v_1 as the vertex which joins C_m with K_1 such that v_1 is adjacent with u of K_1. By the Theorem 3.3.6, $\gamma_{dc}(C_m) = \left\lceil \frac{m}{3} \right\rceil$. Since the vertex v_1 of C_m dominates u of K_1, the dom-chromatic number of $T_{m,1}$ is the same as that of C_m. Hence $\gamma_{dc}(T_{m,1}) = \left\lceil \frac{m}{3} \right\rceil$.

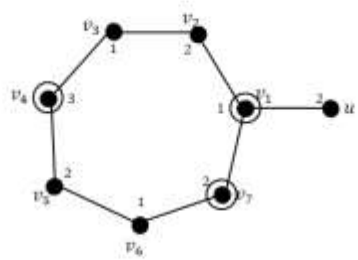

Figure 5. Pan graph $T_{7,1}$.

Theorem 3.3.10. *Let* $L_{m,n}$ *be a lollipop graph with* $m \geq 5$, $n \geq 2$. *Then* $\gamma_{dc}(L_{m,n}) = m + \left\lceil \frac{n-1}{3} \right\rceil$.

Proof. Consider a lollipop $L_{m,n}$. Let $v_1, v_2, \ldots, v_m$ be the vertices of a complete graph K_m. Choose v_1 as the vertex which joins K_m with the path P_n such that v_1 is adjacent with u_1. By Theorems 3.3.5 and 3.3.7, $\gamma_{dc}(K_m) = m$ and $\gamma_{dc}(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Since the vertex v_1 of K_m dominates u_1 of P_1, the dom-chromatic number is obtained for $P_{n-1} : u_2 u_3 \ldots u_n$ starting from vertex v_2. Hence $\gamma_{dc}(P_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil$. Therefore we get $\gamma_{dc}(L_{m,n}) = m + \left\lceil \frac{n-1}{3} \right\rceil$.

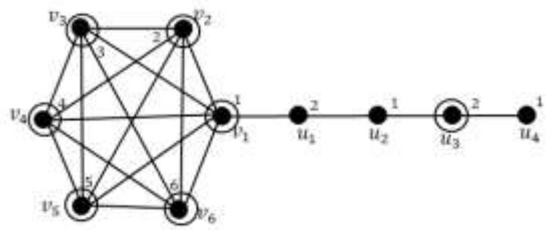

Figure 6. Lollipop graph $L_{6,4}$.

4. Conclusion

In this paper, the concepts of domination and coloring are applied to determine the dom-chromatic number of certain path related graphs like coconut tree, caterpillar, lobster, tadpole, pan graph and lollipop graphs. This can be extended to networks like Butterfly and Benes network which is open for study.

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