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ESTIMATION OF THE PARAMETERS IN PERTURBED WEIBULL MODEL USING EM ALGORITHM

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Abstract

The maximum likelihood estimators of the parameters are obtained through EM algorithm in perturbed Weibull model. A simulation study has been carried out to demonstrate the asymptotic normality of the estimators of the parameters in perturbed Weibull model through three-dimensional graph using *R* software (maxLik/optim).

1. Introduction

There are many applications for the Weibull distribution in statistics. Although it was first identified by Fréchet in 1927, it is named after Waalobi Weibull. Waalobi Weibull was the first to promote the usefulness of this distribution by modelling data sets from various disciplines (Murthy, Xie, and Jiang [4]).

The Weibull distribution, an extreme value distribution, is frequently used to model survival, reliability, and other data. Its flexibility can take off various distributions like the exponential or normal. The Weibull distribution is specifically used to model extreme value data. One example of this is the

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frequent use of the Weibull distribution to model failure time data (Murthy et al. [4]). The extreme values in these analyses would be the unusual longevity of the survival of affected patients. It is widely used in reliability and life data analysis.

Suppose that a company is interested in the duration of the telephone calls going out from its office. Let us assume that a caller from this office does not continue speaking beyond a known time s, if the call goes to a wrong person and the duration of such a call is uniformly distributed over $(0, s)$. On the other hand, let the duration of a calls follows certain life time distributions, in this paper Weibull distribution has been taken into consideration with parameters θ and $\alpha(\alpha$ is assumed to be known) if it goes to a right person. Further suppose that the proportion of calls going to wrong persons from this office is φ.

If a person who makes a telephone calls continue to speaking beyond time s, then definitely the call goes to a right person. On the other hand, if a person does not continue to speaking beyond time s, then it cannot be definitely classified as whether the call goes to a right person or a wrong person. In other words, if the duration of telephone calls X is less than or equal to s, then the information about the call is missing whether it has gone to a right person or a wrong person.

2. Perturbed Weibull Distribution

Suppose that the random variable X has the following probability density function

$$f(x) = \begin{cases} \dfrac{\varphi}{s} + (1-\varphi)\dfrac{\alpha}{\theta^{\alpha}} x^{\alpha-1} \exp\left(-\dfrac{x}{\theta}\right)^{\alpha}, & 0 < x \le s \\ (1-\varphi)\dfrac{\alpha}{\theta^{\alpha}} x^{\alpha-1} \exp\left(-\dfrac{x}{\theta}\right)^{\alpha}, & x > s,\ 0 < \varphi < 1,\ \theta > 0,\ \alpha > 0 \end{cases} \tag{1.1}$$

$$f(x) = \varphi f_1(x) + (1-\varphi) f_2(x),$$

where $f_1(x) = \frac{1}{s},\ 0 < x \le s$ and $f_2(x) = f_2(x; \theta, \alpha) = \frac{\alpha}{\theta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right)^{\alpha}$, $x > 0,\ (\theta, \alpha) > 0$ is Weibull distribution. Here is assumed to be known and equation (1.1) is known as perturbed Weibull density.

3. Maximum Likelihood Estimation

Maximum likelihood estimation and likelihood-based inference are everywhere in statistical theory and applications. Maximum likelihood estimation has gained importance in the frequentist as well as the Bayesian approaches because of its attractive properties. Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be the duration of telephone calls of n calls going out from a company having the density in (1.1). Then, the likelihood of θ and φ given the sample $\underline{X}$ can be written as

$$L(\theta, \varphi; \underline{x})$$

$$= \prod_{j=1}^{n} \left(\frac{\varphi}{s} + (1-\varphi)\frac{\alpha}{\theta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right)^{\alpha} \right)^{1-y_j} \left((1-\varphi)\frac{\alpha}{\theta^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\theta}\right)\alpha \right)^{y_j}, \quad (1.2)$$

where

$$y_j = \begin{cases} 1, & \text{if } X_j > s \\ 0, & \text{if } X_j \le s \end{cases}.$$

It can be easily seen that the likelihood equations $\frac{\partial L}{\partial \theta} = 0$ and $\frac{\partial L}{\partial \varphi} = 0$ do not yield closed form expressions for the maximum likelihood estimators (MLEs) of θ and φ. Hence, for a given sample, the maximum likelihood estimates of θ and φ could be computed using one of the numerical iterative procedures like Newton-Raphson, Fletcher-Reeves etc. But these procedures are analytically and computationally tedious in the case of (1.1). The alternative method is discussed in the next section.

4. The EM Algorithm

When the likelihood functions have complicated structures and their maximization by numerical methods is difficult, the MLEs of the parameters can be computed by the Expectation-Maximization (EM) algorithm. It is popular and remarkably simple to implement. It is an iterative procedure and there are two steps in each of the iterations, namely the Expectation Step (*E*-step) and the Maximization step (*M*-step). The EM algorithm was developed by Dempster, Laird, and Rubin [1] who synthesized an earlier formulation in many particular cases and gave a general method of finding the MLEs in a variety of situations. Since then the EM algorithm has been applied to a variety of statistical problems such as resolution of mixtures, multi-way contingency tables, variance component estimation, and factor analysis. It has also found applications in specialized areas like genetics, medical imaging, and neural networks. McLachlan and Krishnan [2] discuss the EM algorithm and its extensions to a variety of problems in detail while Krishnan (2004) gives a brief introduction to the algorithm with examples.

We now need to rewrite the likelihood so as to accommodate the missing data to apply the EM algorithm to compute the MLEs.

$$\text{Let } Z_j = \begin{cases} 1, & 1, \text{ if the } j\text{-th sample call going to a right person} \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$P(Z_j = 1) = 1 - \varphi = 1 - P(Z_j = 0),\ j = 1, 2, \ldots, n. \tag{1.3}$$

If $\underline{X} = (X_1, X_2, \ldots, X_n)$ is the observed sample on X, then $((X_1, Z_1), (X_2, Z_2), \ldots, (X_n, Z_n))$ becomes the complete sample as $(X_1, X_2, \ldots, X_n)$ is augmented by $(Z_1, Z_2, \ldots, Z_n)$. If $X_j > s$, then $Z_j = 1$ and if $X_j \leq s$, then $Z_j = 0$ or 1. In other words, we have no information on Z_j for $X_j \leq s$. Hence, $\{Z_j : X_j \leq s\}$ can be treated as the missing data.

The likelihood function corresponding to the complete data is then given by

$$L_c(\theta, \varphi \mid \underline{x}, \underline{u}) = \prod_{j=1}^{n} \left((1 - \varphi) \frac{\alpha}{\theta^{\alpha}} x^{\alpha - 1} \exp\left(-\left(\frac{x}{\theta}\right)^{\alpha} \right) \right)^{u_j} \left(\frac{\varphi}{s}\right)^{1 - u_j}, \tag{1.4}$$

where $U_j = 1$ if $X_j > s$ and $U_j = Z_j$, if $X_j \leq s$. Consequently, the log-likelihood can be written as

$$\log L_c(\theta, \varphi \mid X, \underline{u})$$

$$= \sum_{x_j > s} \left[\log(1-\varphi) + \log\alpha - \alpha\log\theta + (\alpha - 1)\log x_j - \left(\frac{x_j}{\theta}\right)^{\alpha} \right] +$$

$$\sum_{x_j \le s} u_j \left[\log(1-\varphi) + \log\alpha - \alpha\log\theta + (\alpha - 1)\log x_j - \left(\frac{x_j}{\theta}\right)^{\alpha} \right] +$$

$$\sum_{x_j \le s} (1 - u_j)[\log\varphi - \log s].$$

The *E*-Step: By taking the expectation of the log-likelihood, we get

$$E[\log L_c(\theta, \varphi \mid X, \underline{u})]$$

$$= \sum_{x_j > s} \left[\log(1-\varphi) + \log\alpha - \alpha\log\theta + (\alpha - 1)\log x_j - \left(\frac{x_j}{\theta}\right)^{\alpha} \right] +$$

$$\sum_{x_j \le s} E(Z_j) \left[\log(1-\varphi) + \log\alpha - \alpha\log\theta + (\alpha - 1)\log x_j - \left(\frac{x_j}{\theta}\right)^{\alpha} \right] +$$

$$\sum_{x_j \le s} (1 - E(Z_j))[\log\varphi - \log s]. \tag{1.5}$$

Now, $E(Z_j)$ is replaced by the conditional expectation $E(Z_j \mid \theta_0, \varphi_0, x_j)$, where θ_0 and φ_0 are initial estimates of θ and φ. From (1.4), we get

$$E(Z_j \mid \theta_0, \varphi_0, x_j) = 0.P(Z_j = 0 \mid \theta_0, \varphi_0, x_j) + 1.P(Z_j = 1 \mid \theta_0, \varphi_0, x_j)$$

$$= P(Z_j = 1 \mid \theta_0, \varphi_0, x_j).$$

Using the Bayes Theorem,

$$P(Z_j = 1 \mid \theta_0, \varphi_0, x_j) = \frac{P(A_j \mid Z_j = 1)P(Z_j = 1)}{P(A_j \mid Z_j = 0)P(Z_j = 0) + P(A_j \mid Z_j = 1)P(Z_j = 1)},$$

where $A_j = [X_j = x_j \mid \theta_0, \varphi_0]$.

That is

$$P(Z_j = 1 \mid (\theta_0, \phi_0, X_j = 0) = \frac{(1-\phi_0)\dfrac{\alpha}{\theta_0^{\alpha}} x^{\alpha-1} \exp\left(-\dfrac{x}{\theta_0}\right)^{\alpha}}{\dfrac{\phi_0}{s} + (1-\phi_0)\dfrac{\alpha}{\theta_0^{\alpha}} x^{\alpha-1} \exp\left(-\dfrac{x}{\theta_0}\right)^{\alpha}} = w_j, \quad \text{say,}$$

$x_j \leq s.$

Substitute w_j in equation (1.5) we get,

$$\sum_{x_j > s} [\log(1-\varphi) + \log \alpha - \alpha \log \theta + (\alpha - 1)\log x_j - (\frac{x_j}{\theta})^{\alpha}] +$$

$$E[\log L_c(\theta, \varphi \mid X, \underline{u}]$$

$$= \sum_{x_j \leq s} w_j \left[\log(1-\varphi) + \log \alpha - \alpha \log \theta + (\alpha - 1)\log x_j \left(\frac{x_j}{\theta}\right)^{\alpha}\right]$$

$$+ \sum_{x_j \leq s} (1 - w_j)[\log \varphi - \log s]. \tag{1.6}$$

The *M*-Step: In this step, $E[\log (L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u}],$ is maximized for θ and φ. If θ_1 and φ_1 are the values of θ and φ which maximize $E[\log (L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u}],$ then the *E*-step is repeated using θ_1 and φ. After each of the iterations, the value of the likelihood $L(\theta, \varphi \mid \underline{x})$ specified in (1.4) can be evaluated and observed whether it is increasing. The iterative procedure is terminated when $L(\theta, \varphi; \underline{x})$ or $\log L(\theta, \varphi; \underline{x})$ converges to a value correct to a desirable number of decimal places.

Since $E[\log (L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u}]$ is differentiable with respect to both θ and φ, the values of θ and φ for which $E[\log (L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u}],$ is a maximum can be obtained by the method of calculus.

From (1.6), we get

$$\frac{\partial E \log L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u})}{\partial \theta} = 0 \Rightarrow \sum_{x_j > s}\left[-\frac{\alpha}{\theta} + \frac{\alpha x_j \alpha}{\theta(\alpha+1)}\right] + \sum_{x_j \leq s} w_j \left[-\frac{\alpha}{\theta} + \frac{\alpha x_j \alpha}{\theta(\alpha+1)}\right] = 0$$

and

$$\frac{\partial E \log L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u})}{\partial \varphi} = 0 \Rightarrow \sum_{x_j > s} \left[\frac{-1}{1-\varphi}\right] + \sum_{x_j \le s} w_j \left[\frac{-1}{1-\varphi}\right] + \sum_{x_j \le s} (1 - w_j) \left[\frac{1}{\varphi}\right] = 0.$$

By solving

$$\frac{\partial E[\log (L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u})]}{\partial \theta} \text{ and } \frac{\partial E[\log (L_c(\theta, \varphi \mid \theta_0, \varphi_0, \underline{x}, \underline{u})]}{\partial \varphi} = 0.$$

We get θ_1 and φ_1 by solving the above equations.

The EM algorithm for computing the MLEs of the parameters in the perturbed Weibull model can be summarized as follows:

(A) Choose the initial estimates θ_0 and φ_0.

(B) Using the realization $(x_1, x_2, \ldots, x_n)$ of the observed sample, compute

$$w_j = \frac{(1-\varphi_0) \frac{\alpha}{\theta_0^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\theta_0}\right)^{\alpha}}{\frac{\varphi_0}{s} + (1-\varphi_0) \frac{\alpha}{\theta_0^{\alpha}} x^{\alpha-1} \exp\left(-\frac{x}{\theta_0}\right)\alpha}, \text{ for } x_j \le s.$$

(C)

$$\text{Take } \theta_1 = \left[\frac{\sum_{x_j > s} x_j\alpha + \sum_{x_j \le s} w_j x_j \alpha}{n_g + \sum_{x_j \le s} w_j}\right]^{\frac{1}{\alpha}} \text{ and } \varphi_1 = \frac{n_l - \sum_{x_j \le s} w_j}{n},$$

where n_g and n_l are defined as number of observations that are greater than '*s*' and less than or equal to '*s*' respectively.

(C) Repeat step (B) by fixing $\theta_0 = \theta_1$ and $\varphi_0 = \varphi_1$ until $L(\theta, \varphi; \underline{x})$ or $\log L(\theta, \varphi; \underline{x})$ converges to a value correct to a desirable number of decimal places.

A reasonable initial estimate of φ is $\frac{n_i}{n}$ and $\frac{x}{\Gamma(1 + \frac{1}{\alpha})}$ can be taken as an initial estimate of θ.

A simulation study has been carried out to demonstrate the asymptotic normality of the estimators of the parameters in perturbed Weibull model using *R* software (maxLik/optim).

Following are the three-dimensional graph of MLEs of parameters θ and φ.

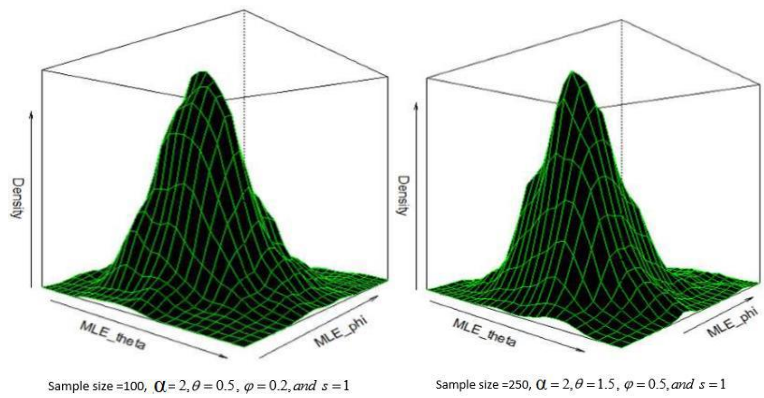

Figure 1. Bivariate density of MLEs of θ and φ for perturbed Weibull model.

2.1. Conclusion

The maximum likelihood estimators of the parameters are computed through EM algorithm for perturbed Weibull model. The simulation study has been carried out by using *R* software to demonstrate the normality of the parameters through three-dimensional graph (bivariate density of MLEs of θ and φ). Similar work can be done for gamma distribution.

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