Advances and Applications in Mathematical Sciences
Volume 21, Issue 3, January 2022, Pages 1075-1087
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NEIGHBORHOOD PRIME LABELING IN PRODUCT DIGRAPHS

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Abstract

Let $D(p, q)$ be a digraph. A function $f : V \to \{1, 2, \ldots, n\}$ is said to a neighborhood prime labeling of D if it is both in and out degree neighborhood prime labeling. In this paper, we investigate the existence of neighborhood prime labeling in product digraphs.

1. Introduction

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. The concept of graph labeling was introduced by Rosa in 1967 [8]. A useful survey on graph labeling by J. A.

2020 Mathematics Subject Classification: 05C78.
Keywords: Neighborhood prime, Labeling, Cartesian, Strong, Product, Digraphs.
Received July 12, 2021; Accepted October 12, 2021

Gallian (2014) can be found in [1]. S. K. Patel and N. P. Shrimali [5] have introduced the neighborhood prime labeling of graphs. A directed graph or digraph D consists of a finite set V of vertices and a collection of ordered pairs of distinct vertices. K. Palani et al. introduced the concept of neighborhood prime digraphs in [7]. In this paper, we investigate some product of digraphs for neighborhood prime labeling.

2. Preliminaries

The following definitions are from [3, 4, 6 and 7].

2.1. Definition. Let $D(p, q)$ be a digraph. A function $f : V(D) \to \{1, 2, \ldots, n\}$ is said to a *neighborhood prime labeling of D* if it is both in and out degree neighborhood prime labeling.

2.2. Observations.

1. If D is a digraph such that $N^{+}(u)$ or $N^{-}(u)$ are either $\emptyset$ or singleton set, then D admits neighborhood prime labeling.

2. A neighborhood prime digraph D in which every vertex of in-degree or out-degree at most 1, is neighborhood prime.

2.3. Definition. The *Cartesian product* of a family of digraphs $D_1, D_2, \ldots, D_n$ denoted by $D_1 \times D_2 \times \ldots \times D_n$ or $\prod_{i=1}^{n} D_i$ where $n \geq 2$ is the digraph D having $V(D) = V(D_1) \times V(D_2) \times \ldots \times V(D_n) = \{(W_1, W_2, \ldots, W_n) : W_i \in V(D_i), i = 1, 2, \ldots, n\}$ and a vertex $(u_1, u_2, \ldots, u_n)$ dominates a vertex $(v_1, v_2, \ldots, v_n)$ of D if and only if there exists an $r \in \{1, 2, \ldots, n\}$ such that $u_r v_r \in A(D_r)$ and $u_i = v_i$ for all $i \in \{1, 2, \ldots, n\} - \{r\}$.

2.4. Definition. Let D and F be digraphs. The product of digraphs D and F have similarly as in graphs, their set of vertices equal to $V(D) \times V(F)$. In the *strong product* $D \boxtimes F$ we have $((d, f), (d', f')) \in A(D \boxtimes F)$ if $((d, d') \in A(D)$ and $f = f')$ or $(d = d'$ and $(f, f') \in A(F))$ or $((d, d') \in AD$ and $(f, f') \in A(F))$.

2.5. Definition. A comb graph $P_n \odot K_1$ in which the path edges are directed in one direction and the pendant edges are oriented away from the end vertices is called an *upcomb* and it is denoted as $\overrightarrow{UpP_n \odot K_1}$.

2.6. Definition. A crown graph $C_n \odot K_1$ in which the edges of the cycle are directed clockwise or anti-clockwise and the pendant edges are directed towards the cycle is called an *incrown* and it is denoted as $\overrightarrow{iC_n \odot K_1}$.

2.7. Definition. A wheel graph W_n in which the edges of the outer cycle are directed clockwise or anti-clockwise and the spoke edges are directed towards the central vertex is called an *inwheel* and it is denoted as $\overrightarrow{iW_n}$.

3. Main Results

3.1. Theorem. *Cartesian product of* $\overrightarrow{K_2}$ *and* $\overrightarrow{UpP_n \odot K_1}$ *is a neighborhood prime digraph.*

Proof. Let u_1, u_2 be the vertices of $\overrightarrow{K_2}$ and let $v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$ be the vertices of $\overrightarrow{UpP_n \odot K_1}$.

Let $V(\overrightarrow{K_2} \times \overrightarrow{UpP_n \odot K_1}) = \{(u_1, v_i) \mid 1 \le i \le n\} \cup \{(u_2, v_i) \mid 1 \le i \le n\} \cup \{(u_1, w_i) \mid 1 \le i \le n\} \cup \{(u_2, w_i) \mid 1 \le i \le n\}$ be the vertex set.

Let $x_i = (u_1, v_i), x_i' = (u_2, v_i), y_i = (u_1, w_i) \,\&\, y_i' = (u_2, w_i)$ for $i = 1, 2, \ldots, n$.

Correspondingly, $V(\overrightarrow{K_2} \times \overrightarrow{UpP_n \odot K_1}) = \{x_i \mid 1 \le i \le n\} \cup \{x_i' \mid 1 \le i \le n\} \cup \{y_i \mid 1 \le i \le n\} \cup \{y_i' \mid 1 \le i \le n\}$

Then $A(\overrightarrow{K_2} \times \overrightarrow{UpP_n \odot K_1}) = \{\overrightarrow{x_i x_{i+1}} \mid 1 \le i \le n-1\} \cup \{\overrightarrow{x_i x_i'} \mid 1 \le i \le n\} \cup \{\overrightarrow{x_i' x_{i+1}'} \mid 1 \le i \le n-1\} \cup \{\overrightarrow{y_i x_i} \mid 1 \le i \le n\} \cup \{\overrightarrow{y_i' x_i'} \mid 1 \le i \le n\} \cup \{\overrightarrow{y_i y_i'} \mid 1 \le i \le n\}$ is the arc set.

Define $f : V(\overrightarrow{K_2} \times \overrightarrow{UpP_n \odot K_1}) \to \{1, 2, \ldots, 4n\}$ by

$f(x_i) = 4i - 3, 1 \le i \le n;$

$f(x_i') = 4i - 1, 1 \le i \le n;$

$f(y_i) = 4i, 1 \le i \le n;$

$f(y_i') = 4i - 2, 1 \le i \le n;$

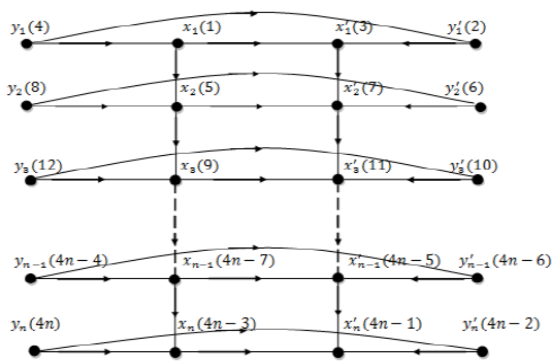

Figure 1. Neighborhood prime labeling of $\overrightarrow{K_2} \times Up\overrightarrow{P_n \odot K_1}$.

Now $d^-(x_i) > 1$ for $i = 2, \ldots, n$ and $d^-(x_i') > 1$ for $i = 2, \ldots, n$.

$$N^-(x_1) = \{y_1\} \tag{1}$$

for $i = 2, \ldots, n$ and so $\gcd\{f(x_{i-1}), f(y_i)\}$ $N^-(x_i) = \{x_{i-1}, y_i\}$ $= \gcd\{4i - 7, 4i\} = 1$ for $i = 2, \ldots, n$

$$\therefore \gcd\{f(p) \mid p \in N^-(x_i)\} = 1 \text{ for } i = 2, \ldots, n \tag{2}$$

Now, $N^-(x_1') = \{x_1, y_1'\}$ and $\gcd\{f(x_1), f(y_1')\} = \gcd\{1, 2\} = 1$

$N^-(x_1') = \{x_i, x_{i-1}, y_1'\}$ for $i = 2, \ldots, n$ and

$\gcd\{f(x_i), f(x_{i-1}'), f(y_1')\} = \gcd\{4i - 3, 4i - 5, 4i - 2\} = 1$ for $i = 2, \ldots, n$

$$\therefore \gcd\{f(p) \mid p \in N^-(x_i')\} = 1 \text{ for } i = 1, 2, \ldots, n \tag{3}$$

$$N^-(y_i) = \emptyset \text{ for } i = 1, 2, \ldots, n \tag{4}$$

$$N^-(y_i') = \{y_i\} \text{ for } i = 1, 2, \ldots, n \tag{5}$$

From (1) to (5), we get f is an in-degree neighborhood prime labeling. Here $d^+(x_i) > 1$ for $i = 1, 2, \dots, n-1$ and $d^+(y_i) > 1$ for $i = 1, 2, \dots, n$.

$N^+(x_i) = \{x_{i+1}, x_i'\}$ for $i = 1, 2, \dots, n-1$

Then $\gcd\{f(x_{i+1}), f(x_i')\} = \gcd\{4i+1, 4i-1\} = 1$ for $i = 1, 2, \dots, n-1$.

$$\therefore \gcd\{f(p) \mid p \in N^+(x_i)\} = 1 \text{ for } i = 1, 2, \dots, n-1 \tag{6}$$

$$N^+(x_n) = \{x_n'\} \tag{7}$$

$$N^+(x_i') = \{x_{i+1}'\} \text{ for } i = 1, 2, \dots, n-1 \tag{8}$$

$$N^+(x_n') = \emptyset \tag{9}$$

$N^+(y_i) = \{x_i, y_i'\}$ for $i = 1, 2, \dots, n$ and

$\gcd\{f(x_i), f(y_i')\} = \gcd\{4i-3, 4i-2\} = 1$ for $i = 1, 2, \dots, n$.

$$\therefore \gcd\{f(p) \mid p \in N^+(y_i)\} = 1 \text{ for } i = 1, 2, \dots, n \tag{10}$$

$$N^+(y_i') = \{x_i'\} \text{ for } i = 1, 2, \dots, n. \tag{11}$$

From (6) to (11) we have, f is an out-degree neighborhood prime labeling $\therefore f$ satisfies both in and out degree neighborhood prime labeling.

Thus, the Cartesian product of $\overrightarrow{K_2}$ and $\overrightarrow{UpP_n \odot K_1}$ is a neighborhood prime digraph.

3.2. Theorem. *Cartesian product of* $\overrightarrow{K_2}$ *and* $i\overrightarrow{W_n}$ *is a neighborhood prime digraph.*

Proof. Let u_1, u_2 and $w, v_1, v_2, \dots, v_n$ be the vertices of $\overrightarrow{K_2}$ and $i\overrightarrow{W_n}$ respectively.

Let $V(\overrightarrow{K_2} \times i\overrightarrow{W_n}) = \{(u_1, v_i) \cup \mid 1 \le i \le n\} \cup \{(u_2, v_i) \mid 1 \le i \le n\} \cup \{(u_i, w) \mid$

$i = 1, 2$ be the vertex set.

Let $x_i = (u_1, v_i)$, $y_i = (u_2, v_i)$ for $i = 1, 2, \ldots, n$ and $z_i = (u_i, w)$ for $i = 1, 2$.

Correspondingly, $V(\overrightarrow{K_2} \times i\overrightarrow{W_n}) = \{x_i \mid 1 \le i \le n\} \cup \{y_i \mid 1 \le i \le n\} \cup \{z_i \mid i = 1, 2\}$.

Then $A(\overrightarrow{K_2} \times i\overrightarrow{W_n}) = \{\overrightarrow{x_i x_{i+1}} \mid 1 \le i \le n-1\} \cup \{\overrightarrow{x_n x_1}\} \cup \{\overrightarrow{x_i z_1} \mid 1 \le i \le n\}$ $\cup \{\overrightarrow{y_i z_2} \mid 1 \le i \le n\} \cup \{\overrightarrow{y_i y_{i+1}} \mid 1 \le i \le n-1\} \cup \{\overrightarrow{y_n y_1}\} \cup \{\overrightarrow{x_i y_i} \mid 1 \le i \le n\}$ is the arc set.

Define $f : V(\overrightarrow{K_2} \times i\overrightarrow{W_n}) \to \{1, 2, \ldots, 2n+2\}$ by

$f(x_1) = 1$, $f(x_i) = 2i$, $2 \le i \le n$;

$f(y_i) = 2i + 1, 1 \le i \le n$;

$f(z_1) = 2n + 2$ and $f(z_2) = 2$.

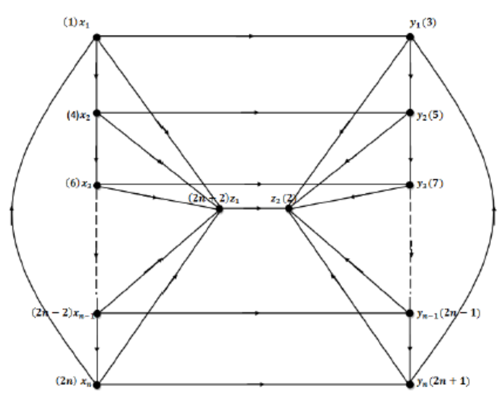

Figure 2. Neighborhood prime labeling of $\overrightarrow{K_2} \times i\overrightarrow{W_n}$.

Now $d^-(z_i) > 1$ for $i = 1, 2$ and $d^-(y_i) > 1$ for $i = 1, 2, \ldots, n$.

$N^-(z_1) = \{x_1, x_2, \ldots, x_n\}$ and $\gcd\{f(x_1), f(x_2), \ldots, f(x_n)\}$ $= \gcd\{1, 4, 6, \ldots, 2n\} = 1$.

$N^-(z_2) = \{z_1, y_1, y_2, \ldots, y_n\}$ and $\gcd\{f(y_1), f(y_2), \ldots, f(y_n), f(z_1)\}$ $= \gcd\{3, 5, \ldots, 2n+1, 2n+2\} = 1$.

$\therefore \gcd\{f(p) \mid p \in N^-(z_i)\} = 1$ for $i = 1, 2$ (1)

Now, $N^-(x_1) = \{x_n\}$ and $N^-(x_i) = \{x_{i-1}\}$ for $i = 2, \ldots, n$ (2)

Then $N^-(y_1) = \{x_1, y_n\}$ and $\gcd\{f(x_1), f(y_n)\} = \gcd\{1, 2n+1\} = 1$

$N^-(y_i) = \{x_i, y_{i-1}\}$ for $i = 1, 2, \ldots, n$ and so $\gcd\{f(x_i), f(y_{i-1})\}$ $= \gcd\{2i, 2i-1\} = 1$ for $i = 2, \ldots, n$

$\therefore \gcd\{f(p) \mid p \in N^-(y_i)\} = 1$ for $i = 1, 2, \ldots, n$ (3)

From (1) to (3) we get f is an in-degree neighborhood prime labeling.

Here $d^+(x_i) > 1$ and $d^+(y_i) > 1$ for $i = 1, 2, \ldots, n$.

Now $N^+(x_i) = \{x_{i+1}, y_i, z_1\}$ for $i = 1, 2, \ldots, n-1$

Then $\gcd\{f(x_{i+1}), f(y_i), f(z_1)\} = \gcd\{2i+2, 2i+1, 2n+2\} = 1$ for $i = 1, 2, \ldots, n-1$.

$N^+(x_n) = \{x_1, y_n, z_1\}$ and $\gcd\{f(x_1), f(y_n), f(z_1)\} = \gcd\{1, 2n+1, 2n+2\} = 1$.

$\therefore \gcd\{f(p) \mid p \in N^+(x_i)\} = 1$ for $i = 1, 2, \ldots, n$ (4)

Now $N^+(y_i) = \{y_{i+1}, z_2\}$ for $i = 1, 2, \ldots, n-1$ and

$\gcd\{f(y_{i+1}), f(z_2)\} = \gcd\{2i+3, 2\} = 1$ for $i = 1, 2, \ldots, n-1$.

$N^+(y_n) = \{y_1, z_2\}$ and $\gcd\{f(y_1), f(z_2)\} = \gcd\{3, 2\} = 1$.

$\therefore \gcd\{f(p) \mid p \in N^+(y_i)\} = 1$ for $i = 1, 2, \ldots, n$ (5)

Then $N^{+}(z_1)=\{z_2\}$ and $N^{+}(z_2)=\emptyset.$ (6)

From (4) to (6) we get, f is an out-degree neighborhood prime labeling

$\therefore f$ satisfies both in and out degree neighborhood prime labeling. Thus, the Cartesian product of $\overrightarrow{K_2}$ and $i\overrightarrow{W_n}$ is a neighborhood prime digraph.

3.3. Theorem. *Cartesian product of* $\overrightarrow{K_2}$ *and* $i\overrightarrow{C_n \odot K_1}$ *is a neighborhood prime digraph.*

Proof. Let u_1, u_2 and $v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$ be the vertices of $\overrightarrow{K_2}$ and $i\overrightarrow{C_n \odot K_1}$ respectively.

Let $V(\overrightarrow{K_2} \times i\overrightarrow{C_n \odot K_1}) = \{(u_1, v_i) \bigcup | 1 \le i \le n\} \bigcup \{(u_2, v_i) | 1 \le i \le n\}$ $\bigcup \{(u_1, w_i) | 1 \le i \le n\} \bigcup \{(u_2, w_i) | 1 \le i \le n\}$ be the vertex set.

Let $x_i = (u_1, v_i), x_i' = (u_2, v_i), y_i = (u_1, w_i) \,\&\, y_i' = (u_2, w_i)$ for $i = 1, 2, \ldots, n.$

Correspondingly, $V(\overrightarrow{K_2} \times i\overrightarrow{C_n \odot K_1}) = \{x_i \,|\, 1 \le i \le n\} \bigcup \{x_i' \,|\, 1 \le i \le n\} \{y_i \,|\, 1 \le i \le n\}$ $\bigcup \{y_i' \,|\, 1 \le i \le n\}$

Then $A(\overrightarrow{K_2} \times i\overrightarrow{C_n \odot K_1}) = \{\overrightarrow{x_i x_{i+1}} \,|\, 1 \le i \le n-1\} \bigcup \{\overrightarrow{x_n x_1}\} \bigcup \{\overrightarrow{x_i x_i'} \,|\, 1 \le i \le n\}$ $\bigcup \{\overrightarrow{x_i' x_{i+1}'}\} \,|\, 1 \le i \le n-1\} \bigcup \{\overrightarrow{x_n' x_1'}\} \bigcup \{\overrightarrow{y_i x_i} \,|\, 1 \le i \le n\} \bigcup \{\overrightarrow{y_i' x_i'} \,|\, 1 \le i \le n\}$ $\bigcup \{\overrightarrow{y_i y_i'} \,|\, 1 \le i \le n\}$ is the arc set.

Define $f : V(\overrightarrow{K_2} \times i\overrightarrow{C_n \odot K_1}) \to \{1, 2, \ldots, 4n\}$ by

$f(x_i) = 4i - 3, 1 \le i \le n;$

$f(x_i') = 4i - 1, 1 \le i \le n;$

$f(y_1) = 4i, 1 \le i \le n;$

$f(y_i') = 4i - 2, 1 \le i \le n;$

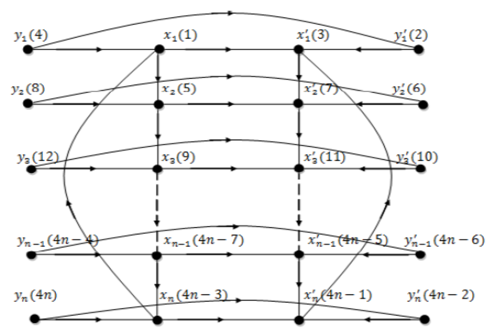

Figure 3. Neighborhood prime labeling of $\overrightarrow{K_2} \times i\overrightarrow{C_n \odot K_1}$.

Now $d^-(x_i) > 1$ for $i = 1, 2, \ldots, n$ and $d^-(x_i') > 1$ for $i = 1, 2, \ldots, n$.

$N^-(x_1) = \{x_n, y_1\}$ and so $\gcd\{f(x_n), f(y_1)\} = \gcd\{4n-3, 4\} = 1$.

$N^-(x_i) = \{x_{i-1}, y_i\}$ for $i = 1, 2, \ldots, n$ and

$\gcd\{f(x_{i-1}), f(y_i)\} = \gcd\{4i-7, 4i\} = 1$ for $i = 2, \ldots, n$

$\therefore \gcd\{f(p) \mid p \in N^-(x_i)\} = 1$ for $i = 1, 2, \ldots, n$ (1)

Now, $N^-(x_1') = \{x_1, x_n', y_1'\}$ and $\gcd\{f(x_1), f(x_n'), f(y_1')\} = \gcd\{1, 4n-1, 2\} = 1$.

$N^-(x_i') = \{x_i, x_{i-1}', y_i'\}$ for $i = 1, 2, \ldots, n$ and

$\gcd\{f(x_1), f(x_{i-1}'), f(y_i')\} = \gcd\{4i-3, 4i-5, 4i-2\} = 1$ for $i = 2, \ldots, n$

$\therefore \gcd\{f(p) \mid p \in N^-(x_i')\} = 1$ for $i = 1, 2, \ldots, n$ (2)

$N^-(y_i) = \emptyset$ for $i = 1, 2, \ldots, n$ (3)

$N^-(y_i') = \{y_i\}$ for $i = 1, 2, \ldots, n$ (4)

From (1) to (4) we get *f* is an in-degree neighborhood prime labeling.

Here $d^+(x_i) > 1$ and $d^+(y_i) > 1$ for $i = 1, 2, \ldots, n$.

Now $N^+(x_i) = \{x_{i+1}, x_i'\}$ for $i = 1, 2, \ldots, n-1$

Then $\gcd\{f(x_{i+1}), f(x_i')\} = \gcd\{4i+1, 4i-1\} = 1$ for $i = 1, 2, \ldots, n-1$.

$N^+(x_n) = \{x_n', x_1\}$ and $\gcd\{f(x_n'), f(x_1)\} = \gcd\{4n-1, 1\} = 1$

$$\therefore \gcd\{f(p) \mid p \in N^+(x_i)\} = 1 \text{ for } i = 1, 2, \ldots, n \quad (5)$$

$$N^+(x_i') = \{x_{i+1}'\} \text{ for } i = 1, 2, \ldots, n-1 \text{ and } N^+(x_n') = \{x_1'\} \quad (6)$$

$N^+(y_i) = \{x_i, y_i'\}$ and $\gcd\{f(x_i), f(y_i')\} = \gcd\{4i-3, 4i-2\} = 1$ for $i = 1, 2, \ldots, n$

$$\therefore \gcd\{f(p) \mid p \in N^+(y_i)\} = 1 \text{ for } i = 1, 2, \ldots, n \quad (7)$$

$$N^+(y_i') = \{x_i'\} \text{ for } i = 1, 2, \ldots, n \quad (8)$$

From (5) to (8) we have, f is an out-degree neighborhood prime labeling $\therefore f$ satisfies both in and out degree neighborhood prime labeling.

Thus, the Cartesian product of $\overrightarrow{K_2}$ and $\overrightarrow{iC_n \odot K_2}$ is a neighborhood prime digraph.

3.4. Theorem. *Strong product of $\overrightarrow{P_2}$ and $\overrightarrow{UpP_n \odot K_1}$ is a neighborhood prime digraph.*

Proof. Let u_1, u_2 and $v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n$ be the vertices of $\overrightarrow{P_2}$ and $\overrightarrow{UpP_n \odot K_1}$ respectively.

Let $V(\overrightarrow{P_2} \boxtimes \overrightarrow{UpP_n \odot K_1}) = \{(u_1, v_i) \bigcup \mid 1 \le i \le n\} \bigcup \{(u_2, v_i) \mid 1 \le i \le n\} \bigcup \{(u_1, w_i) \mid 1 \le i \le n\} \bigcup \{(u_2, w_i) \mid 1 \le i \le n\}$.

Let $x_i = (u_1, v_i), x_i' = (u_2, v_i), y_i = (u_1, w_i) \,\&\, y_i' = (u_2, w_i)$ for $i = 1, 2, \ldots, n$.

Correspondingly, $V(\overrightarrow{P_2} \boxtimes \overrightarrow{UpP_n \odot K_1}) = \{x_i \mid 1 \le i \le n\} \bigcup \{x_i' \mid 1 \le i \le n\}$ $\{y_i \mid 1 \le i \le n\} \bigcup \{y_i' \mid 1 \le i \le n\}$

Then $A(\overrightarrow{P_2} \boxtimes \overrightarrow{UpP_n \odot K_1}) = \{\overrightarrow{x_i x_{i+1}} \mid 1 \le i \le n-1\} \bigcup \{\overrightarrow{x_i x_i'} \mid 1 \le i \le n\}$ $\bigcup \{\overrightarrow{x_i' x_{i+1}'}\} \mid 1 \le i \le n-1\} \bigcup \{\overrightarrow{x_i x_{i+1}'} \mid 1 \le i \le n-1\} \bigcup \{\overrightarrow{y_i x_i} \mid 1 \le i \le n\}$

$\cup\{\overrightarrow{y_i'x_i} \mid 1 \le i \le n\} \cup \{\overrightarrow{y_i'x_i'} \mid 1 \le i \le n\} \cup \{\overrightarrow{y_iy_i'} \mid 1 \le i \le n\}$ is the arc set.

Define $f : V(\overrightarrow{P_2} \boxtimes \overrightarrow{UpP_n \odot K_1}) \to \{1, 2, \ldots, 4n\}$ by

$f(x_i) = 4i - 3, 1 \le i \le n;$

$f(x_i') = 4i - 1, 1 \le i \le n;$

$f(y_i) = 4n - 2, f(y_i) = 4i - 6, 2 \le i \le n;$

$f(y_i') = 4i, \ 1 \le i \le n;$

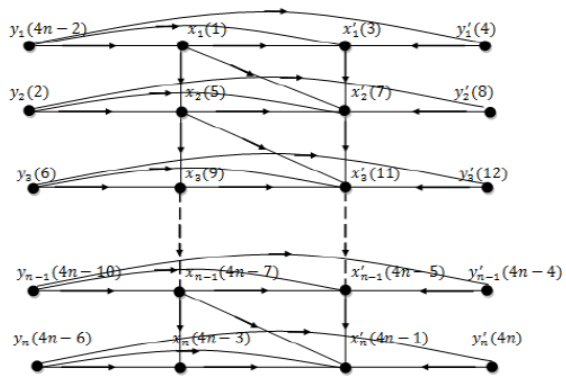

Figure 4. Neighborhood prime labeling of $\overrightarrow{P_2} \boxtimes \overrightarrow{UpP_n \odot K_1}$.

Now $d^-(x_i) > 1$ for $i = 2, \ldots, n$ and $d^-(x_i') > 1$ for $i = 1, 2, \ldots, n.$

$$N^-(x_1) = \{y_1\} \quad (1)$$

$N^-(x_i) = \{x_{i-1}, y_i\}$ for $i = 2, \ldots, n$ and so $\gcd\{f(x_{i-1}), f(y_i)\}$ $= \gcd\{4i - 7, 4i - 6\} = 1$ for $i = 2, \ldots, n$

$$\therefore \gcd\{f(p) \mid p \in N^-(x_i)\} = 1 \text{ for } i = 2, \ldots, n \quad (2)$$

Now, $N^-(x_1') = \{x_1, y_1, y_1'\}$ and $\gcd\{f(x_1), f(y_1), f(y_1')\} = \gcd\{1, 4n - 2, 4\} = 1.$

$N^-(x_i') = \{x_i, x_{i-1}, y_i, x_{i-1}'y_1'\}$ for $i = 2, \ldots, n$ and

$\gcd\{f(x_i), f(x_{i-1}), f(y_i), f(x_{i-1}'), f(y_i')\}$

$= \gcd\{4i-3, 4i-7, 4i-6, 4i-5, 4i\} = 1$ for $i = 2, \ldots, n$

$\therefore \gcd\{f(p) \mid p \in N^-(x_i')\} = 1$ for $i = 2, \ldots, n$ (3)

$N^-(y_i) = \emptyset$ for $i = 1, 2, \ldots, n$ (4)

$N^-(y_i') = \{y_i\}$ for $i = 1, 2, \ldots, n$ (5)

From (1) to (5), we get f is an in-degree neighborhood prime labeling.

Here $d^+(x_i) > 1$ for $i = 1, 2, \ldots, n-1$ and $d^+(y_i) > 1$ for $i = 1, 2, \ldots, n$.

$N^+(x_i) = \{x_{i+1}, x_{i+1}', x_i'\}$ for $i = 1, 2, \ldots, n-1$ and

$\gcd\{f(x_{i+1}), f(x_{i+1}'), f(x_i')\} = \gcd\{4i+1, 4i+3, 4i-1\} = 1$ for $i = 1, 2, \ldots, n-1$.

$\therefore \gcd\{f(p) \mid p \in N^+(x_i)\} = 1$ for $i = 1, 2, \ldots, n-1$ (6)

$N^+(x_n) = \{x_n'\}$ (7)

$N^+(x_i') = \{x_{i+1}'\}$ for $i = 1, 2, \ldots, n-1$ (8)

$N^+(x_n') = \emptyset$ (9)

$N^+(y_i) = \{x_i, x_i', y_i'\}$ for $i = 1, 2, \ldots, n$ and

$\gcd\{f(x_i), f(x_i'), f(y_i')\} = \gcd\{4i-3, 4i-1, 4i\} = 1$ for $i = 1, 2, \ldots, n$

$\therefore \gcd\{f(p) \mid p \in N^+(y_i)\} = 1$ for $i = 1, 2, \ldots, n$ (10)

$N^+(y_i') = \{x_i'\}$ for $i = 1, 2, \ldots, n$ (11)

From (6) to (11) we have, f is an out-degree neighborhood prime labeling.

$\therefore f$ satisfies both in and out degree neighborhood prime labeling.

Thus, the strong product of $\overrightarrow{P_2}$ and $Up\overrightarrow{P_2 \odot K_1}$ is a neighborhood prime digraph.

References

[1] J. A. Gallian, A dynamic survey of labeling, The Electronics Journal of Combinatorics 17 (2014).

[2] F. Harary, Graph Theory, Addition Wesley, Massachusetts, (1972).

[3] Iztok Peterin and Ismael G. Yero, Efficient closed domination in digraph products, Journal of Combinatorial optimization 38(1) (2019), 130-149.

[4] Jorgen Bang-Jensen and Gregory Gutin, Digraphs Theory, Algorithms and Applications, Springer Verlag, (2007).

[5] S. K. Patel and N. P. Shrimali, Neighborhood-prime labeling, International Journal of Mathematics and Soft Computing 5(2) (2015), 135-143.

[6] K. Palani and G. Suganya, Prime Pair Labeling of Directed Graphs, Journal of Emerging Technologies and Innovative Research (JETIR) 6(2) (2019), 513-517.

[7] K. Palani and A. Shunmugapriya, Neighborhood prime labeling in some product and power digraphs, Journal of Physics: Conference Series 1947 (2021) 012040.

[8] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Intl. Symp. Rome 1966), Gordon and Breach, Dunod, Paris (1967), 349-355.