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CYCLIC DECOMPOSITION OF UNICYCLIC GRAPHS

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Abstract

A decomposition H of a graph G into subgraphs $H_1, H_2, \ldots, H_n$ is said to be cyclic if there exists an isomorphism f of G which induces a cyclic permutation f_V of the set $V = V(G)$ and satisfies the following implication: if $H_i \in H$, then $fH_i \in H$ for some subgraph H_i of H. Here fH_i is the subgraph of G with vertex set $\{f_u : u \in V(H_i)$ and edge set $\{f_w : w \in E(H_i)\}$. In this paper we concentrate the cyclic decomposition of unicyclic graphs into stars and paths.

1. Introduction

The method of cyclic decompositions which we are going to expound has been used in many papers, see e.g. Kotzig (1965), Rosa (1966) and Hartnell (1975). The treatment of cyclic decomposition and its extend has appeared in a book "Decomposition of Graphs" authored by Juraj Bosak. Terms that not defined here are used in the sense of Harary [4] and Juraj Bosak [5].

2. Related Results and Examples

Notation 2.1. Let G be a graph which admit a cyclic decomposition. Notice that $H_1, H_2, \ldots, H_n$ are the cyclic decompositions of G where

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$n = | H_i |$ and we say that G can be cyclically decomposed into n copies of H_i.

A cyclic decomposition is called a cyclic H_i-decomposition with respect to the suitable subgraph H_i of G where H_i can be any connected subgraph of G.

Remark 2.2. Consider the vertex set of a graph $G, V(G) = \{1, 2, 3, \ldots, v\}, v \geq 3$ and

$$f = \begin{pmatrix} 1 \; 2 \; 3, \ldots, v-1 \; v \\ 2 \; 3 \; 4, \ldots v \; 1 \end{pmatrix}$$

The mapping f will be called a (simple) rotation, and its i-th iteration an i-fold rotation $(i = 1, 2, 3, \ldots, n)$. In cyclic decomposition, the term rotation can be applied to subgraphs or their elements. Therefore, we shall speak of a rotation of subgraph, vertex or an edge. This terminology is justified by a representation of G in the plane so that the vertices of G coincide with the vertices of an N-gon, $N \geq 3$ and the edges of G are the sides and diagonals of the N-gon. Rotations of subgraphs, vertices, or edges then become rotations around the center of the N-gon giving this word its natural meaning.

Example 2.3. Consider the graph G given in Figure 2.2 where the cyclic decomposition of G is shown by different colours.

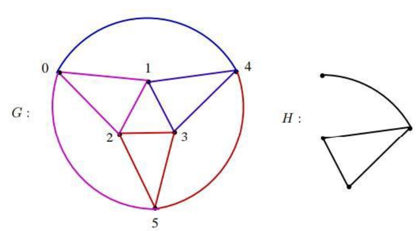
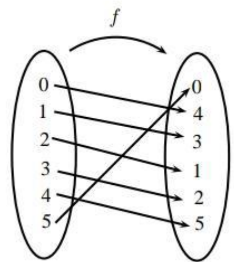


Figure 2.1

We shall write the permutation f of the set $V(G) = \{0, 1, 2, 3, 4, 5\}$ as

$$f = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 & 0 \end{pmatrix}$$

is a cyclic permutation which is represented by the mapping given below



where $f(0) = 4$, $f(1) = 3$, $f(2) = 1$, $f(3) = 2$, $f(4) = 5$, $f(5) = 0$. Using the mapping f we get the edges of respective H_i's for all $i = 1, 2, 3$ Consider $E(H_1) = \{0\ 4, 1\ 4, 1\ 3, 4\ 3\}$. The following are the edges of H_2, $f(0\ 4) = 4\ 5$, $f(1\ 4) = 3\ 5$, $f(1\ 3) = 3\ 2$, $f(4\ 3) = 5\ 2$, the following are the edges of H_3, $f(4\ 5) = 5\ 0$, $f(3\ 5) = 2\ 0$, $f(3\ 2) = 2\ 1$, $f(5\ 2) = 01$ and the following are the edges of H_1, $f(5\ 0) = 0\ 4$, $f(2\ 0) = 1\ 4$, $f(2\ 1) = 1\ 3$, $f(0\ 1) = 4\ 3$.

2. Cyclic Decomposition of Unicyclic Graphs

If U admits a cyclic H_i-decomposition then H_i is any one of the tree. In this section we concentrate the cyclic decomposition of U into stars and paths.

Notation 3.1. Let U be a unicyclic graph and r be the length of the cycle in U. Let $\{v_1, v_2, v_3, \ldots, v_r\}$ be the consecutive vertices of C_r in U.

Lemma 3.2. *If U admits a cyclic decomposition, then* $C(U) \subseteq V(C_r)$.

Proof. It follows from that Lemma 2.3, $C(U) \subseteq V(C_r)$.

Lemma 3.3. *If U admits a cyclic decomposition, then* $r \cong 0 (\bmod n)$

Proof. Let U admit a cyclic decomposition. It follows from that Lemma 3.2, $C(U) \subseteq V(C_r)$. We claim that C_r contributes exactly r/n edges to each H_i. Consider the following cases.

Case (i) C_r contributes more than r/n edges to each H_i.

Then we can find at least one H_i which contains $< r/n$ edges from C_r. Since the decomposition is cyclic, the rotation of H_i around C_r produces U. Then G does not admit a cyclic decomposition since the improper rotation of some H_i around C_r, which contradicts our assumption.

Case (ii) C_r contributes less than r/n edges to each H_i.

Then we can find atleast one H_i which contains $> r/n$ edges from C_r. It follows from that case (i), we get contradiction since the improper rotation of some H_i around C_r.

Then from the above cases C_r contributes exactly r/n edges to each H_i so that $r \equiv 0(\bmod n)$.

Theorem 3.4. *U admits a cyclic P_m decomposition if and only if U is any one of the following unicyclic graphs*

1. $U \cong C_r$, r *is composite.*

2. $U - V(C_r) \cong nP_j$, $j = m - r/n - 1$ *and* P_j *is the component of* $U - \{v_{i\frac{r}{n}}\}$ *for all* $1 \leq i \leq n$.

3. $U - V(C_r) \cong nP_j \cup nP_k$, $j + k = m - r/n - 1$ *and* P_j, P_k *are the components of* $U - \{v_{i\frac{r}{n}}\}$ *for all* $1 \leq i \leq n$.

Proof. Let U admit a cyclic P_m decomposition. Let u_i and w_j be the end vertices of H_i. Consider the following cases.

Suppose $u_i, w_j \in V(C_r)$. Without loss of generality we may assume that $u_i = v_1, w_j = v_m$ are the end vertices of H_1 where H_1 is a $v_1 - v_m$ path. Since the decomposition is cyclic, v_m, v_{2m-1} are the end vertices of H_2 where H_2 is a $v_m - v_{2m-1}$ path. Similarly, $v_{2m-1}v_{3m-1}$ are the end vertices of H_3 where H_3 is a $v_{2m-1} - v_{3m-1}$ path. Proceeding like this we get $v_{(n-1)m-(n-2)}$ and $v_{nm-(n-1)}(= v_1)$ are the end vertices of H_n where H_n is a $v_{(n-1m-(n-2))} - v_1$ path. Then we can define that $U \cong C_r$, r is composite. Hence (1).

Suppose $u_i \in V(C_r)$ and $w_i \notin V(C_r)$. Then u_i is one of the end vertex of some H_i and belongs to exactly two H_i where $d(u_i) = 3$ and $w_j \in U - V(C_r)$. Then C_r contributes exactly r/n edges to each H_i. Let

$u_i = v_i$ where $V(C_r) = \{v_1, v_2, v_3, \ldots, v_{\frac{r}{n}}, v_{1+\frac{r}{n}}, v_{2+\frac{r}{n}}, \ldots, v_{2\frac{r}{n}}, v_{1+2\frac{r}{n}}, v_{2+2\frac{r}{n}}, \ldots, v_{n\frac{r}{n}}\}$. Without loss of generality we may assume that $w_1, v_{\frac{r}{n}}$ are the end vertices of H_1. Since the decomposition is cyclic, $w_2, v_{2\frac{r}{n}}$ are the end vertices of H_2 Similarly $w_3, v_{3\frac{r}{n}}$ are the end vertices of H_3. Proceeding like this we get $w_n, v_{n\frac{r}{n}} (= v_r)$ are the end vertices of H_n. Then H_n is a $w_i - v_{i\frac{r}{n}}$ path for all $1 \leq i \leq n$ and $\{v_{i\frac{r}{n}} : 1 \leq i \leq n\}$ is the set of all cut points of U where $d(v_n \frac{r}{n}) = 3$. Then we can define that $U - V(C_r) \cong nP_j$, $j = m - r/n - 1$ and P_j is the component of $U - \{v_{i\frac{r}{n}}\}$ for all $1 \leq i \leq n$. Hence (2).

Suppose $u_j, w_k \notin V(C_r)$. Then $u_j, w_k \in U - V(C_r)$ and C_r contributes exactly r/n internal edges to each H_i. Let $V(C_r) = \{v_1, v_2, v_3, \ldots, v_{\frac{r}{n}}, v_{1+\frac{r}{n}}, v_{2+\frac{r}{n}}, \ldots, v_{2\frac{r}{n}}, v_{1+2\frac{r}{n}}, v_{2+2\frac{r}{n}}, \ldots, v_{n\frac{r}{n}}\}$ where $V(C\ r)$ is the internal vertices of H_i for all $1 \leq i \leq n$. Without loss of generality we may assume that u_1, w_1 are the end vertices of H_1 where H_1 is a $u_1 - w_1$ path. Since the decomposition is cyclic, H_2 is a $u_2 - w_2$ path and H_3 is a $u_3 - w_3$ path. Proceeding like this we get H_n is a $u_n - w_n$ path. Then H_i is a $u_j - w_k$ path for some $u_j \in P_j$ and $w_k \in P_k$ and $\{v_{i\frac{r}{n}} : 1 \leq i \leq n\}$ is the set of all cut points of U where $d(v_{n\frac{r}{n}}) = 4$. Then we can define that $U - V(C_r) \cong nP_j \cup nP_k$, $j + k = m - r/n - 1$ and P_j, P_k are the components of $U - \{v_{i\frac{r}{n}}\}$ for all $1 \leq i \leq n$. Hence (3).

Conversely, let (1), (2) and (3).

Consider (1). Then $H_i \cong P_{1+\frac{r}{n}}$ for all $1 \leq i \leq n$ is a cyclic P_m

decomposition of U.

Consider (2). Then $H_i \cong P_{j+1+\frac{r}{n}}$ is a cyclic P_m decomposition of U where C_r contributes $1+\frac{r}{n}$ consecutive vertices to each H_i.

Consider (3). Then $H_i \cong P_{j+k+1+\frac{r}{n}}$ is a cyclic decomposition of U where contributes $1+\frac{r}{n}$ consecutive vertices to each H_i.

Lemma 3.5. *If U admits a cyclic S_m decomposition, then either $n = r$ or $n = \frac{r}{2}$.*

Proof. It follows from that Lemma 3.3, $r \equiv 0 (\operatorname{mod} n)$. Clearly C_r contributes exactly $\frac{r}{n}$ edges to each H_i. We claim that $\frac{r}{n} \leq 2$. Suppose C_r contributes at least three edges to each H_i. Then H_i can contain more than one internal vertex, which contradicts our assumption. Thus $\frac{r}{n} \leq 2$. Then either $\frac{r}{n} = 1$ or $\frac{r}{n} = 2$. If $\frac{r}{n} = 1$ then $r = n$ and if $\frac{r}{n} = 2$, then $n = \frac{r}{n}$. Hence the lemma.

Theorem 3.6. *U admits a cyclic S_m, $m \geq 2$ decomposition if and only if U is any one of the following unicyclic graphs*

1. $U \cong C_{2r}$.

2. $U \cong C_r \odot K_{m-1}^c$.

3. *U is a graph obtained by subdividing the edges of C_r in $U \cong C_r \odot K_{m-1}^c$.*

Proof. Let U admit a cyclic S_m decomposition. It follows from that Lemma 3.5, C_r contributes at most two edges to each H_i and $n = r$ or $n = \frac{r}{2}$. Since $m \geq 2$, we start to concentrate the case of $m = 2$. Suppose that

C_r contributes the $m = 2$ edges to each H_i. Without loss of generality we may assume that v_1, v_3 are the end vertices of H_1 and v_2 is the internal vertex of H_1. Since the decomposition is cyclic, v_3, v_5 are the end vertices of H_2 and v_4 is the internal vertex of H_2. Similarly v_5, v_7 are the end vertices of H_3 and v_6 is the internal vertex of H_3. Proceeding like this we get $v_{2r-1}, v_{2r+1}(= v_1)$ are the end vertices of H_n and v_{2r} is the internal vertex of H_n. Then we can define that $U \cong C_r$. Hence (1).

Suppose C_r contributes exactly one edge to each H_i. Without loss of generality we may assume that v_1 is the internal vertex of H_1 and v_2 is an end vertex of H_1. Since the decomposition is cyclic, v_2 is the internal vertex of H_2 and v_3 is an end vertex of H_2. Similarly v_3 is the internal vertex of H_3 and v_4 is an end vertex of H_3. Proceeding like this we get v_r is the internal vertex of H_n and $v_{r+1}(= v_1)$ is an end vertex of H_n. Then C_r contributes an edge $v_i v_{i+1}$ for all $1 \le i \le r$ to each subgraph where $v_{r+1} = v_1$. Since $H_i \cong S_m$, there are $m - 1$ vertices in $U - V(C_r)$ which uniquely incident with $\{v_i\}$ for all $1 \le i \le n, \in V(C_r)$ to produce the remaining $m - 1$ edges of each subgraph. Then we can define $U \cong C_r \odot K^c_{m-1}$, $r = n$. Hence (2).

Suppose C_r contributes exactly two edges to each H_i. Without loss of generality we may assume that $v_{2r}v_1, v_1v_2 \in V(H_1)$. Since the decomposition is cyclic, $v_2v_3, v_3v_4 \in V(H_2)$. Similarly $v_4v_5, v_5v_6 \in V(H_3)$. Proceeding like this we get $v_{2r-2}v_{2r-1}, v_{2r-1}v_{2r} \in V(H_n)$. Then $v_{j-1}v_j, v_{j+1} \in V(H_i)$ for all i where $d(v_j) \ge 3$, $j = 1, 3, 5, \ldots, 2r - 1$. Since $H_i \cong S_m$, there are $m - 2$ vertices in $U - V(C_r)$ which uniquely incident with $\{v_j\}$ for some $v_j \in V(C_r)$ where $j = 1, 3, 5, \ldots, 2r - 1$ to produce the remaining $m - 2$ edges of each H_i for all $1 \le i \le n$, $r = 2n$. Hence (3).

Conversely, let (1), (2) and (3).

Consider (1). Then $H_i \cong S_2$, is a cyclic S_m decomposition of U.

Consider (2). Let $V(U) = \{v_1, v_2, v_3, \ldots, v_r\} \cup \{x_1, x_2, x_3, \ldots, x_{m-1}, x_{(m-1)+1}, x_{(m-1)+2}, \ldots, x_{2(m-1)}, x_{2(m-1)+1}, \ldots, x_{n(m-1)}\}$ where $\{v_1, v_2, v_3, \ldots, v_r\} \in V(C_r)$ and $\{x_1, x_2, x_3, \ldots, x_{n(m-1)}\} \in V(K_{m-1}^c)$. Then $H_i = \langle v_i, v_{i+1}, x_j, x_{j+2}, x_{j+3}, \ldots, x_{i(m-1)} \rangle$ for all $i = 1, 2, 3, \ldots, r$ and $j = 1, (m-1)+1, 2(m-1)+1, \ldots, (n-1)(m-1)+1$ is a cyclic S_m decomposition of U where for $i = r, v_{i+1} = v_1$.

Consider (3). Let $V(U) = \{v_1, v_2, v_3, \ldots, vr\} \cup \{u_1, u_2, u, \ldots, u_r\} \cup \{x_1, x_2, x_3, \ldots, x_{m-2}, x_{(m-2)+1}, x_{(m-2)+2}, \ldots, x_{2(m-2)+1}, \ldots, x_{n(m-2)}\}$ where $\{v_1, v_2, v_3, \ldots, v_r\} \in V(C_r)$, $\{x_1, x_2, x_3, \ldots, x_{n(m-2)}\} \in V(K_{m-2}^c)$ and $\{u_1, u_2, u, \ldots, u_r\}$ be the vertex set subdividing the edges of C_r. Then $H_i = \langle v_i, u_i, u_{i-1}, x_j, x_{j+2}, x_{j+3}, \ldots, x_{i(m-2)} \rangle$ for all $i = 1, 2, 3, \ldots, r$ and $j = 1, (m-2)+1, 2(m-2)+1, \ldots, (n-1)(m-2)+1$ is a cyclic S_m decomposition of U where for $i = 1, u_{i-1} = u_r$.

Conclusion

In this paper we have dealt with some general properties and construction methods of cyclic decomposition of unicyclic graphs. In future we plan to find the cyclic decomposition of various graphs.

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