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SOME INEQUALITIES INVOLVING THE COMPLEX FOURIER TRANSFORM

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Abstract

Fourier transform developed and pioneered by Joseph Fourier is a very powerful tool and it is extensively used in many areas of Mathematics such as the study of partial and ordinary differential equations, which are critical for modelling engineering problems. Also the Fourier transform is used in the studies of signals and in the area of signal processing. On the theoretical side, Fourier Analysis is a significant and concrete part of Mathematical Analysis. In this article, we will focus on the theoretical part and obtain bounds of the Fourier transform of a real valued function in the conjugate complex variable. We will impose certain assumptions on the real valued function and derive bounds of its complex Fourier transform where the Fourier number is a complex number modified. We also obtain bounds of the Fourier transform of the derivative of the function using modified Fourier number.

1. Introduction

Fourier transform is a powerful and elegant tool in Mathematical Analysis devised by the French Mathematician Joseph Fourier [1] who used his approaches for studying heat conduction. This tool is usually employed for studying problems in the area of Analysis of Partial Differential equations and also Number theory. There is extensive literature in the topic that we can refer to [2], [3], [4], [5], [6], [7], [8], [9]. Also the Fourier transform is an ideal tool for signal processing studying them as superposition of trigonometric functions. There is a whole area called Wavelet theory which studies signal processing extensively. In our article, we concern ourselves with the complex Fourier transform and we obtain certain bounds for the Fourier transformed

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functions where the Fourier variable is complex number and modified. We also obtain bounds for the Fourier transform of the derivative of the function where the Fourier number is a modified complex number. We notice that in our results, the bounds associate the modulus of the Fourier transformed functions in certain Fourier variables with the 2 norm of the function $u(t)$ multiplied by quantities depending on Real and Imaginary part of the Fourier variables. The assumptions on the function $u = u(t)$ are the following:

$$u = u(t) : \Delta_{t_+} \to U(\Delta_{t_+}) \Subset \mathbb{R}$$

where $\Delta_{t_+} =]0, +\infty[$. Also the function u belongs to the class of continuous functions and it has continuous derivative. $C^0(\Delta_{t_+})$ is the space that denotes the continuity of the function $u(t)$ and $C^1(\Delta_{t_+})$ is the space denoting the continuity of $u'(t)$. Another assumption for the function $u(t)$ is that belongs to the space of the square integrable functions with finite $L_2(\Delta_{t_+})$ norm. More precisely, the $L_2(\Delta_{t_+})$ function space is defined as

$$L_2(\Delta_{t_+}) = \left\{ u : \Delta_{t_+} \to U(\Delta_{t_+}) \mid \left(\int_0^{+\infty} | u(t) |^2 dt \right)^{\frac{1}{2}} < +\infty \right\}.$$

The Complex Fourier transform, is an integral transform defined as

$$\hat{u}(\xi) = \int_0^{+\infty} u(t) K(\xi, t)\, dt$$

where $K(\xi, t) = \exp(-i\, \xi t)$ is the Kernel function, with $\xi \in \mathbb{C}$.

2. Remarks on Theorem 1

The first theorem associates the modulus of the Fourier transform of the function $u(t)$ where the Fourier variable is complex conjugate, with the L_2 norm of the function multiplied by a constant depending on the size of the Imaginary part of the Fourier number ξ. These two components are associated with each other by the corresponding inequality.

Theorem 1. *For* $0 < \Im(\xi) < +\infty$, $0 < \Re(\xi) < +\infty$, *the following estimate holds*:

$$\left| \hat{u}(\xi^*) \right| \leq \frac{\sqrt{2}}{2} \|u\|_{L_2(\Delta_{t_+})} \mathcal{O}(\Im(\xi)^{-\frac{1}{2}}),$$

where $\xi^* = \Re(\xi) - i\Im(\xi)$.

Proof.

$$| \hat{u}(\xi^*) | = | \int_0^{+\infty} u(t) K(\xi^*, t)\, dt |$$

$$= | \int_0^{+\infty} u(t) \exp(-i\xi^* t)\, dt |$$

$$= | \int_0^{+\infty} u(t) \exp(-i\Re(\xi) t) \exp(-\Im(\xi) t)\, dt |$$

$$\leq \int_0^{+\infty} | u(t) | \, | \exp(-i\Re(\xi) t) | \, | \exp(-\Im(\xi) t)\, dt |$$

$$\leq \int_0^{+\infty} | u(t) \| \exp(-\Im(\xi) t) | dt$$

$$\leq \left(\int_0^{+\infty} |u(t)|^2 dt| \right)^{\frac{1}{2}} \left(\int_0^{+\infty} | \exp(-\Im(\xi) t) |^2 dt| \right)^{\frac{1}{2}}$$

(Schwarz-Cauchy inequality)

$$\leq \frac{\sqrt{2}}{2} \| u \|_{L_2(\Delta_{t_+})} (\Im(\xi))^{\frac{-1}{2}}$$

$$\leq \frac{\sqrt{2}}{2} \| u \|_{L_2(\Delta_{t_+})} \mathcal{O}\left(\Im(\xi)^{-\frac{1}{2}} \right).$$

3. Remarks on Theorem 2

The theorem associates the modulus of the Fourier transform of $u(t)$ where the new Fourier symbol has real part the imaginary part of ξ and imaginary part the real part of ξ, with the norm of u in L_2 multiplied by a constant depending on the size of the Real part of ξ. These two elements are associated together with the appropriate inequality.

Theorem 2. *For* $0 < \Im(\xi) < +\infty,\ 0 < \Re(\xi) < +\infty,$ *the following estimate holds*:

$$|\hat{u}(w^*)| \leq \frac{\sqrt{2}}{2} \| u \|_{L_2(\Delta_{t_+})} \mathcal{O}(\Re(\xi)^{-\frac{1}{2}}),$$

where $w^* = \Im(\xi) - i\Re(\xi)$.

Proof. $|\hat{u}(w^*)| = \left| \int_0^{+\infty} u(t)\, K(w^*, t)\, dt \right|$

$$= \left| \int_0^{+\infty} u(t) \exp(-iw^* t)\, dt \right|$$

$$= \left| \int_0^{+\infty} u(t) \exp(-i\Im(\xi)t) \exp(-\Re(\xi)t) dt \right|$$

$$= \left| \int_0^{+\infty} |u(t)|\, |\exp(-i\Im(\xi)t)|\, |\exp(-\Re(\xi)t)| dt \right.$$

$$\leq \int_0^{+\infty} |u(t)|\, |\exp(-\Re(\xi)t)| dt$$

$$\leq \left(\int_0^{+\infty} |u(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^{+\infty} |\exp(-\Re(\xi)t|^2 dt \right)^{\frac{1}{2}}$$

(Schwarz-Cauchy inequality)

$$\leq \frac{\sqrt{2}}{2} \| u \|_{L_2(\Delta_{t_+})} (\Re(\xi))^{\frac{-1}{2}}$$

$$\leq \frac{\sqrt{2}}{2} \| u \|_{L_2(\Delta_{t_+})} \mathcal{O}\left(\Re(\xi)^{-\frac{1}{2}} \right).$$

4. Remarks on Theorem 3

In Theorem 3, we obtain an estimate for the modulus of the sum of two Fourier transforms of $u(t)$ in different Fourier numbers, and we bound it by the L_2 norm of $u(t)$ multiplied by a constant depending both on Real and Imaginary part of ξ.

Theorem 3. *For* $0 < \Im(\xi) < +\infty,\ 0 < \Re(\xi) < +\infty,$ *the following estimate holds*:

$$|\hat{u}(\xi^*) + \hat{u}(w^*)| \leq \| u \|_{L_{2(\Delta_{t_+})}} \frac{\sqrt{2}}{2}\left(\frac{\Re(\xi)\sqrt{\Im(\xi)} + \Im(\xi)\sqrt{\Re(\xi)}}{\Im(\xi)\,\Re(\xi)}\right)$$

where $w^* = \Im(\xi) - i\Re(\xi),\ \xi = \Re(\xi) + i\Im(\xi).$

Proof. $|\hat{u}(\xi^*) + \hat{u}(w^*)| = |\int_0^{+\infty} u(t) K(\xi^*, t)\, dt + \int_0^{+\infty} u(t) K(w^*, t)\, dt|$

$$\leq |\int_0^{+\infty} u(t) K(\xi^*, t)\, dt| + |\int_0^{+\infty} u(t) K(w^*, t)\, dt|$$

$$\leq |\int_0^{+\infty} |u(t)|\,|K(\xi^*, t)| dt + \int_0^{+\infty} |u(t)|\,|K(w^*, t)| dt$$

$$\leq \frac{\sqrt{2}}{2} \| u \|_{L_{2(\Delta_{t_+})}} (\Im(\xi))^{\frac{-1}{2}} + \frac{\sqrt{2}}{2} \| u \|_{L_{2(\Delta_{t_+})}} (\Re(\xi))^{\frac{-1}{2}}$$

$$\leq \| u \|_{L_{2(\Delta_{t_+})}} \frac{\sqrt{2}}{2}\left(\frac{\Re(\xi)\sqrt{\Im(\xi)} + \Im(\xi)\sqrt{\Re(\xi)}}{\Im(\xi)\,\Re(\xi)}\right).$$

5. Remarks on Theorem 4

In theorem 4, we obtain an inequality for the modulus of the product of two Fourier transforms of $u(t)$ in different Fourier numbers, and we bound it by the L_2 norm of $u(t)$ multiplied by a constant depending on the product of the Real and Imaginary part of ξ.

Theorem 4. *For* $0 < \Im(\xi) < +\infty,\ 0 < \Re(\xi) < +\infty,$ *the following estimate holds*:

$$|\hat{u}(\xi^*)\ \hat{u}(w^*)| \leq \frac{1}{2} \| u \|^2_{L_{2(\Delta_{t_+})}} \mathcal{O}\left((\Re(\xi)\ \Im(\xi))^{\frac{-1}{2}}\right).$$

where $w^* = \Im(\xi) - i\Re(\xi),\ \xi^* = \Re(\xi) - i\Im(\xi).$

Proof. $|\hat{u}(\xi^*)\hat{u}(w^*)| = |\hat{u}(\xi^*)||\hat{u}(w^*)|$

$$= \left|\int_0^{+\infty} u(t)K(\xi^*, t)\,dt\right| \left|\int_0^{+\infty} u(t)K(w^*, t)\,dt\right|$$

$$\leq \left(\int_0^{+\infty} |u(t)||K(\xi^*, t)|dt\right)\left(\int_0^{+\infty} |u(t)||K(w^*, t)|dt\right)$$

$$\leq \left(\frac{\sqrt{2}}{2}\|u\|_{L2(\Delta_{t_+})}(\Im(\xi))^{\frac{-1}{2}}\right)\left(\frac{\sqrt{2}}{2}\|u\|_{L2(\Delta_{t_+})}(\Re(\xi))^{\frac{-1}{2}}\right)$$

$$\leq \frac{1}{2}\|u\|^2_{L2(\Delta_{t_+})}(\Re(\xi)\,\Im(\xi))^{\frac{-1}{2}}$$

$$\leq \frac{1}{2}\|u\|^2_{L2(\Delta_{t_+})}\,\mathcal{O}\left((\Re(\xi)\,\Im(\xi))^{\frac{-1}{2}}\right).$$

6. Remarks on Theorem 5

In theorem 5, we obtain a bound of the modulus of the Fourier transform of $u(t)$ in the Fourier variable $(\xi^*)^n$, and the bound depends on the L_2 norm of u multiplied by a number, and also depends on modulus of ξ^* to a negative power times a trigonometric quantity to the negative power of 1/2.

Theorem 5. *The following estimate holds*:

$$|\hat{u}((\xi^*)^n) \leq \|u\|_{L_2(\Delta_{t_+})}\frac{\sqrt{2}}{2}|\xi^*|^{-\frac{n}{2}}\,\mathcal{O}\left(\sin\left(n\tan^{-1}\left(\frac{\Re(\xi)}{\Im(\xi)}\right)\right)^{-\frac{1}{2}}\right)$$

under the following restrictions:

$$\Im(\xi) > 0,\ \Re(\xi) > 0,\ \sin\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right) > 0,\ \cos\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right) > 0,\ n \in \mathbb{Z}^+.$$

Proof.

$$|\hat{u}((\xi^*)^n)| = \left|\int_0^{+\infty} u(t)K((\xi^*)^n, t)\,dt\right|$$

$$\leq \int_0^{+\infty} |u(t)||K((\xi^*)^n, t)|\,dt$$

$$\leq \int_0^{+\infty} |\, u(t) \,|\, |\, \exp(-i(\xi^*)^n, t) \,|\, dt$$

$$\leq \int_0^{+\infty} |\, u(t) \,|\, |\, \exp\left(-i \,|\, \xi^* \,|^n \cos\left(n \tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right) t\right) |$$

$$|\, \exp\left(-|\, \xi^* \,|^n \sin\left(n \tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right) t\right) | dt$$

$$\leq \int_0^{+\infty} |\, u(t) \,|\, |\, \exp\left(-\,|\, \xi^* \,|^n \sin\left(n \tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right) t\right) |\, dt$$

$$\leq \| \, u \, \|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2} |\, \xi^* \,|^{-\frac{n}{2}} \sin\left(n \tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right)^{-\frac{1}{2}}$$

$$\leq \| \, u \, \|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2} |\, \xi^* \,|^{-\frac{n}{2}} \, \mathcal{O}\left(\sin\left(n \tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right)^{-\frac{1}{2}}\right).$$

7. Remarks on Theorem 6

A bound is obtained for the modulus of the Fourier transform of $u(t)$ where the Fourier variable is $\xi^* w$, and it is bounded by the 2 norm of u multiplied by a constant depending on the square root of the square difference between the Imaginary and Real part of ξ.

Theorem 6. *For* $0 < \Im(\xi) < +\infty$, $0 < \Re(\xi) < +\infty$, $\left(\frac{\Im(\xi)}{\Re(\xi)}\right)^2 > 1$, *the following bound holds*:

$$|\hat{u}(\xi^* w)|\,| \leq \| \, u \, \|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2} \, \mathcal{O}\left(\left(\Im(\xi)^2 - \Re(\xi)^2\right)^{-\frac{1}{2}}\right).$$

Proof.

$$|\, \hat{u}(\xi^* w) \,| = \left| \int_0^{+\infty} u(t) K(\xi^* w, t) \, dt \,\right|$$

$$\leq \int_0^{+\infty} |\, u(t) \,|\, |\, K(\xi^* w, t) \,| dt$$

$$\leq \left(\int_0^{+\infty} |u(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{+\infty} |\, K(\xi^* w, t)\,|^2 dt\right)^{\frac{1}{2}}$$

$$\leq \|\, u \,\|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Im\,(\xi)^2 - \Re\,(\xi)^2\,)^{-\frac{1}{2}}$$

$$\leq \|\, u \,\|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2}\, \mathcal{O}\left((\Im(\xi)^2 - \Re\,(\xi)^2)^{-\frac{1}{2}}\right).$$

8. Remarks on Theorem 7

A bound is obtained for the modulus of the Fourier transform of $u(t)$ where the Fourier variable is ξw^*, and it is bounded by the 2 norm of u multiplied by a constant depending on the square root of the square difference between the Real and Imaginary part of ξ.

Theorem 7. *For* $0 < \Im\,(\xi) < +\infty$, $0 < \Re\,(\xi) < +\infty$, $\left(\frac{\Re\,(\xi)}{\Im\,(\xi)}\right)^2 > 1$, *the following inequality holds*

$$|\hat{u}(\xi w^*)| \leq \|\, u \,\|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2}\, \mathcal{O}\left((\Re\,(\xi)^2 - \Im\,(\xi)^2)^{-\frac{1}{2}}\right).$$

Proof.

$$|\, \hat{u}\,(\xi w^*)\,| = \left|\int_0^{+\infty} u\,(t)\, K(\xi w^*, t)\, dt\,\right|$$

$$\leq \int_0^{+\infty} |\, u\,(t)\,|\,|\, K(\xi w^*, t)\,|\, dt$$

$$\leq \left(\int_0^{+\infty} |\, u(t)\,|^2 dt\right)^{\frac{1}{2}} \left(\int_0^{+\infty} |\, K(\xi w^*, t)\,|^2 dt\right)^{\frac{1}{2}}$$

$$\leq \|\, u \,\|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Re\,(\xi)^2 - \Im\,(\xi)^2\,)^{-\frac{1}{2}}$$

$$\leq \|\, u \,\|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2}\, \mathcal{O}\left((\Re(\xi)^2 - \Im\,(\xi)^2)^{-\frac{1}{2}}\right).$$

9. Remarks on Theorem 8

In this theorem, we obtain the bound of the modulus the Fourier transform of $u'(t)$ in Fourier number ξ^*, and the modulus is upper bounded by the 2 norm of $u(t)$ multiplied by a quantity depending on Real and Imaginary part of ξ plus the $|u(0)|$ (initial data of the function in absolute value).

Theorem 8. *For* $0 < \Im(\xi) < +\infty$, $0 < \Re(\xi) < +\infty$, *the following inequality holds*:

$$|\hat{u}'(\xi^*)| \leq \| u \|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Im(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + |u(0)|$$

Proof.

$$\begin{aligned} |\hat{u}'(\xi^*)| &= \left|\int_0^{+\infty} u'(t) K(\xi^*, t)\, dt\right| \\ &\leq \left| i\xi^* \int_0^{+\infty} u(t) \exp(-i\xi^* t)\, dt - u(0)\right| \\ &\leq |\xi^*| \int_0^{+\infty} |u(t)|\, |\exp(-i\xi^* t)|\, dt + |u(0)| \\ &\leq \| u \|_{L_2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Im(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + |u(0)|. \end{aligned}$$

10. Remarks on Theorem 9

In this theorem, we obtain the bound of the modulus the Fourier transform of $u'(t)$ in Fourier number w^*, and the modulus is upper bounded by the 2 norm of $u(t)$ multiplied by a quantity depending on Real and Imaginary part of ξ plus the $|u(0)|$ (initial data of the function in absolute value).

Theorem 9. *For* $0 < \Im(\xi) < +\infty$, $0 < \Re(\xi) < +\infty$, *the following inequality holds*:

$$|\,\hat{u}'(\xi^*)\,| \leq \|\,u\,\|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Re(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + |\,u(0)\,|.$$

Proof.

$$|\,\hat{u}'(w^*)\,| = \left|\int_0^{+\infty} u'(t) K(w^*, t)\, dt\,\right|$$

$$\leq |\, iw^* \int_0^{+\infty} u(t) \exp(-i w^* t)\, dt - u(0)\,|$$

$$\leq |\, w^*\,| \int_0^{+\infty} |\,u(t)\,|\,|\exp(-iw^* t)\,|\, dt + |\,u(0)\,|$$

$$\leq \|\,u\,\|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Re(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + |\,u(0)\,|.$$

11. Remarks on Theorem 10

In theorem 10, we bound the modulus of the sum of the Fourier transforms of the $u'(t)$ in Fourier variables ξ^* and w^*. The statement of the theorem follows below.

Theorem 10. *For* $0 < \Im(\xi) < +\infty$, $0 < \Re(\xi) < +\infty$, *the following inequality holds*:

$$|\,\hat{u}'(\xi^*) + \hat{u}'(w^*)\,| \leq \|\,u\,\|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} \left((\Im(\xi))^{\frac{-1}{2}} + (\Re(\xi))^{\frac{-1}{2}} \right) \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + 2\,|\,u(0)\,|$$

Proof.

$$|\,\hat{u}'(\xi^*) + \hat{u}'(w^*)\,| = \left|\int_0^{+\infty} u'(t) K(w^*, t)\, dt + \int_0^{+\infty} u'(t) K(w^*, t)\, dt\,\right|$$

$$\leq \left|\int_0^{+\infty} u'(t) K(\xi^*, t)\, dt\,\right| + \left|\int_0^{+\infty} u'(t) K(w^*, t)\, dt\,\right|$$

$$\leq \|\,u\,\|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Im(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + |\,u(0)\,|$$

$$+\| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Re(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + | u(0) |$$

$$\leq \| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} \left((\Im(\xi))^{\frac{-1}{2}} + (\Re(\xi))^{\frac{-1}{2}} \right) \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + 2| u(0) |.$$

12. Remarks on Theorem 11

In the Theorem 11, we bound the modulus of the product of the Fourier transforms of u' in the Fourier numbers ξ^* and w^*. The theorem follows below.

Theorem 11.

$$| \hat{u}'(\xi^*) \hat{u}'(w^*) | \leq \left(\| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Im(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + | u(0) | \right)$$

$$\times \left(\| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Im(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + | u(0) | \right).$$

Proof.

$$| \hat{u}'(\xi^*) \, \hat{u}'(w^*) | = | \hat{u}'(\xi^*) \, \hat{u}'(w^*) |$$

$$\leq \left| \int_0^{+\infty} u'(t) K(\xi^*, t) dt \right| \left| \int_0^{+\infty} u'(t) K(w^*, t) dt \right|$$

$$\leq \left(\int_0^{+\infty} | u'(t) | | K(\xi^*, t) | dt \right) \left(\int_0^{+\infty} | u'(t) | | K(w^*, t) | \right)$$

$$\leq \left(\| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Im(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + | u(0) | \right)$$

$$\times \left(\| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} (\Re(\xi))^{\frac{-1}{2}} \sqrt{\Re(\xi)^2 + \Im(\xi)^2} + | u(0) | \right).$$

13. Remarks on Theorem 12

We bound the modulus of the Fourier transform of $u'(t)$ in the Fourier number $(\xi^*)^n$. The theorem follows below.

Theorem 12. *The following estimate holds*:

$$|\hat{u}'((\xi^*)^n)| \leq |(\xi^*)^n| \, \|u\|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} |\xi^*|^{-\frac{n}{2}} \frac{\sqrt{\sin\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right)}}{\sin\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right)} + |u(0)|,$$

under the following restrictions:

$$\Im(\xi) > 0,\, \Re(\xi) > 0,\, \sin\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right) > 0,\, \cos\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right) > 0,\, n \in \mathbb{Z}^+.$$

Proof.

$$|\hat{u}'((\xi^*)^n)| = \left|\int_0^{+\infty} u'(t) K((\xi^*)^n, t) dt\right|$$

$$= \left| i(\xi^*)^n \int_0^{+\infty} u(t) \exp(-i(\xi^*)^n t) dt - u(0) \right|$$

$$\leq |(\xi^*)^n| \int_0^{+\infty} |u(t)| \, |\exp(-i(\xi^*)^n t)| \, dt + |u(0)|$$

$$\leq |(\xi^*)^n| \, \|u\|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} |\xi^*|^{-\frac{n}{2}} \frac{\sqrt{\sin\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right)}}{\sin\left(n\tan^{-1}\left(\frac{\Im(\xi)}{\Re(\xi)}\right)\right)} + |u(0)|.$$

14. Remarks on Theorem 13

We bound the modulus of the Fourier transform $u'(t)$ in the Fourier number $\xi^* w$.

Theorem 13. *The following estimate holds*

$$|\hat{u}'(\xi^* w)| \leq \|u\|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} \frac{\Im(\xi)^2 + \Re(\xi)^2}{\Im(\xi)^2 - \Re(\xi)^2} \sqrt{\Im(\xi)^2 - \Re(\xi)^2} + |u(0)|,$$

provided that $\left(\frac{\Im(\xi)}{\Re(\xi)}\right)^2 > 1.$

Proof.

$$| \hat{u}' (\xi^* w) | = | \int_0^{+\infty} u' (t) K (\xi^* w, t) dt |$$

$$= | i\xi^* w \int_0^{+\infty} u(t) \exp (- i(\xi^* wt) dt - u(0) |$$

$$\leq | \xi^* | | w | \int_0^{+\infty} | u(t) | | \exp (- i(\xi^* wt) | dt + | u(0) |$$

$$\leq \| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} \frac{\Im (\xi)^2 + \Re (\xi)^2}{\Im (\xi)^2 - \Re (\xi)^2} \sqrt{\Im(\xi)^2 - \Re (\xi)^2} + | u(0) |.$$

15. Remarks on Theorem 14

We bound the modulus of the Fourier transform $u'(t)$ in the Fourier number ξw^*.

Theorem 14. *The following bound holds*

$$| \hat{u}'(\xi w^*) | \leq \| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} \frac{\Im (\xi)^2 + \Re (\xi)^2}{-\Im (\xi)^2 + \Re (\xi)^2} \sqrt{-\Im(\xi)^2 + \Re (\xi)^2} + | u(0) |$$

under the assumption that $\left(\frac{\Re (\xi)}{\Im (\xi)}\right)^2 > 1.$

Proof.

$$| \hat{u}'(\xi w^*) | = | \int_0^{+\infty} u'(t) K(\xi w^*, t)^n, t) dt |$$

$$= |i\xi w^* \int_0^{+\infty} u(t) \exp (- i \xi w^* t) dt - u(0)|$$

$$\leq | \xi | |w^*| \int_0^{+\infty} | u(t) | | \exp (- i \xi w^* t) | dt + | u(0) |$$

$$\leq \| u \|_{L2(\Delta_{t_+})} \frac{\sqrt{2}}{2} \frac{\Im (\xi)^2 + \Re (\xi)^2}{-\Im (\xi)^2 + \Re (\xi)^2} \sqrt{-\Im(\xi)^2 + \Re (\xi)^2} + | u(0) |.$$

16. Remarks on Theorem 15

We bound the modulus of the Fourier transform of the function $g(t) = t^{\frac{m}{2}} u(t)$ Fourier variable ξ^{*}.

Theorem 15.

$$| \hat{g}(\xi^{*}) | \leq \| u \|_{L2(\Delta_{t_+})} \sqrt{\Gamma(m+1)} 2^{-\frac{(m+1)}{2}} (\Im(\xi))^{-\frac{(m+1)}{2}},$$

where $m \in \mathbb{Z}^{+}$.

Proof.

$$| \hat{g}\, (\xi^{*}) | = | \int_0^{+\infty} t^{\frac{m}{2}}\, u(t) K\, (\xi^{*}, t) dt |$$

$$\leq \int_0^{+\infty} | t^{\frac{m}{2}}\, u(t) | \, | K\, (\xi^{*}, t) | dt$$

$$\leq \left(\int_0^{+\infty} | u(t) |^2 dt \right)^{\frac{1}{2}} \left(\int_0^{+\infty} t^m \exp(-2\, \Im(\xi) t) dt \right)^{\frac{1}{2}}$$

$$\leq \| u \|_{L2(\Delta_{t_+})} \sqrt{\Gamma(m+1)} 2^{-\frac{(m+1)}{2}} (\Im(\xi))^{-\frac{(m+1)}{2}}.$$

17. Conclusions and Future Work

In this article we have derived bounds for the Fourier transform of u and u' in various Fourier variables. We notice that the bounds involve the 2 norm of the function u multiplied by a quantity which depends on Real and Imaginary parts of each Fourier number. In the last theorem we obtained a bound of the Fourier transform of a monomial times the function u. Similar bounds can be obtained when using modified Fourier transform or exponential transform where the Kernel function is $K(\xi, t) = a^{-i\xi t}$, $a \in (0, +\infty)$ [10], [11], [12]. This analysis will appear in future work in form of an article.

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