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I-STATISTICAL CONVERGENCE IN PARANORMED SPACE

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Abstract

In this paper, we deal with the ideal statistical convergent and ideal statistical Cauchy sequence in paranormed spaces. Let $I \subset 2^N$ be a non-trivial ideal in *N*. A sequence $x = (x_k)$ is said to be *I*-statistically convergent to ξ in (X, g) if for every $\varepsilon > 0$ and $\delta > 0$, $\left\{n \in N : \frac{1}{n} |\{k \leq n : g(x_k - \xi) \leq \varepsilon\}| \geq \delta\right\} \in I$. ξ is called (g, I)- Statistical limit of the sequence (x_k) and we write $(g, I) - st \lim x_k = \xi$. We discuss some properties of these concepts and some inclusion relations between the spaces.

1. Introduction

The notion of statistical convergence was introduced by Fast [1] and Steinhaus independently in the same year 1951 and since then several generalizations and applications of this notion have been investigated by various authors. In [2], Kostyrko et al. introduced the concept of *I*-Convergence of sequences in a metric space and studied some properties of

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such convergence. Note that I-Convergence is an interesting generalization of statistical convergence. The notion of ideal statistical convergence and ideal statistical Cauchy sequence are introduced in a_2-normed space [3]. Motivated by this fact, in this paper, the notion of ideal statistical convergence and ideal statistical Cauchy sequence in paranormed space and some important results are established.

2. Definitions and Preliminaries

Definition 2.1. A paranorm is a function $g : X \to \mathbb{R}$ defined on a linear space X such that for all $x, y, z \in X$.

(i) $g(x) = 0$ if $x = \theta$.

(ii) $g(-x) = g(x)$

(iii) $g(x + y) \leq g(x) + g(y)$

(iv) If (α_n) is a sequence of scalars with $\alpha_n \to \alpha_0 (n \to \infty)$ and $x_n, a \in X$ with $x_n \to a(n \to \infty)$ in the sense that $g(x_n - a) \to 0(n \to \infty)$, then $\alpha_n x_n \to \alpha_0 a(n \to \infty)$ in the sense that $g(\alpha_n x_n - \alpha_0 a) \to 0(n \to \infty)$. A paranorm g for which $g(x) = 0$ implies $x = \theta$ is called a total paranorm on X, and the pair (X, g) is called a total paranormed space.

Definition 2.2. A family $I \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Y if (i) $\varphi \in I$, (ii) $A, B \in I$ imply $A \bigcup B \in I$, (iii) $A \in I, B \subset A$ imply $B \in I$.

Definition 2.3. Let X be a non empty set. A family of subsets $I \subset 2^{\mathbb{N}}$ is called a filter on X if and only if (i) $\varphi \notin I$, (ii) $A, B \in I$ imply $A, B \in I$, (iii) $A \in I, A \subset B$ imply $B \in I$.

Definition 2.4. An ideal I is called non trivial if $I \neq \varphi$ and $X \notin I$.

The filter $I = \{X - A : A \in I\}$ is called the filter associated with the ideal I.

A nontrivial ideal $I \subset 2^{\mathbb{N}}$ is called an admissible ideal in X if and only if

$I \supset \{\{x\} : x \in X\}$.

Let I_f be the family of all finite subsets of N. Then I_f is an admissible ideal in N and I-convergence coincides with the usual convergence.

Definition 2.5. A sequence (x_k) is called Statistically convergent to L in a paranormed space (X, g) if for each $\varepsilon > 0$, $\lim \frac{1}{n} | \{k \leq n : g(x_k - L) > \varepsilon\} | = 0$. It is written by $g(st) - \lim x_k = L$.

3. Main Result

Definition 3.1. Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence (x_k) of (X, g) is said to be I-Convergent to x_0 if for each $e > 0$, the set $A(e) = \{k \in N : g(x_n - x_0) \geq \varepsilon\} \in I$. x_0 is called the (g, I)-limit of the sequence (x_k) and we write $(g, I) - \lim x_n = x_0$.

Definition 3.2. Let $I \subset 2^{\mathbb{N}}$ be a non trivial ideal in N. A sequence (x_k) is said to be I-Statistically Convergent to ξ in (X, g), if for every $\varepsilon > 0$ and every $\delta > 0$, $\{n \in N : 1/n| \{k \leq n : g(x_k - \xi) \geq \delta\} \in I$. ξ is called (g, I)-Statistical limit of the sequence (x_k) and we write $(g, I) - st \lim x_k = \xi$.

Theorem 3.3. *If (x_k) be a sequence such that $(g, I) - st \lim x_k = \xi$ then ξ is determined uniquely.*

Proof. If possible let the sequence (x_k) be (g, I)-statistically convergent to two different numbers ξ_1 and ξ_2. That is for any $\varepsilon > 0, \delta > 0$ we have

$$A_1 = \left\{n \in N : \frac{1}{n} | \{k \leq n : g(x_k - \xi_1) \geq \varepsilon\} | < \delta\right\} \in F(I)$$

and

$$A_2 = \{n \in N : 1/n| \{k \leq n : g(x_k - \xi_2) \geq \varepsilon\} | < \delta\} \in F(I).$$

Therefore $A_1 \cap A_2 \neq \theta$, since $A_1 \cap A_2 \in F(I)$. Let $m \in A_1 \cap A_2$ and take

$\varepsilon = \frac{g(\xi_1 - \xi_2)}{3} > 0$, so $\frac{1}{m} | \{k \le m : g(x_k - \xi_1) \ge \varepsilon\} | < \delta$ and $\frac{1}{m} | \{k \le m :$ $g(x_k - \xi_2) \ge \varepsilon\} | < \delta$, for maximum $k \le m$ will satisfy $g(x_k - \xi_1) < \varepsilon$ and $g(x_k - \xi_2) < \varepsilon$ for a very small $\delta > 0$. Thus we must have $\{k \le m : g(x_k - \xi_2) \ge \varepsilon\} \cap \{k \le m : g(x_k - \xi_2) < \varepsilon\} \ne \phi$, which is contradiction as the neighbourhood of ξ_1 and ξ_2 are disjoint. Hence theorem is proved.

Theorem 3.4. *Let I be an admissible ideal. Then for any sequence* (x_k), $(g, I) - \lim x_k = \xi$ *implies* $(g, I) - st \lim x_k = \xi$.

Proof. Let $(g, I) - st \lim x_k = \xi$. Then for each $\varepsilon > 0$ and $\delta > 0$ $A(e) = \{k \in \mathbb{N} : g(x_k - \xi) \ge \varepsilon\} \in I$, so for every $\varepsilon > 0$ and $\delta > 0 \left\{n \in N : \frac{1}{n} | \{k \le n : g(x_k - \xi) \ge \varepsilon\} \ge \delta\right\}$ is a finite set and therefore belongs to *I*, as *I* is an admissible ideal. Hence $(g, I) - st \lim X_k = \Xi$. But converse is not true.

Example 3.5. Take $I = I_F$. The sequence (x_k) where $x_n = \begin{cases} 0 & \mathrm{n} = \mathrm{k}^2, \mathrm{k} \in \mathrm{N} \\ 1 & \text{otherwise} \end{cases}$ is (g, I)- Statistically convergent to 1. But (x_k) is not (g, I)- Convergent.

Theorem 3.6. *Let I be an admissible ideal. Then for any sequence* (x_k), $st - \lim x_k = \xi$ *in paranormed space implies* $(g, I) - st \lim x_k = \xi$.

Proof. Similar to proof of theorem 3.4. But converse is not true.

Example 3.7. Let $(g, I) = \zeta$ be the class of $A \subset \mathbb{N}$ that intersect a finite number of $\Delta_j's$ where $\mathbb{N} = \bigcup_{j=1}^{\infty} \Delta_j$ and $\Delta_i \cap \Delta_j = \phi$ for $i \ne j$. Let $x_n = \frac{1}{n}$ and so $\lim g(x_n - 0) = 0$. Put $\varepsilon_n = g(x_k - 0)$ for $n \in \mathbb{N}$. Now define a sequence (y_n) by $y_n = x_j$ if $n \in \Delta_j$. Let $\eta > 0$. Choose $\gamma \in \mathbb{N}$ such that $\varepsilon_\gamma < \eta$. Then $A(\eta) = \{n \in N : g(y_n - 0) \ge \eta\} \subset \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_\gamma \in \zeta$. Now, $\{k \le n : g(y_k - 0) \ge \eta\} \subseteq \{n \in N : g(y_n - 0) \ge \eta\}$. That is $\frac{1}{n} | \{k \le n :$

$g(y_k - 0) \geq \eta\} | \leq \{n \in N : g(y_n - 0) \geq \eta\}$. So for any $\delta > 0$ $\left\{n \in N : \frac{1}{n} | \{k \leq n :\right.$ $g(y_k - 0) \geq \eta\} | \geq \delta\} \subseteq \{n \in N : g(y_n - 0) \geq \eta\} \in \zeta$. Therefore (y_n) is ζ-statistically convergent to 0. But (y_n) is not statistically convergent.

Theorem 3.9. *Let I be an admissible ideal. Then for each subsequence of (x_n) is (g, I)-statistically convergent to ξ then (x_n) is also (g, I)-statistically convergent to ξ.*

Proof. Suppose (x_n) is not (g, I)-statistically convergent to ξ, then there exists $\varepsilon > 0$ and $\delta > 0$ $A = \left\{n \in N : \frac{1}{n} | \{k \leq n : g(x_k - \xi) \geq \varepsilon\} \geq \delta\right\}$ $\notin I$. Since I is an admissible ideal so A must be an infinite set. Let $A = \{n_1 < n_2 < \ldots < n_m\}$. Let $(y_m) = x_{nm}$ for $m \in \mathbb{N}$. Then $(y_m)_{m \in \mathbb{N}}$ is a subsequence of (x_n) which is not (g, I)-statistically convergent to ξ. Which is a contradiction.

Hence the theorem is proved. But converse is not true. We can easily show this from example 3.5.

Definition 3.10. A sequence $x = (x_n)_{n \in \mathbb{N}}$ of elements of X is said to be $(g, I)^*$-Statistical convergent to $\xi \in X$ if and only if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in F(I)$ such that $st - \lim g(x_m k - \xi) = 0$.

Theorem 3.11. *If* $(g, I)^* - st \lim x_n = \xi$ *then* $(g, I) - st \lim x_n = \xi$.

Proof. Let $(g, I)^* - st \lim x_n = \xi$. By assumption there exists a set $B \in I$ such that for $M = N/B = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ we have $g(st) - \lim x_{mk} = \xi$. That is $1/n| \{m_k \leq n : g(x_{mk} - \xi) \geq \varepsilon\} | = 0$. So for any $\delta > 0$, $\{n \in \mathbb{N} : 1/n| \{m_k \leq n : g(x_{mk} - \xi) \geq \varepsilon\} | \geq \delta\} \in I$, since I is an admissible ideal. Now $A(\varepsilon, \delta) = \{n \in \mathbb{N} : 1/n| \{k \leq n : g(x_k - \xi) \geq \varepsilon | \geq \delta\}$ $\subset B \cup \{n \in \mathbb{N} : 1/n| \{k \leq n : g(x_{mk} - \xi) \geq \varepsilon\} | \geq \delta\} \in I$. That is $(g, I) - st \lim x_n = \xi$.

But the converse may not be true.

Example 3.12. From example 3.7, we have $\xi - st \lim y_n = 0$. Suppose that $\zeta^* - st \lim y_n = 0$. Then there exists a set $B \in \zeta$ such that for $M = N/B = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ we have $g(st) - \lim y_{mk} = 0$. By definition of ζ there exists a $q \in N$ such that $B \subset \bigcup_{i=n}^{q} \Delta_i$. But $\Delta_{q+1} \subset M$. So for infinitely many $m_k \in \Delta_{q+1} \cdot |\{m_k \in \Delta_{q+1} : g(y_{mk} - 0) \geq \eta\}| = 2^{-(q+1)} > 0$ for $0 < \eta < 1/(q+1)$. That is $|\{m_k \in \Delta_{q+1} : g(y_{mk} - 0) \geq \eta\}| \neq 0$ which contradicts to $g(st) - \lim y_{mk} = 0$. Hence $\zeta^* - st \lim y_n \neq 0$.

Definition 3.13. A sequence (x_k) is said to be I-statistically Cauchy sequence in (X, g) if for every $\varepsilon > 0, \delta > 0$, there exists a number $N = N(e)$ such that $\left\{n \in \mathbb{N} : \frac{1}{n} |\{k : g(x_k - x_N) \geq \varepsilon\}| \geq \delta\right\} \in I$. It can be written as (g, I)- statistically Cauchy.

Theorem 3.14. *A sequence (x_k) in a paranormed space (X, g) is (g, I)- statistically convergent if and only if it is (g, I)- statistically Cauchy.*

Proof. Assume that $st - \lim x_k = \xi$. Then for each $\varepsilon > 0$, the set $A(\varepsilon) = \{k \leq n : g(x_k - \xi) \geq \varepsilon/2\}$ has density zero. That is $\delta(A(\varepsilon)) = 0$. This implies that $\delta(N/A(\varepsilon)) = \delta(\{k \leq n : g(x_k - \xi) \geq \varepsilon/2\}) \neq 0$. Let $m, n \notin A(\varepsilon)$, then $g(x_m - x_n) < \varepsilon$. Let $B(\varepsilon) = \{k \leq n : g(x_m - x_n) < \varepsilon\}$, for a fixed $m \notin A(\varepsilon)$. Then $N/A(\varepsilon) \subset B(\varepsilon)$. Hence $0 \neq \delta(N/A(\varepsilon)) \leq \delta(B(\varepsilon)) \neq 0$. This imply $\delta(N/B(\varepsilon)) = 0$. Then for any $\delta > 0$ we have $N/B(\varepsilon) = \{k \leq n : g(x_m - x_n) \geq \varepsilon\} \geq \delta\}$. This implies that (x_n) is (g, I)- statistically Cauchy sequence.

Conversely let as assume that (x_k) is (g, I)- statistically Cauchy sequence, but not (g, I)- statistically convergent in (X, g). Then we have $m \in N$ such that $\delta(A(\varepsilon)) = 0$ where $A(\varepsilon) = \{n \in N : g(x_n - x_m) \geq \varepsilon\}$ and $\delta(B(\varepsilon)) = 0$ where $B(\varepsilon) = \{n \in N : g(x_n - \xi) \geq \varepsilon/2\}$, that is $\delta(B^c(\varepsilon)) \neq 0$. $g(x_n - \xi) < \varepsilon/2$ then $g(x_n - x_m) \leq g(x_n - \xi) < \varepsilon$. Therefore $\delta(A^c(\varepsilon)) = 0$. This implies that $\delta(A(\varepsilon)) \neq 0$, which is a contradiction, since (x_k) was

(g, I)-Statistically Cauchy. Hence (x_k) must be (g, I)-Statistically convergent.

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