Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 393-398

CO-SECURE SET DOMINATION IN LINE GRAPHS

D. BHUVANESWARI and S. MEENAKSHI

Department of Mathematics
Vels Institute of Science, Technology
and Advanced Studies
Chennai-600117, Tamil Nadu, India
E-mail: bhuvanamaths15@gmail.com
meenakshikarthikeyan@yahoo.co.in

Abstract

Throughout this paper, $G = (V, E)$ is a connected graph and we determine the co-secure set domination number of a Line graph of some standard graphs and also for its complement graphs.

1. Introduction

Let $G = (V, E)$ be a finite, connected graph without loops and multiple edges. The order and size of the graph G is $|V(G)|$ and $|E(G)|$ respectively. For a vertex $v \in G,$ the number of edges incident to a vertex v is the degree of a vertex v and is denoted by $d(v)$ or $\deg(v)$. The path of n vertices is denoted by P_n and cycle of n vertices is denoted as C_n. The complete graph of n vertices is a graph having every vertices of degree m-1 and is denoted by K_n. The wheel of n vertices is a graph obtained by joining all the vertices of cycle C_{n-1} to the vertex at the center of the cycle and is denoted by W_n, $n \geq 4$. The graph $K_{1,n}$ is star graph. The Line graph $L(G)$ of a graph $G = (V, E)$ is also a graph with vertex set as $E(G)$ and the two vertices e_i, e_j are adjacent in $L(G)$ if and only if the edges e_i and e_j are adjacent in a graph G.

2020 Mathematics Subject Classification: 05C69; 05C76, 05C38.
Keywords: Co-secure dominating set; Co-secure set; Dominating set; Line graph; Set dominating set.
Received May 17, 2021; Accepted June 7, 2021

The set $D \subseteq V(G)$ is a set dominating set of G if every set $T \subseteq V - D$ there exist a non-empty set S of D such that the induced sub graph $\langle T \cup S \rangle$ is connected. The minimum cardinality of a set dominating set of G is said to be set domination number and is denoted by $\gamma_s(G)$ and is abbreviated as a SD-set (set dominating set) [7]. A set $D \subseteq V(G)$ is a co-secure dominating set of G if D is a dominating set and forever vertex $u' \in D$ there exists a vertex $v' \in V - D$ such that $u'v' \in E(G)$ and $(D - \{u'\}) \cup \{v'\}$ is a dominating set and is abbreviated as CSD-set of G. The minimum cardinality of co-secure dominating set is the co-secure domination number and is denoted by $\gamma_{cs}(G)$ [1]. A co-secure dominating set of G is said to be a co-secure set dominating set of G if for every set $T \subseteq V - D$ there exist a non-empty set S of D such that the induced sub graph $\langle T \cup S \rangle$ is connected and is abbreviated as a CSSD-set of G. The minimum cardinality of co-secure set dominating set of G is the co-secure set domination number of G and is denoted by $\gamma_{cs}^{s}(G)$.

The co-secure domination set was introduced and determined the cosecure domination number for some standard graphs by Arumugam S., Karam Ebadi and Martin Manrique and they also find the sharp bounds for that parameter [1]. Aleena Joseph and Sangeetha investigated the co-secure domination number of Friendship graph, Jahangir graph and Helm graph and also obtained the bounds [2]. The co-secure set dominating set was introduced and we investigated the co-secure domination number of some standard graphs and some sharp bounds for it [3]. A graph G with isolated vertices does not have a CSD-set and CSSD-set for G. The undefined terms used in this paper are in [5].

2. Main Results

In this paper, we are investigating the co-secure set domination number for the Line graph of path, cycle, complete graph, star graph and wheel graph and also determine the co-secure set domination number for complement graph of $L(P_n)$, $L(C_n)$.

Observation 2.1. For $L(P_n)$, $\gamma_{cs}^{s}(L(P_n)) = \begin{cases} n-2, & n = 3 \text{ and } 4, \\ n-3, & 5 \leq n \leq 7, \\ \text{dose not exist} & n \geq 8. \end{cases}$

Example 2.2. For $n = 8$,

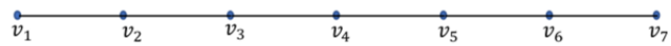

Figure 1. The line graph of P_8.

Consider $D = \{v_1, v_3, v_5, v_7\}$ is also a dominating set of $L(P_8)$ and $V - D = \{v_2, v_4, v_6\}$. This D is not a set dominating set but it is a co-secure dominating set of $L(P_8)$. So, for any D, the $L(P_8)$ does not have a co-secure set domination number. For the same reason, the co-secure set domination number does not exist for $n \geq 8$.

Observation 2.3. For $L(C_n)$,

$$\gamma_{cs}^{s}(L(C_n)) = \begin{cases} n-2, & \text{for } n = 3 \text{ and } 4, \\ n-3, & \text{for } 5 \leq n \leq 9, \\ \text{dose not exist}, & n \geq 10 \end{cases}.$$

Example 2.4. For a Line graph of cycle graph, $L(C_{10})$, Consider the set $D = \{v_1, v_2, v_4, v_5, v_6, v_8, v_9\}$ is also a dominating set of $L(C_{10})$, and $V - D = \{v_3, v_7, v_{10}\}$. For this, D is a set dominating set but it is not a co-secure dominating set. So, D is not a co-secure set dominating set. For any D, the $\gamma_{cs}^{s}(L(C_{10}))$ does not exist.

Theorem 2.5. *If K_n is a complete graph with n vertices, then*

$$\gamma_{cs}^{s}(L(K_n)) = \begin{cases} n-2, & n = 3 \text{ and } 4, \\ n-3, & n \geq 5. \end{cases}$$

Proof of Theorem 2.5. Let us consider $e_1, e_2, \ldots, e_{\frac{n(n-1)}{2}}$ be a vertex of $L(K_n)$ with order ${}^{n}C_2$ and it's a $2n$-4 -regular graph for $n \geq 3$.

For $n = 3$, $D = \{e_1\}$ or $\{e_2\}$ or $\{e_3\}$, are the CSSD-set of $L(K_3)$. Hence,

$\gamma_{cs}^{s}(L(K_3)) = 1 = n - 2.$

For $n = 4$, $D = \{e_5, e_6\}$ is the only CSSD-set of $L(K_4)$. Hence,

$\gamma_{cs}^{s}(L(K_4)) = 2 = n - 2.$

For $n \geq 5$, consider the dominating set D with cardinality $n - 3$. Every vertex in D and $V - D$ is of degree $2n - 4$. So, every e_i in D can be replaced by e_j in $V - D$ with the condition that $e_i e_j \in L(K_n)$ and the new set $(D - \{e_i\}) \cup \{e_j\}$ will also a dominating set. Therefore, D is a CSD-set of $L(K_n)$. Since for every subset M in D we can form a set N in $V - D$ so that the sub graph formed from $\langle M \cup N \rangle$ will be connected. Thus, D is also a SD-set of $L(K_n)$. Hence D is a CSSD-set of $L(K_n)$.

To check the minimality of D, assume that D is the CSSD-set with cardinality n-4. So that the vertex in D is not adjacent to all the vertex in $L(K_n)$. and that D is not a dominating set of $L(K_n)$. Therefore, D is a smallest CSSD-set of $L(K_n)$. with cardinality $n - 3$.

Hence, $\gamma_{cs}^{s}(L(K_n)) = \begin{cases} n - 2, & n = 3 \text{ and } 4, \\ n - 3, & n \geq 5, \end{cases}.$

Theorem 2.6. *For a graph* $G = K_{1, n-1}$, *then* $\gamma_{cs}^{s}(L(K_{1, n-1})) = 1.$

Proof of Theorem 2.6. Since $L(K_{1, n-1})) \cong K_n$ and hence by the theorem E in [3], we have the result $\gamma_{cs}^{s}(L(K_{1, n-1})) = 1.$

Theorem 2.7. *If* W_n *is a wheel graph with n vertices, then*

$$\gamma_{cs}^{s}(L(W_n)) = \begin{cases} 2, & n = 4 \\ n - 3, & n \geq 5 \end{cases}.$$

Proof of Theorem 2.7. Let us consider $L(W_n) : e_1, e_2, e_3, \ldots, e_{2n-2}$ with order $2n$-2 and $d(e_i) = \begin{cases} 4, & 1 \leq i \leq n - 1 \\ n, & n \leq i \leq 2n - 2 \end{cases}.$

For $n = 4$, $D = \{e_1, e_2\}$ or $\{e_5, e_6\}$ or $\{e_2, e_4\}$ or $\{e_3, e_4\}$ is the CSSD-set of $L(W_n)$. Hence $\gamma_{cs}^s(L(W_n)) = 2$.

For $n \geq 5$, consider the dominating set D with cardinality $n - 3$ (out of that at most one vertex is of degree n and the remaining vertex is of degree 4). So that every e_i in D can be replaced by e_j in $V - D$ with the condition that $e_ie_j \in L(W_n)$ and the new set $(D - \{e_i\}) \cup \{e_j\}$ will also a dominating set.

Therefore, D is a CSD-set of $L(W_n)$. Since for every subset M in D we can form a set N in $V - D$ so that the sub graph formed from $\langle M \cup N \rangle$ will be connected. Thus, D is also a SD-set of $L(W_n)$. Hence D is a CSSD-set of $L(W_n)$. To verify the minimality of D, assume that D is the CSSD-set with n-4 vertices of $L(W_n)$. So that the vertex in D is not adjacent to all the vertex in $L(W_n)$ and that D is not a dominating set of $L(W_n)$. Therefore, D is a smallest CSSD-set of $L(W_n)$ with cardinality $n - 3$. Hence,

$$\gamma_{cs}^s(L(W_n)) = \begin{cases} 2, & n = 4 \\ n - 3, & n \geq 5 \end{cases}.$$

Next, we are going to determine the γ_{cs}^s of complements of $L(P_n)$ and $L(C_n)$ because for large values of n this number does not exist.

Theorem 2.8. *For a graph* $G = \overline{L(P_n)}$. *then* $\gamma_{cs}^s(G) = 2$ *for* $n \geq 5$.

Proof of Theorem 2.8. Let us consider $\overline{L(P_n)} : e_1, e_2, e_3, \ldots, e_{n-1}$ with order $n - 1$ and $d(e_1) = d(e_{n-1}) = n - 3$ and the degree of the remaining vertices, $d(e_i) = n - 4$ for $2 \leq i \leq n - 2$.

Consider the dominating set $D = \{e_1, e_2\}$ (or $D = \{e_1, e_{n-1}\}$) of $\overline{L(P_n)}$, and $d(e_1) = n - 3$ and $d(e_2) = n - 4$. So that every e_1 or e_2 in D can be replaced by e_i, for $i \geq 3$ in $V - D$ with the condition that $e_1e_i \in \overline{L(P_n)}$. and the new set $(D - \{e_1\}) \cup \{e_i\}$ will also a dominating set of $\overline{L(P_n)}$. Therefore, D is a CSD-set of $\overline{L(P_n)}$. Since for every subset M in D we can form a set N in $V - D$ so that the sub graph constructed from $M \cup N$ will be connected.

Therefore, D is also a SD-set of $\overline{L(P_n)}$. Hence D is a CSSD-set of $\overline{L(P_n)}$.

To verify the minimality of D, assume that D is the CSSD-set with only one vertex of $\overline{L(P_n)}$. So that D is not a dominating set of $\overline{L(P_n)}$. Thus, D is a smallest CSSD-set of $\overline{L(P_n)}$ with cardinality 2. Hence, $\gamma_{cs}^{s}(G) = 2$ for $n \geq 5$.

Theorem 2.9. *For a graph* $G = \overline{L(C_n)}$, *then* $\gamma_{cs}^{s}(G) = 2$ *for* $n \geq 5$.

Proof of Theorem 2.9. Let us consider $\overline{L(C_n)} : e_1, e_2, e_3, \ldots, e_n$ with order and $d(e_i) = n - 3$ for $1 \leq i \leq n$. The proof is similar to the proof theorem 2.8.

3. Conclusion

For a line graph of a path P_n, $n \geq 8$ and a line graph of cycle C_n, $n \geq 10$, the co-secure set domination number does not exist. But the complement graphs of that graph have the co-secure set domination number equal to 2.

References

[1] S. Arumugan, Karam Ebadi and Martin Manrique, Co-secure and secure domination in graphs, Util. Math. 94 (2014), 167-182.

[2] Aleena Joseph and V. Sangeetha, Bounds on cosecure domination in graphs, International Journal of Mathematics Trends and Technology (IJMTT) 5 (2018), 158-164.

[3] D. Bhuvaneswari and S. Meenakshi, Co-secure set domination in graphs, International Journal of Recent Technology and Engineering (IJRTE) 8(4) S5 (2019), 66-68.

[4] E. Sampathkumar and Puspha Latha, Set domination in graphs, J. Graph Theory 18(5) (1994), 489-495.

[5] W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, Inc., New York, 1998.