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MADDENING SOLUTIONS FOR AN EXPONENTIAL DIOPHANTINE EQUATION CONCERNING DEFINITE PRIME NUMBERS $(2P+1)^u + (P-1)^v - 2 = w^2$

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Abstract

In this calligraphy, an exponential Diophantine equation $(2P+1)^u + (P-1)^v - 2 = w^2$ where P is a prime number of the form $6n+1, n \in N,$ the set of all-natural numbers and n is not a multiple of prime numbers 2, 3, 5, 7 etc. and u, v, w are non-negative integers is studied for all the choices of $u+v = 1, 2, 3.$ Also, it is evinced that the crucial solutions are $(u, v, w) = (1, 2, P)$ and there is no solution for all other choices of u, v by retaining elementary concepts of congruencies.

Introduction

A Diophantine equation is an equation generally connecting two or more variables such that its solutions are integers. A Diophantine equation has further a variable or variables as exponents, it is recognized as an exponential Diophantine equation [1-4]. In [5-7], the authors clarify the solution of different types of exponential Diophantine equations involving prime numbers. In provision of comprehensive study one can relegate [8-11]. In this paper, an exponential Diophantine equation $(2P+1)^u + (P-1)^v - 2$

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$= w^2$ where P is a prime number of the category $6n + 1,\ n \in N,$ the set of all-natural numbers and n is not a multiple of prime numbers 2, 3, 5, 7 etc. and u, v, w are non-negative integers is considered for all the preferences of $u + v = 1, 2, 3.$ Also, it is revealed that all possible solutions are $(u, v, w) = (1, 2, P)$ and there is no other solution by employing basic perceptions of congruences.

Lemma 1. *If* $P \geq 7$ *is a prime number of the form* $6n + 1$ *where* $n \in N$ *and n is not a manifold of prime numbers* 2, 3, 5, 7 *etc, then the non-linear Diophantine equation* $(2P + 1)^u - 1 = w^2$ *has no solution where u and w are non-negative integers.*

Proof. It is clear that any square number is congruence to either 0 $(\text{mod } 3)$ or $1(\text{mod } 3).$

On the other hand, for any prime number $P \geq 7$ is of the form $P = 6n + 1$ where $n \in N$ and n is not a multiple of prime numbers 2, 3, 5, 7 etc, $P \equiv 1 \,(\text{mod } 3)$

Consequently

$$2P + 1 \equiv 0 \,(\text{mod } 3)$$

$$\Rightarrow (2P + 1)^u \equiv 0 \,(\text{mod } 3)$$

$$\Rightarrow (2P + 1)^u - 1 \equiv 2 \,(\text{mod } 3).$$

This contradiction shows that, there is no solution to the desired equation.

Lemma 2. *If* $P > 3$ *is a prime number of the form* $6n + 1$ *where* $n \in N$ *and* n *is not a multiple of prime numbers* 2, 3, 5, 7 *etc, then the equation* $(P - 1)^v - 1 = w^2$ *has no integer solution where v and w are non-negative integers.*

Proof. As the explanation given in lemma 1, $P \equiv 1 \,(\text{mod } 3)$

Thus,

$$P - 1 \equiv 0 \pmod 3$$

$$\Rightarrow (P-1)^u \equiv 0 \pmod 3$$

$$\Rightarrow (P-1)^u - 1 \equiv 2 \pmod 3.$$

Note that, the left-hand side expression of the original equation is congruence to $2 \pmod 3$ where as the right side is congruence to either $0 \pmod 3$ or $1 \pmod 3$.

This inconsistency indicates that there exists no integer solution to an equation.

Theorem. *Let* $P = 6n + 1$ *be a prime number where* $n \in N$, *the set of all-natural numbers and* n *is not a multiple of prime numbers* 2, 3, 5, 7 *etc. If* $u + v = 1, 2, 3,$ *then an exponential Diophantine equation* $(2P + 1)^u + (P-1)^v - 2 = w^2$ *has solutions* $(u, v, w) = (1, 2, P)$ *where* u, v, w *are whole numbers.*

Proof. The rule that, any square number satisfies the succeeding conditions.

$$w^2 \equiv \begin{cases} 0 \pmod 3 \text{ if } n \text{ is odd} \\ 1 \pmod 3 \text{ if } n \text{ is even} \end{cases} \text{ and } w^2 \equiv \begin{cases} 1 \pmod 4 \text{ if } n \text{ is odd} \\ 0 \pmod 4 \text{ if } n \text{ is even} \end{cases} \tag{1}$$

Since $P = 6n + 1$ where $n \in N$ and n is not a multiple of prime numbers 2, 3, 5, 7 etc, $P \equiv 1 \pmod 3$

Also, $P \equiv \begin{cases} 3 \pmod 4 \text{ if } n \text{ is odd} \\ 1 \pmod 4 \text{ if } n \text{ is even} \end{cases}$

Let us analyse the theorem for the following three cases

Case 1. $u + v = 1$

Sub case 1(i). $u = 0, v = 1$

Then, the inventive solutions to an equation can be reduced into $P - 2 = w^2$

Since, $P \equiv 1 \pmod 3$, $P - 2 \equiv 2 \pmod 3$ (2)

Sub case 1 (ii). $u = 1, v = 0$

These options modify an equation to $2P = w^2$

Here, $2P \equiv 2 \pmod 3$ (3)

Case 2. $u + v = 2$

Sub case 2(i). $u = 0, v = 2$

The corresponding equation becomes for these values of u and v as

$$(P-1)^2 - 1 = w^2$$

It is identified that $(P-1)^2 \equiv 0 \pmod 3$

$$\Rightarrow (P-1)^2 - 1 \equiv 2 \pmod 3 \tag{4}$$

Sub case 2(ii). $u = 1, v = 1$

The equation for examining solution is customized into $3P - 2 = w^2$

It is profoundly scrutinized that for all prime number P sustaining our hypothesis

$$3P \equiv \begin{cases} 1 \pmod 4 \text{ if } n \text{ is odd} \\ 3 \pmod 4 \text{ if } n \text{ is even} \end{cases}$$

$$\Rightarrow 3P - 2 \equiv \begin{cases} 3 \pmod 4 \text{ if } n \text{ is odd} \\ 1 \pmod 4 \text{ if } n \text{ is even} \end{cases} \tag{5}$$

Sub case 2(iii). $u = 2, v = 0$

The deliberated equation for these choices can be condensed to $(2P+1)^2 - 1 = w^2$

Since $2P + 1 \equiv 0 \pmod 3$, $(2P+1)^2 - 1 \equiv 2 \pmod 3$ (6)

Case 3. $u + v = 3$

Sub case 3(i). $u = 0, v = 3$

The preferred equation is revised into $(P-1)^3 - 1 = w^2$

But, $(P-1)^3 \equiv 0 \pmod 3$ provides the following condition

$$(P-1)^3 - 1 \equiv 2 \pmod 3 \tag{7}$$

Sub case 3(ii). $u = 2, v = 1$

The equivalent form of the chosen equation can be simplified by $(2P+1)^2 + P - 3 = w^2$

The two already analysed conditions

$$(2P+1)^2 \equiv \begin{cases} 1 \pmod 4 \text{ if } n \text{ is odd} \\ 1 \pmod 4 \text{ if } n \text{ is even} \end{cases} \text{ and } P \equiv \begin{cases} 3 \pmod 4 \text{ if } n \text{ is odd} \\ 1 \pmod 4 \text{ if } n \text{ is even} \end{cases}$$

together gives the succeeding declaration

$$(2P+1)^2 + P - 3 \equiv \begin{cases} 1 \pmod 4 \text{ if } n \text{ is odd} \\ 3 \pmod 4 \text{ if } n \text{ is even} \end{cases} \tag{8}$$

Sub case 3(iii). $u = 3, v = 0$

These constraints condense the equation that the solution to be discovered as

$$(2P+1)^3 - 1 = w^2$$

It is deeply observed that $2P + 1 \equiv 0 \pmod 3$ and subsequently

$$(2P+1)^3 - 1 \equiv 2 \pmod 3 \tag{9}$$

It is noticed that all the conclusions exposed from (2) to (9) for various selections of u and v are the left-hand side expressions for the inspected equation which contradicts the conditions of the right-hand side of the same equation specified in (1).

Hence, there is no solution in integer for all above choices of u an v.

Sub case 3(iv). $u = 1, v = 2$

Then, the original equation is converted into $P^2 = w^2 \Rightarrow P = w$

Hence, the required solutions to the equation are $(u, v, w) = (1, 2, P)$.

Corollary 1. *The Diophantine equation* $(2P+1)^u + (P-1)^v + 2 = w^2$ *has no integer solution where the exponents* u, v *are non-negative integers.*

Corollary 2. *The Diophantine equation* $(2P-1)^u + (P-1)^v + 2 = w^2$ *has no solution in integer where* u, v *are whole numbers*.

Corollary 3. *The exponential Diophantine equation* $(2P+1)^u + (P+1)^v + 2 = w^2$ *has solutions* $(u, v, w) = (1, 2, P+2)$.

Corollary 4. *The solutions of the exponential Diophantine equation* $(P-1)^u + (2P+1)^v - 2 = w^2$ *are* $(u, v, w) = (2, 1, P)$.

Conclusion

In this inscription, especially integer solutions to a particular exponential equation $(2P+1)^u + (P-1)^v - 2 = w^2$ where P is a prime number of the form $6n+1, n \in N$ and n is not a manifold of prime numbers 2, 3, 5, 7 etc. and u, v, w are whole numbers are calculated for all the picks of $u + v = 1, 2, 3$. In this way, one may also explore solutions of assumed equation for $u = v > 3$.

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