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PYTHAGOREAN FUZZY GRADATION OF OPENNESS

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Abstract

In this paper the concept of Pythagorean fuzzy gradation of openness and Pythagorean fuzzy gradation of closedness are introduced and analysed.

1. Introduction

In Yager [5] introduced the notion of Pythagorean fuzzy set which has many effective applications in natural and social sciences. A Pythagorean fuzzy set in a non-empty set X is a pair $A = (\mu_A, \upsilon_A)$ of a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\upsilon_A : X \to [0, 1]$ with the condition that $0 \leq {\mu_A}^2(x) + {\upsilon_A}^2(x) \leq 1,$ for every $x \in X.$ In Olgun [4] introduced the notion of Pythagorean fuzzy topological space in the sense of Chang [1]. In Hazra, et al. [3] gave a new definition of fuzzy topology by introducing the concept of gradation of openness of fuzzy subsets. In Chattopadhyay, et al. [2] modified the definition of gradation function and studied subspace of fuzzy topological spaces and gradation preserving maps. In this paper, the concept of Pythagorean fuzzy gradation of openness is introduced and studied.

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2. Preliminary Definitions

Definition 2.1 [4]. A Pythagorean fuzzy subset A of a non-empty set X is a pair (μ_A, υ_A) of a membership function $\mu_A : X \to [0, 1]$ and a non-membership function $\upsilon_A : X \to [0, 1]$ with the condition that $0 \leq \mu_A{}^2(x) + \upsilon_A{}^2(x) \leq 1$, for every $x \in X$.

Definition 2.2 [4]. Let X be a non-empty set. Let $A = (\mu_A, \upsilon_A)$ and $B = (\mu_B, \upsilon_B)$ be two Pythagorean fuzzy subsets of X. Then

(i) A is a *subset* of B or B contains A denoted by $A \subset B$ or $B \supset A$ if $\mu_A \leq \mu_B$ and $\upsilon_A \geq \upsilon_B$.

(ii) The *complement* of A denoted by A^c is a Pythagorean fuzzy set in X defined by $A^c = (\upsilon_A, \mu_A)$.

(iii) The *intersection* of A and B is defined by $A \cap B = (\min\{\mu_A, \mu_B\}, \max\{\upsilon_A, \upsilon_B\})$.

(iv) The *union* of A and B is defined by $A \cup B = (\max\{\mu_A, \mu_B\}, \min\{\upsilon_A, \upsilon_B\})$.

(v) The *intersection* of $(A_i)_{i \in I}$, a collection of Pythagorean fuzzy subsets of X, denoted by $\bigcap_{i \in I} A_i$ is a Pythagorean fuzzy set in X defined by $\bigcap_{i \in I} A_i = (\wedge_{i \in I} \mu_{A_i}, \vee_{i \in I} \upsilon_{A_i})$ for each $x \in X$ and the union of $(A_i)_{i \in I}$, a collection of Pythagorean fuzzy subsets of X, denoted by $\bigcup_{i \in I} A_i$ is a Pythagorean fuzzy set in X defined by $\bigcup_{i \in I} A_i = (\vee_{i \in I} \mu_{A_i}, \wedge_{i \in I} \upsilon_{A_i})$.

Definition 2.3 [4]. The *Pythagorean fuzzy whole set* $\dot{1}$ is defined by $\dot{1} = (\mu_{\dot{1}}, \upsilon_{\dot{1}})$ where $\mu_i(x) = 1$ and $\upsilon_{\dot{1}}(x) = 0$, for each $x \in X$.

Definition 2.4 [4]. The Pythagorean fuzzy null set $\dot{0}$ is defined by $\dot{0} = (\mu_{\dot{0}}, \upsilon_{\dot{0}})$ where $\mu_{\dot{0}}(x) = 0$ and $\upsilon_{\dot{0}}, (x) = 1$ for each $x \in X$.

Definition 2.5 [4]. Let X be a non-empty set and let τ be a family of Pythagorean fuzzy subsets of X. If

(i) $\dot{1}, \dot{0} \in \tau$

(ii) For any $A_1, A_2 \in \tau$, $A_1 \cap A_2 \in \tau$

(iii) For a collection $\{A_i\}_{i \in I} \subset \tau$, $\bigcup_{i \in I} A_i \in \tau$

then τ is called a *Pythagorean fuzzy topology* on X and (X, τ) is called a *Pythagorean fuzzy topological space*.

Definition 2.6. Let (X, τ) be a Pythagorean fuzzy topological space. Let $Y \subseteq X$. Let $A = (\mu_A, v_A) \in \tau$.

Define $A/Y = (\mu_{A/Y}, v_{A/Y})$ as $(\mu_{A/Y})(z) = \mu_A(z)$ and $(v_{A/Y})(z) = v_A(z)$, for all $z \in Y$.

Define $(\tau/Y) = \{A/Y/A \in \tau\}$ Then (τ/Y) is called the Pythagorean fuzzy subspace topology on Y and $(Y, \tau/Y)$ is called a Pythagorean fuzzy subspace of (X, τ) (or) simply Y is called a Pythagorean fuzzy subspace of X.

Definition 2.7 [4]**.** Let X and Y be no-empty sets, Let $\theta : X \to Y$ be a function and let A and B be two Pythagorean fuzzy sets of X and Y respectively. Then,

(i) The image of A under θ, denoted by $\theta(A) = (\mu_{\theta(A)}, v_{\theta(A)})$ is a Pythagorean fuzzy set in Y defined as follows: for each $y \in Y$

$$\mu_{\theta(\hat{A})}(y) = \begin{cases} \vee_{x \in \theta^{-1}(y)} \mu_{\hat{A}}(x), & \text{if } \theta^{-1}(y) \text{ is non-empty} \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$$v_{\theta(\hat{A})}(y) = \begin{cases} \wedge_{x \in \theta^{-1}(y)} v_{\hat{A}}(x), & \text{if } \theta^{-1}(y) \text{ is non-empty} \\ 0, & \text{otherwise} \end{cases}$$

(ii) The pre-image of B under θ, denoted by $\theta^{-1}(B) = (\mu_{\theta^{-1}(B)}, v_{\theta^{-1}(B)})$ is a Pythagorean fuzzy set in X defined as follows: for each $x \in X$ $\mu_{\theta^{-1}(B)}(x) = \mu_B(\theta(x))$ and $v_{\theta^{-1}(B)} = v_B(\theta(x))$.

Definition 2.8 [4]**.** Let (X, τ_1) and (y, τ_2) be two Pythagorean fuzzy topological spaces. A mapping $\theta : X \to Y$ is called Pythagorean fuzzy continuous if for all $A \in \tau_2$, $\theta^{-1}(A) \in \tau_1$.

3. Pythagorean Fuzzy Gradation of Openness

Definition 3.1. Let X be a non-empty set. A mapping $\mathcal{G} : PYF(X) \to I$ is said to be a Pythagorean fuzzy gradation of openness on X iff the following conditions are satisfied:

(PYFGO1) $\mathcal{G}(\dot{0}) = \varsigma(\dot{1}) = 1$

(PYFGO2) $\mathcal{G}(A_i) > 0$, for $i = 1, 2, \ldots, m$

$\mathcal{G}(\bigcap_{i=1}^{m} A_i) > 0$

(PYFGO3) $\mathcal{G}(A_\lambda) > 0$, $\lambda \in \Lambda$

$\mathcal{G}(\bigcup_{\lambda \in \Lambda} A_\lambda) > 0$

The pair (X, ς) is called a Pythagorean fuzzy gradation space.

Definition 3.2. Let $(X, \mathcal{G})$ be a Pythagorean fuzzy gradation space. Then the Pythagorean fuzzy topology on X induced by $\mathcal{G}$ is given by $\tau(\mathcal{G}) = \{A \in PYF(X)/\mathcal{G}(A) > 0\}$.

Definition 3.3. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Pythagorean fuzzy gradations of openness on X. Then $\mathcal{G}_1 \geq \mathcal{G}_2$ if $\mathcal{G}_1(A) \geq \mathcal{G}_2(A)$ for all $A \in PYF(X)$.

Definition 3.4. Let $(X, \mathcal{G}_1)$, $(Y, \mathcal{G}_2)$ be two Pythagorean fuzzy gradation spaces. Then a map $\theta : X \to Y$ is called

(1) A Pythagorean fuzzy gradation preserving map, if $\mathcal{G}_2(A) \leq \mathcal{G}_1(\theta^{-1}(A))$, for each $A \in PYF(Y)$.

(2) A Pythagorean fuzzy strongly gradation preserving map, if $\mathcal{G}_2(A) \leq \mathcal{G}_1(\theta^{-1}(A))$, for each $A \in PYF(Y)$.

(3) A Pythagorean fuzzy weakly gradation preserving map, if

$\mathcal{G}_2(A) > 0 \Rightarrow \mathcal{G}_1(\theta^{-1}(A)) > 0,$ for each $A \in PYF(Y).$

Definition 3.5. Let X be a non-empty set. A mapping $\mathfrak{F} : PYF(X) \to I$ is said to be a Pythagorean fuzzy gradation of closedness on X iff the following conditions are satisfied:

(PYFGC1) $\mathfrak{F}(\dot{0}) = \mathfrak{F}(\dot{1}) = 1$

(PYFGC2) $\mathfrak{F}(A_i) > 0,$ for $i = 1, 2, \ldots, m$

$$\Rightarrow \mathfrak{F}(\cup_{i=1}^{m} A_i) > 0$$

(PYFGC3) $\mathfrak{F}(A_\lambda) > 0,$ for $\lambda \in \Lambda$

$$\Rightarrow \mathfrak{F}(\cap_{\lambda \in \Lambda} A_\lambda) > 0.$$

Definition 3.6. Let $(X, \mathcal{G})$ be a Pythagorean fuzzy gradation space and $A \in PYF(X)$. Then $\mathcal{G}$-closure of A, denoted by $\mathcal{G}cl(A)$ and is defined as $\mathcal{G}cl(A) = \cap\{B \in PYF(X); \mathfrak{F}_{\mathcal{G}}(B) > 0, B \supseteq A\}.$

Note: From the definition it follows that

(i) $\mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A)) > 0$

(ii) For every $A, B \in PYF(X), A \supseteq B$ implies that $\mathcal{G}cl(A) \supseteq \mathcal{G}cl(B).$

Definition 3.7. Let X be a non-empty set. A mapping $\mathcal{G}_* : 2^x \to I$ is said to be crisp gradation of openness on X iff the following conditions are satisfied:

(i) $\mathcal{G}_*(\phi) = \mathcal{G}_*(X) = 1$

(ii) $\mathcal{G}_*(\dot{A}_i) > 0, i = 1, 2, \ldots, m$

$$\Rightarrow \mathcal{G}_*(\cap_{i=1}^{m} \dot{A}_i) > 0$$

(iii) $\mathcal{G}_*(\dot{A}_\lambda) > 0,$ for $\lambda \in \Lambda$

$$\Rightarrow \mathcal{G}_*(\cup_{\lambda \in \Lambda} \dot{A}_\lambda) > 0$$

Therefore $(X, \mathcal{G}_*)$ is crisp gradation space. Then the topology on X induced by $\mathcal{G}_*$ is $\tau(\mathcal{G}_*) = \{\dot{A} \in 2^x / \mathcal{G}_*(\dot{A}) > 0\}$.

Definition 3.8. Given $\dot{A} = 2^x$ define $A = (\mu_{A,}\, v_A)$ as $\mu_A : X \to [0, 1]$ such that $\mu_A(x) > 0$, if $x \in \dot{A}$ and $\mu_A(x) = 0$, if $x \notin \dot{A}$ and $v_A : X \to [0, 1]$ such that $v_A(x) > 0$, if $x \in \dot{A}$ and $v_A(x) = 0$, if $x \notin \dot{A}$. Therefore $A \in PYF(X)$.

Definition 3.9. Let $\mathcal{G} : PYF(X) \to I$ be a Pythagorean fuzzy gradation of openness on X. Define $\mathcal{G}_* : 2^x \to I$ such that $\mathcal{G}_*(\dot{A}) = \mathcal{G}(A)$. Then $\mathcal{G}_*$ is a crisp gradation of openness on X.

Theorem 3.10. *Let $\mathcal{G}$ be a Pythagorean fuzzy gradation of openness on X and $\mathfrak{F}_{\mathcal{G}} : PYF(X) \to I$ be a mapping by $\mathfrak{F}_{\mathcal{G}}(A) = \mathcal{G}(A^c)$. Then $\mathfrak{F}_{\mathcal{G}}$ is a Pythagorean fuzzy gradation of closedness on X.*

Proof. Let $\mathcal{G}$ be a Pythagorean fuzzy gradation of openness on X.

To prove: $\mathfrak{F}_{\mathcal{G}}$ is a Pythagorean fuzzy gradation of closedness on X.

(i) $\mathfrak{F}_{\mathcal{G}}(\dot{0}) = \mathcal{G}(\dot{0}^c) = \mathcal{G}(\dot{1}) = 1$

$\mathfrak{F}_{\mathcal{G}}(\dot{1}) = \mathcal{G}(\dot{1}^c) = \mathcal{G}(\dot{0}) = 1$

Therefore $\mathfrak{F}_{\mathcal{G}}(\dot{0}) = \mathfrak{F}(\dot{1}) = 1$

(ii) $\mathfrak{F}_{\mathcal{G}}(\bigcup_{i=1}^m A_i) = \mathcal{G}(\bigcup_{i=1}^m A_i)^c$

$$= \mathcal{G}(\bigcap_{i=1}^m (A_i)^c) > 0$$

$\mathfrak{F}_{\mathcal{G}}(\bigcup_{i=1}^m A_i) > 0$

(iii) $\mathfrak{F}_{\mathcal{G}}(\bigcap_{\lambda \in \Lambda} A_\lambda) = \mathcal{G}(\bigcap_{\lambda \in \Lambda} A_\lambda)^c$

$$= \mathcal{G}(\bigcup_{\lambda \in \Lambda} (A_\lambda)^c) > 0$$

$\mathfrak{F}(\bigcap_{\lambda \in \Lambda} A_{\lambda}) > 0$

Therefore $\mathfrak{F}_{\mathcal{G}}$ is a Pythagorean fuzzy gradation of closedness on *X*.

Theorem 3.11. *Let $\mathfrak{F}$ be a Pythagorean fuzzy gradation of closedness on X and $\mathcal{G}_{\mathfrak{F}} : PYF(X) \to I$ be a mapping defined by $\mathcal{G}_{\mathfrak{F}}(A) = \mathfrak{F}(A^c)$. Then $\mathcal{G}_{\mathfrak{F}}$ is a Pythagorean fuzzy gradation of openness on X.*

Proof. Let $\mathfrak{F}$ be a Pythagorean fuzzy gradation of closedness on *X*.

To prove $\mathcal{G}_{\mathfrak{F}}$ is a Pythagorean fuzzy gradation of openness on *X*.

(i) $\mathcal{G}_{\mathfrak{F}}(\dot{0}) = \mathfrak{F}(\dot{0}^c) = \mathfrak{F}(i) = 1$

$\mathcal{G}_{\mathfrak{F}}(\dot{1}) = \mathfrak{F}(i^c) = \mathfrak{F}(\dot{0}) = 1$

Therefore $\mathcal{G}_{\mathfrak{F}}(\dot{0}) = \mathcal{G}_{\mathfrak{F}}(\dot{1}) = 1$

(ii) $\mathcal{G}_{\mathfrak{F}}(\bigcap_{i=1}^{m} A_i) = \mathfrak{F}(\bigcap_{i=1}^{m} A_i)^c$

$= \mathfrak{F}(\bigcup_{i=1}^{m} (A_i)^c) > 0$

Therefore $\mathcal{G}_{\mathfrak{F}}(\bigcap_{i=1}^{m} A_i) > 0$

(iii) $\mathcal{G}_{\mathfrak{F}}(\bigcup_{\lambda \in \Lambda} A_{\lambda}) = \mathfrak{F}(\bigcup_{\lambda \in \Lambda} A_{\lambda})^c$

$= \mathfrak{F}(\bigcap_{\lambda \in \Lambda} (A_{\lambda})^c) > 0$

Therefore $\mathcal{G}_{\mathfrak{F}}(\bigcup_{\lambda \in \Lambda} A_{\lambda}) > 0$

Therefore $\mathcal{G}_{\mathfrak{F}}$ is a Pythagorean fuzzy gradation of openness on *X*.

Theorem 3.12. *Let $\mathcal{G}$, $\mathfrak{F}$ be a Pythagorean fuzzy gradations of closedness and openness respectively on X. Then $\mathcal{G}_{\mathfrak{F}_{\mathcal{G}}} = \mathcal{G}$, $\mathfrak{F}_{\mathcal{G}_{\mathfrak{F}}} = \mathfrak{F}$.*

The proof is follows from Theorem 3.1, Theorem 3.2.

Theorem 3.13. *Let $(X, \mathcal{G})$ be a Pythagorean fuzzy topological space. Then*

(i) $\mathcal{G}cl(\dot{0}) = \dot{0}$

(ii) $\mathcal{G}cl(A) \supseteq A$

(iii) $\mathcal{G}cl(A_1 \cup A_2) = \mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2)$

(iv) $\mathcal{G}cl(\mathcal{G}cl(A)) = \mathcal{G}cl(A)$

Proof. (i) and (ii) are obvious.

(iii) Let $A_1, A_2 \in PYF(X)$

From (ii), it is clear that $\mathcal{G}cl(A_1) \supseteq A_1$ and $\mathcal{G}cl(A_2) \supseteq A_2$

$\mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2) \supseteq A_1 \cup A_2$

By the definition of Pythagorean fuzzy gradation of closedness,

$\mathfrak{F}_{\mathcal{G}}(A_i) > 0, i = 1, 2$

$\Rightarrow \mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2)) > 0$

Since $\mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2)) > 0$ and $\mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2) \supseteq A_1 \cup A_2$, then

$$\mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2) \supseteq \mathcal{G}cl(A_1 \cup A_2), \ldots, (1)$$

Now to prove $\mathcal{G}cl(A_1 \cup A_2) \supseteq \mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2)$

We know that, $A_1 \subseteq A_1 \cup A_2$ and $A_2 \subseteq A_1 \cup A_2$

$\Rightarrow \mathcal{G}cl(A_1) \subseteq \mathcal{G}cl(A_1 \cup A_2)$ and $\mathcal{G}cl(A_2) \subseteq \mathcal{G}cl(A_1, A_2)$

$\Rightarrow \mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2) \subseteq \mathcal{G}cl(A_1 \cup A_2)$

That is $\mathcal{G}cl(A_1 \cup A_2) \supseteq \mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2), \ldots, (2)$

From (1) and (2),

$\mathcal{G}cl(A_1 \cup A_2) = \mathcal{G}cl(A_1) \cup \mathcal{G}cl(A_2).$

Proof of (iv) is obvious.

Theorem 3.14. *Let* $(X, \mathcal{G})$ *be a Pythagorean fuzzy topological space. Then for each* $A \in PYF(X)$, $\mathfrak{F}_{\mathcal{G}}(A) > 0$ *iff* $A = \mathcal{G}cl(A)$.

Proof. Let $(X, \mathcal{G})$ be a Pythagorean fuzzy gradation space.

Assume $\mathfrak{F}_{\mathcal{G}}(A) > 0,$ for each $A \in PYF(X)$.

To prove: $A = \mathcal{G}cl(A)$

$\mathcal{G}cl(A) = \bigcap \{B \in PYF(X) / \mathfrak{F}_{\mathcal{G}}(B) > 0, B \supseteq A\}$

$= A$ (since $\mathfrak{F}_{\mathcal{G}}(A) > 0,$ then A is a member of the above collections and every member contains A)

Therefore $\mathcal{G}cl(A) = A$.

Conversely, assume $\mathcal{G}cl(A) = A, \ldots,$ (1)

To prove: $\mathfrak{F}_{\mathcal{G}}(A) > 0$

From the definition of $\mathcal{G}$-closure of A, $\mathcal{G}cl(A) = \bigcap \{B \in PYF(X);$ $\mathfrak{F}_{\mathcal{G}}(B) > 0, B \supseteq A\}$.

It is clear that $\mathfrak{F}_{\mathcal{G}}(\mathcal{G}cl(A)) > 0$

$\Rightarrow \mathfrak{F}_{\mathcal{G}}(A) > 0$ (since by (1))

Hence proved.

Theorem 3.15. *Let $\{\mathcal{G}_k; k = 1, 2, \ldots, n\}$ be a finite family of Pythagorean fuzzy gradation of openness on X. Then $\mathcal{G} = \bigcap_{k=1}^{n} \mathcal{G}_k$ is a Pythagorean fuzzy gradation of openness on X.*

Proof. Let $\{\mathcal{G}_k; k = 1, 2, \ldots, n\}$ be a finite family of Pythagorean fuzzy gradation of openness on X and $\mathcal{G} = \bigcap_{k=1}^{n} \mathcal{G}_k$

To prove: $\mathcal{G}$ is a Pythagorean fuzzy gradation of openness on X.

(i) $\mathcal{G}(\dot{0}) = \bigcap_{k=1}^{n} \mathcal{G}_k(\dot{0})$

$= \mathcal{G}_1(\dot{0}) \wedge \mathcal{G}_2(\dot{0}) \wedge \ldots \wedge \mathcal{G}_n(\dot{0})$

$= 1$

(since for each $\{\mathcal{G}_k; k = 1, 2, \ldots, n\}$ is (PYFGO1))

$\mathcal{G}(\mathrm{i}) = \bigcap_{k=1}^{n} \mathcal{G}_k(\mathrm{i})$

$= \mathcal{G}(\mathrm{i}) \wedge \mathcal{G}_2(\mathrm{i}) \wedge \ldots \wedge \mathcal{G}_n(\mathrm{i})$

$= 1$

(since for each $\mathcal{G}_k; k = 1, 2, \ldots, n$ is (PYFGO1))

(ii) $\mathcal{G}(A_i) = \bigcap_{k=1}^{n} \mathcal{G}_k(A_i)$, for $i = 1$ to m

$= \mathcal{G}_1(A_i) \wedge \mathcal{G}_2(A_i) \wedge \ldots \wedge \mathcal{G}_n(A_i)$, for $i = 1, 2, \ldots, m$

$\mathcal{G}(A_i) > 0$, for $i = 1, 2, \ldots, m$ (since for each $\mathcal{G}_k(A_i) > 0$, for $i = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, n$)

$\Rightarrow \mathcal{G}(\bigcap_{i=1}^{m} A_i) = \bigcap_{k=1}^{n} \mathcal{G}_k(\bigcap_{i=1}^{m} A_i)$

$= \mathcal{G}_1(\bigcap_{i=1}^{m} A_i) \wedge \mathcal{G}_2(\bigcap_{i=1}^{m} A_i) \wedge \ldots \wedge \mathcal{G}_n(\bigcap_{i=1}^{m} A_i)$

$\Rightarrow \mathcal{G}(\bigcap_{i=1}^{m} A_i) > 0$ (since for each $\mathcal{G}_k(\bigcap_{i=1}^{m} A_i) > 0$, $k = 1, 2, \ldots, n$)

(iii) $\mathcal{G}(A_\lambda) = \bigcap_{k=1}^{n} \mathcal{G}_k(A_\lambda)$, for $\lambda \in \Lambda$

$= \mathcal{G}_1(A_\lambda) \wedge \mathcal{G}_2(A_\lambda) \wedge \ldots \wedge \mathcal{G}_n(A_\lambda)$, for $\lambda \in \Lambda$

$\mathcal{G}(A_\lambda) > 0$, for $\lambda \in \Lambda$ (since for each $\mathcal{G}_k(A_\lambda) > 0$, for $\lambda \in \Lambda$, $k = 1, 2, \ldots, n$)

$\Rightarrow \mathcal{G}(\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcap_{k=1}^{n} \mathcal{G}_k(\bigcup_{\lambda \in \Lambda} A_\lambda)$

$= \mathcal{G}_1(\bigcup_{\lambda \in \Lambda} A_\lambda) \wedge \mathcal{G}_2(\bigcup_{\lambda \in \Lambda} A_\lambda) \wedge \ldots \wedge \mathcal{G}_n(\bigcup_{\lambda = \Lambda} A_\lambda)$

$\mathcal{G}_1(\bigcup_{\lambda \in \Lambda} A_\lambda) > 0$ (since for each $\mathcal{G}_k(\bigcup_{\lambda \in \Lambda} A_\lambda) > 0$, $k = 1, 2, \ldots, n$)

Therefore $\mathcal{G} \bigcap_{k=1}^{n} \mathcal{G}_k$ is a Pythagorean fuzzy gradation of openness on X.

Theorem 3.16. *Let* $(X, \mathcal{G}_1)$, $(Y, \mathcal{G}_2)$ *be two Pythagorean fuzzy topological* *spaces and* $\theta : X \to Y$ *be a function. Then the following are equivalent:*

(i) θ *is a Pythagorean fuzzy weakly gradation preserving map*

(ii) $\theta(\mathcal{G}cl(A)) \leq \mathcal{G}cl(\theta(A))$, *for* $A \in PYF(X)$.

Proof. Suppose (i) holds then for each $A \in PYF(X)$.

$$\theta^{-1}(\mathcal{G}cl(\theta(A))) = \theta^{-1}[\bigcap \{B \in PYF(Y)/\mathfrak{F}_{\mathcal{G}_2}(B) > 0,\ B \supseteq \theta(A)\}]$$

$$= \theta^{-1}[\bigcap \{B \in PYF(Y)/\mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B)) > 0,\ B \supseteq \theta(A)\}]$$

$$\supseteq \bigcap \{\theta^{-1}(B) \in PYF(X)/\mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B)) > 0,\ \theta^{-1}(B)) \supseteq (\theta^{-1}(A))\}$$

$$\supseteq \bigcap\{\theta^{-1}(B) \in PYF(X)/\mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B)) > 0,\ \theta^{-1}(B) \supseteq A\}$$

$$\theta^{-1}(\mathcal{G}cl(\theta(A))) \supseteq \mathcal{G}cl(A)$$

$(\mathcal{G}cl(\theta(A))) \supseteq \theta(\mathcal{G}cl(A))$, for every $A \in PYF(X)$

Therefore (i) $\Rightarrow$ (ii)

To prove: (ii) $\Rightarrow$ (i)

Suppose (ii) holds, that is $\theta(\mathcal{G}cl(A)) \subseteq \mathcal{G}cl(\theta(A))$ for all $A \in PYF(X)$

From the theorem, we've for each $B \in PYF(Y), \mathcal{G}_2(B) > 0$ $\Rightarrow \mathfrak{F}_{\mathcal{G}_2}(B^c) > 0$ iff $\mathcal{G}cl(B^c) = B^c$.

Since, $\theta(\mathcal{G}cl(\theta^{-1}(B))) \subseteq \mathcal{G}cl(\theta(\theta^{-1}(B^c))) \subseteq \mathcal{G}cl(B^c) = B^c$

$$\theta(\mathcal{G}cl(\theta^{-1}(B^c))) \subseteq B^c$$

$$\mathcal{G}cl(\theta^{-1}(B^c)) \subseteq \theta^{-1}(B^c)$$

Hence,

$$\mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B^c)) > 0$$

$$\Rightarrow \mathfrak{F}_{\mathcal{G}_1}(\theta^{-1}(B))^c > 0$$

$\Rightarrow \mathcal{G}_1((\theta^{-1}(B))^c)^c > 0$

$\Rightarrow \mathcal{G}_1(\theta^{-1}(B)) > 0$, for every $B \in PYF(Y)$.

Therefore θ is a Pythagorean fuzzy weakly gradation preserving map.

Theorem 3.17. *$\theta : (X, \mathcal{G}_1) \to (y, \mathcal{G}_2)$ is a Pythagorean fuzzy weakly gradation preserving map iff $\theta : (X, \tau(\mathcal{G}_1)) \to (Y, \tau(\mathcal{G}_2))$ is a Pythagorean fuzzy continuous.*

$A \in \tau(\mathcal{G}_2)$

$\Rightarrow \mathcal{G}_2(A) > 0$

$\Rightarrow \mathcal{G}_1(\theta^{-1}(A)) > 0$

$\Rightarrow \theta^{-1}(A) \in \tau(\mathcal{G}_1)$

Therefore $\theta : (X, \tau(\mathcal{G}_1)) \to (Y, \tau(\mathcal{G}_2))$ is a Pythagorean fuzzy continuous.

Conversely, assume $\theta : (X, \tau(\mathcal{G}_1)) \to (Y, \tau(\mathcal{G}_2))$ is a Pythagorean fuzzy continuous.

To prove: $\theta : (X, \mathcal{G}_1) \to (y, \mathcal{G}_2)$ is a Pythagorean fuzzy weakly gradation preserving map.

$\mathcal{G}_2(A) > 0$

$\Rightarrow A \in \tau(\mathcal{G}_2)$

$\Rightarrow \theta^{-1}(A) \in \tau(\mathcal{G}_1)$

$\Rightarrow \mathcal{G}_1(\theta^{-1}(A)) > 0$

Therefore $\theta : (X, \mathcal{G}_1) \to (Y, \mathcal{G}_2)$ is a Pythagorean fuzzy weakly gradation preserving map.

Theorem 3.18. *For $\dot{A} \in 2^I$, $A \subseteq [0, 1] = I$, define $\dot{A}_I = (\mu_{\dot{A}_I}, v_{\dot{A}_I})$ as $\mu_{\dot{A}_I} : I \to [0, 1]$ and $v_{\dot{A}_I} : I \to [0, 1]$ such that $\mu_{\dot{A}_I}(x) = A$, for every $x \in X$*

and $v_{\dot{A}_I}(x) = 0$, *for every* $x \in X$. *Therefore* $\dot{A}_I \in PYF(I)$.

Let $\mathcal{G}$ *be a Pythagorean fuzzy gradation of openness on X. Define* $(\mathcal{G}_I)_* : 2^I \to I$ *such that* $(\mathcal{G}_I)_*(A) = \mathcal{G}(\dot{A}_I)$. *Then* $(\mathcal{G}_I)_*$ *is a crisp gradation of openness on X.*

Theorem 3.19. *Given* $A \in PYF(I)$, *where* $\mu_A : I \to [0, 1]$. *and* $v_A : I \to [0, 1]$. *Fix* $\alpha \in I$. *Define* $(A_\alpha) \subseteq I$ *such that* $A_\alpha = \{\mu_{A_\alpha}(\alpha)\}$, *singleton set. Let* $\mathcal{G}_* : 2^I \to I$ *be a crisp gradation of openness on I. Define* $\mathcal{G}_\alpha : PYF(I) \to I$ *such that* $\mathcal{G}_\alpha(A) = \mathcal{G}_*(A_\alpha)$. *Then* $\mathcal{G}_\alpha$ *is Pythagorean fuzzy gradation of openness on X.*

Theorem 3.20. *Let* $\mathcal{G}_* : 2^I \to I$ *be a crisp gradation of openness on I. Let X be any non-empty set. Fix* $x \in X$, *define* $\mathcal{G}_x : PYF(X) \to I$ *such that* $\mathcal{G}_x(A) = \mathcal{G}_*(\{\mu_A(x)\})$. *Then* $\mathcal{G}_x$ *is a Pythagorean fuzzy gradation of openness on X.*

Theorem 3.21. *Let* $\mathcal{G} : PYF(X) \to I$ *be a Pythagorean fuzzy gradation of openness on X. Let* $\mathcal{G}_c : PYF(X) \to I$ *such that* $\mathcal{G}_c(A) = \mathcal{G}(A^c)$. *Then* $\mathcal{G}_c$ *is a Pythagorean fuzzy gradation of openness on X.*

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