Advances and Applications in Mathematical Sciences
Volume 21, Issue 9, July 2022, Pages 5469-5494

THE SPLIT AND ANNIHILATOR DOMINANCE OF STRONG PRODUCT GRAPHS

P. APARNA, K. V. SURYANARAYANA RAO and E. KESHAVA REDDY

Research Scholar
Department of Mathematics
Jawaharlal Nehru Technological
University Anantapur, India

Department of Mathematics
RGM College of Engineering
and Technology, Nandyal, India

Department of Mathematics
JNTUA, Anantapuramu, India

Abstract

The paper concentrates on the theory of Annihilator domination in Strong Product graphs. The split domination in graphs was introduced by Kulli and Janakiram [4]. K. V. Suryanarayana Rao and V. Sreenivasan have investigated some properties of the split domination number of some product graphs and obtained several interesting results [7]. In this paper, we have investigated some properties of the split and Annihilator domination number of Strong Product graphs. Few significant and interesting results were studied on different graphs.

Introduction

One of the most active fields of modern mathematics is graph theory. Due to its extensive applicability to discrete optimization issues, combinatorial difficulties, and classical algebraic problems, graph theory has grown dramatically during the last 30 years. It has several applications in disciplines like as engineering, physical, social, and biological sciences, languages, and so on. In recent years, the idea of dominance has been at the centre of graph theory study. This is because to a number of additional

2020 Mathematics Subject Classification: 05Cxx.
Keywords: Split domination set, Split domination number, Annihilator domination number, Strong product of graphs.
Received November 2, 2021; Accepted February 4, 2022

factors that may be derived from the fundamental concept of dominance. The NP-completeness of the fundamental dominance issues, as well as their strong connection to other NP-completeness problems, has fueled a massive increase in domination theory research. The extensive coverage of "Topics on dominance in graph" in the 86th issue of the Journal of Discrete Mathematics (1990) demonstrates that the theory of domination is a very prominent topic of graph theory research effort. The majority of the paper's vocabulary was derived from sources [1], [2]. Kulli and Janakiram [4] first proposed split dominance in graphs. They defined the split dominating set and the split domination number, as well as a number of noteworthy conclusions for the split domination number of various typical graphs. On tensor products of graphs, Sampathkumar [3] discovered some interesting findings. K. V. Suryanarayana Rao and V. Sreenivasan [5] and [8] developed split and annihilator domination in arithmetic graphs, as well as several standard graphs. We consider product graphs and recall the results associated to the product graphs. K. V. Suryanarayana Rao and V. Sreenivasan [7] obtained several interesting results on Kronecker, Cartesian and Lexico product of graphs. Some findings on the Kronecker Product of two graphs were achieved by Dr. P. Bhaskarudu [6]. Arithmetic Graphs of Split and Annihilator Domination Number has been studied by P. Aparna, K. V. Suryanarayana Rao, and E. Keshava Reddy [9].

We looked at certain properties of a Split and Annihilator domination number of Strong product graphs, which was inspired by the research of dominance and split domination.

Basic definitions:

Dominating set. A subset D of V is said to be a dominating set of G if every vertex in $V \setminus D$ is adjacent to a vertex in D.

Dominating number. The dominating number $\gamma(G)$ of G is the minimum cardinality of a dominating set.

Split dominating set. A dominating set D of a graph G is called a split dominating set, if the induced subgraph $\langle V - D \rangle$ is disconnected.

Split domination number. The split dominating number $\gamma_s(G)$ of G is the minimum cardinality of the split dominating set.

Annihilator Domination set. The dominant set D of graph G is said to be the Array of annihilator dominant, if the induced subgraph $\langle V - D \rangle$ is a graph with isolated vertices or a graph with independent vertices.

Annihilator Domination Number. The annihilator domination number $\gamma_a(G)$ of G is the minimum cardinality of an annihilator dominating set.

Strong Product of two Graphs. The strong product $G \boxtimes H$ of graphs G and H is a graph such that

(i) The vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; and

(ii) Distinct vertices (u, u') and (v, v') are adjacent in $G \boxtimes H$ if and only if

- $u = v$ and u' is adjacent to v', or
- $u' = v'$ and u is adjacent to v, or
- u is adjacent to v and u' is adjacent to v'.

It is the union of the Cartesian product and the Tensor product. It is also called as Normal product or the AND product.

Regular Graph. A regular graph is a graph where each vertex has the same number of neighbors; i.e. every vertex has the same degree. A regular graph with vertices of degree k is called a k-regular graph or regular graph of degree k.

Some results on Split domination of Product graphs [7]

Theorem 1 [7]. *If* $G1, G2$ *are any two graphs, then* $\gamma s[G1(k)G2] \leq \min[\gamma s(G1) \cdot |V2|, |V1| \cdot \gamma s(G2)]$ (*Kronecker Product of two graphs*)

Theorem 2 [7]. *If* $G1, G2$ *are any two graphs without isolated vertices, then* $\gamma s[G1(C)G2] \leq \gamma s(G1) \cdot |V2|$ (*Cartesian product of two graphs*)

Theorem 3 [7]. *If* $G1, G2$ *are any two graphs without isolated vertices, then* $\gamma s[G1(L)G2] \leq \gamma s(G1) \cdot |V2|$ (*Lexico product of two graphs*)

In this Paper we have concentrated on Split and Annihilator domination of Strong Product Graphs and obtained several results.

Split and Annihilator domination of Strong Product Graphs

- The following results are identified for Paths in Strong Product of Graphs:

Number of vertices in G and H $[G(s)H]$	Split Domination number $\gamma_s[G(S)H]$	Annihilator Domination number $\gamma_a[G(S)H]$
1. $[G(S)H]$ with 3×3 Vertices	$\gamma_s[G(S)H] = 3$	$\gamma_a[G(S)H] = 5(3+2)$
2. $[G(S)H]$ with 4×4 Vertices	$\gamma_s[G(S)H] = 8$	$\gamma_a[G(S)H] = 12(8+4)$
3. $[G(S)H]$ with 5×5 Vertices	$\gamma_s[G(S)H] = 10$	$\gamma_a[G(S)H] = 16(10+6)$
4. $[G(S)H]$ with 6×6 Vertices	$\gamma_s[G(S)H] = 18$	$\gamma_a[G(S)H] = 27(18+9)$
5. $[G(S)H]$ with 7×7 Vertices	$\gamma_s[G(S)H] = 21$	$\gamma_a[G(S)H] = 33(21+12)$
6. $[G(S)H]$ with 8×8 Vertices	$\gamma_s[G(S)H] = 32$	$\gamma_a[G(S)H] = 48(32+16)$
7. $[G(S)H]$ with 9×9 Vertices	$\gamma_s[G(S)H] = 36$	$\gamma_a[G(S)H] = 56(36+20)$
8. $[G(S)H]$ with 10×10 Vertices	$\gamma_s[G(S)H] = 50$	$\gamma_a[G(S)H] = 75(50+25)$

And so on…

Suppose we have a $G(s)H$ Path graph of order $n \times n$ where $n > 2$ then we have investigated the following results on Split and Annihilator Domination for Strong Product Graph.

Theorem 1. *If* G, H *are any two path graphs with* G_n, H_n *vertices where* $n > 2$ *respectively then the Strong product of* $G(s)H$ *[The Strong* *product of two graphs G and H] Satisfies the following results.*

(i) $\gamma_s[G(s)H] \leq \gamma_s(G) \cdot |V_2|$

(ii) $\gamma_s[G(s)H] \leq D_s + \{|\langle V_1 - D_1\rangle| \times |D_2|\}$.

Proof. (i) Suppose G be a Path graph with n-vertices and H be another Path graph with n-vertices

Let $V(G) = \{u_1, u_2, u_3, \dots, u_n\} = V_1(\text{say})$, $n > 2$ and

$V(H) = \{v_1, v_2, v_3, \dots, v_n\} = V_2(\text{say})$, $n > 2$

Now the Cartesian product of $V(G) \times V(H)$ is

$V[G(s)H] = \{(u_1, v_1), (u_1, v_2), \dots, (u_n, v_n), (u_2, v_1), (u_2, v_2), \dots, (u_2, v_n),$

$(u_3, v_1), (u_3, v_2), \dots, (u_3, v_n), \dots, (u_n, v_1), (u_n, v_2), \dots, (u_n, v_n)\}$

Let $D_1 = \{u_{d_1}, u_{d_2}, \dots, u_{d_n}\}$ and $D_2 = \{v_{d_1}, v_{d_2}, \dots, v_{d_n}\}$ are split dominating sets of G and H respectively. Here the vertices $u_{d_1}, u_{d_2}, \dots, u_{d_n}$ are adjacent with at least one vertex in $\langle V_1 - D_1\rangle$. Similarly the vertices $v_{d_1}, v_{d_2}, \dots, v_{d_n}$ are adjacent with at least one vertex in $\langle V_2 - D_2\rangle$.

Consider $D_s = D_1 \times V_2 = \{u_{d_1}, u_{d_2}, \dots, u_{d_n}\} \times \{v_{d_1}, v_{d_2}, \dots, v_{d_n}\}$

$$= \{(u_{d_1}, v_1), (u_{d_1}, v_2), \dots, (u_{d_1}, v_n), (u_{d_2}, v_1), (u_{d_2}, v_2), \dots,$$

$$(u_{d_2}, v_n), \dots, (u_{d_n}, v_1), (u_{d_n}, v_2), \dots, (u_{d_n}, v_n)\}$$

is the dominating set of $G(s)H$ as shown in figure 1.

Removal of vertex set D_s in V we get the induced subgraph $\langle V[G(s)H] - D_s\rangle$ is disconnected as shown in figure 1(a).

Suppose (u, v) is any vertex in D_s then we have the first component u is in D_1.

Let the first component $u = u_{d_i}$ for some i in D_1 and the second component v is in $\langle V_2 - D_2\rangle$ and will be adjacent to some vertex in $D_2 = \{v_{d_1}, v_{d_2}, \dots, v_{d_n}\}$.

For the sake of definiteness v is adjacent with v_{d_j} for some j then $(u, v) = (u_{d_i}, v)$ is adjacent with (u_{d_i}, v_{d_j}). Thus D_s is a dominating set.

Further we have to prove that D_s is a split dominating set.

Suppose (u, v) be any vertex in $\langle V - D_s \rangle$, where u is in $\langle V_1 - D_1 \rangle$ and v is any vertex in H.

Let, u_i, u_j be two different components of the induced sub graph $\langle V_1 - D_1 \rangle$ of G.

By the definition of split domination and from the definition of strong product of graph the vertices $(u_i, v), (u_j, v)$ will be different components of $\langle V - D_s \rangle$ of $G(s)H$.

Removal of D_s from the vertex set V we obtain the induced subgraph $\langle V - D_s \rangle$ is disconnected.

Thus D_s is a split dominating set of $G(s)H$ as illustrated in figure 1(a).

Hence $\gamma_s[G(s)H] \leq | D_s |$

and so $\gamma_s[G(s)H] \leq \gamma_s(G) \cdot | V_2 |$

(ii) Let $D_a =$ The Cartesian product of $\langle V_1 - D_1 \rangle$ and D_2

$\Rightarrow D_a = \{\langle V_1 - D_1 \rangle\} \times \{D_2\}$

$\Rightarrow D_a = \{(u_{a_1}, v_{d_1}), (u_{a_1}, v_{d_2}), \ldots, (u_{a_1}, v_{d_n}), (u_{a_2}, v_{d_1}), (u_{a_2}, v_{d_2}), \ldots,$

$(u_{a_2}, v_{d_n}), \ldots, (u_{a_n}, v_{d_1}), (u_{a_n}, v_{d_2}), \ldots, (u_{a_n}, v_{d_n})\}$

be the annihilator domination set of $\langle G(s)H - D_s \rangle$.

Let (u_{a_j}, v_{d_j}) be any vertex in D_a. Here u_{a_i} is in $\langle V_1 - D_1 \rangle$ for some i and v_{a_j} is in D_2.

Now u_{a_i} is in $\langle V_1 - D_1 \rangle$ and will be adjacent to some vertex in D_1, v_{d_j} is in D_2 and will be adjacent to some vertex in $\langle V_2 - D_2 \rangle$.

Therefore (u_{a_j}, v_{d_j}) is adjacent with every vertex in $\langle G(s)H - D_s \rangle$.

The removal of D_a from $\langle G(s)H - D_s \rangle$ then we obtain the graph with isolated vertices called as annihilator domination graph.

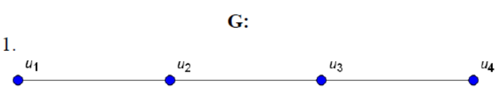
The minimal cardinality of D_a is called the annihilator domination number.

Now total vertices we have to remove is $D_s = \{| \langle V_1 - D_1 \rangle |\} \times \{D_2\}$ then we obtain annihilator graph of $G(s)H$ as illustrated in figure 1(b).

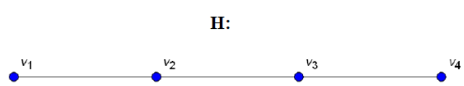
Hence $\gamma_a[G(s)H] \leq D_s = \{| \langle V_1 - D_1 \rangle |\} \times \{D_2\}$

Illustrations.

1.



Split dominating set of $G = D_1 = \{u_1, u_3\}$



Split dominating set of $H = D_2 = \{v_1, v_3\}$.

By the definition of Strong product of two graphs G and H we have

Graph of $[G(s)H]$

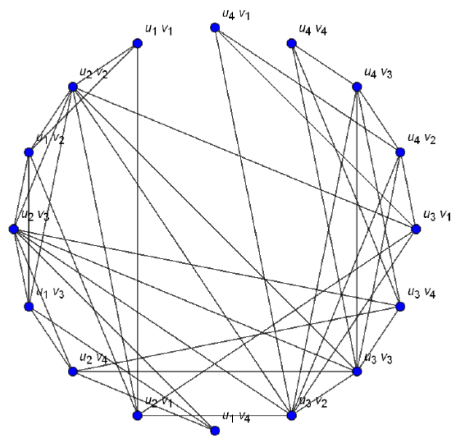


Figure 1.

The split dominating set is $D_s = D_1 \times V_2 = \{u_1, u_3\} \times \{v_1, v_2, v_3, v_4\}$

$= \{(u_1, v_1), (u_1, v_2), (u_1, v_3), (u_1, v_4), (u_3, v_1), (u_3, v_2), (u_3, v_3), (u_3, v_4)\} = 8$ is the minimal split domination set.

Now removal of D_s from V we get the graph is splitted.

Graph of $\langle V[G(s)H] - D_s \rangle$

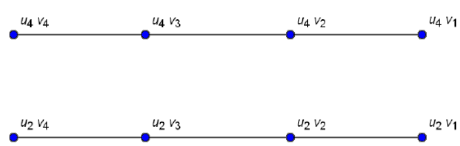


Figure 1(a).

D_s is a Split dominating set

$| D_s | = \gamma_s(G) \cdot | V_2 | = 2 \cdot 4 = 8.$

Hence $\gamma_s[G(s)H] \leq \gamma_s(G) \cdot | V_2 | = 2 \cdot 4 = 8.$

Now the annihilator dominating set is $D_a = \{\langle V_1 - D_1 \rangle\} \times \{D_2\}$ $= \{u_2, u_4\} \times \{v_1, v_3\} = \{(u_2 v_1), (u_2 v_3), (u_4 v_1), (u_4 v_3)\} = 4.$

Now removal of D_a from $\langle V[G(s)H] - D_s\rangle$ we get the graph with isolated vertices i.e. the graph is obtained with independent vertices is called as annihilator domination graph of $G(s)H$.

Graph of $\langle V[G(s)H] - D_a\rangle$

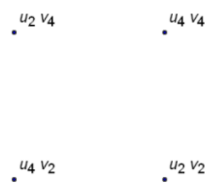
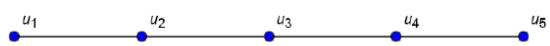


Figure 1(b).

Now total vertices we have removed in Graph of $G(s)H$ to obtain annihilator graph is $\gamma_a[G(s)H] = D_s + \{|\langle V_1 - D_1\rangle| \times |D_2|\} = 8 + 2 \cdot 2 = 12$

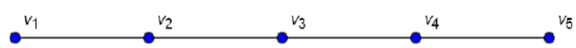
2.

G:



Split dominating set of $G = D_1 = \{u_2, u_4\}$

H:



Split dominating set of $H = D_2 = \{v_2, v_4\}$.

By the definition of Strong product of two graphs G and H we have

Graph of $[G(s)H]$

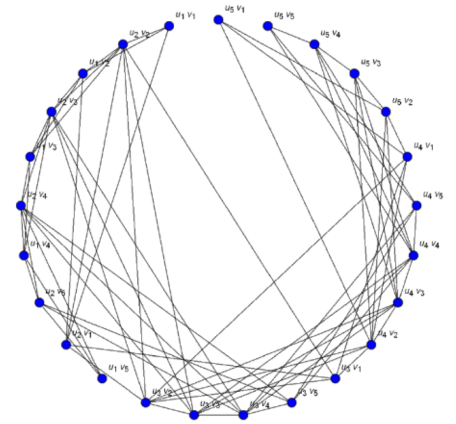


Figure 2.

The split dominating set is $D_s = D_1 \times V_2 = \{u_2, u_4\} \times \{v_1, v_2, v_3, v_4, v_5\}$

$= \{(u_2, v_1), (u_2, v_2), (u_2, v_3), (u_2, v_4), (u_2, v_5), (u_4, v_1), (u_4, v_2), (u_4, v_3),$ $(u_4, v_4), (u_4, v_5)\} = 10$ is the minimal split domination set.

Now removal of D_s from V we get the graph is splitted.

Graph of $\langle V[G(s)H] - D_s \rangle$

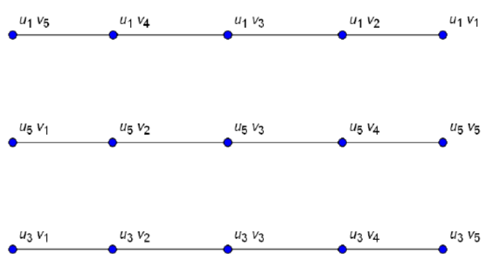


Figure 2(a).

D_s is a Split dominating set

$| D_s | = \gamma_s(G) \cdot | V_2 | = 2 \cdot 5 = 10$

Hence $\gamma_s[G(s)H] \leq \gamma_s(G) \cdot | V_2 | = 2 \cdot 5 = 10$

Now the annihilator dominating set is $D_a = \{\langle V_1 - D_1 \rangle\} \times \{D_2\}$ $= \{u_1, u_3, u_5\} \times \{v_2, v_3\} = \{(u_1 v_2), (u_1 v_4), (u_3 v_2), (u_3 v_4), (u_5 v_2), (u_5 v_4)\} = 6.$

Now removal of D_a from $\langle V[G(s)H] - D_s \rangle$ we get the graph with isolated vertices i.e. the graph is obtained with independent vertices is called as annihilator domination graph of $G(s)H$.

Graph of $\langle V[G(s)H] - D_a \rangle$

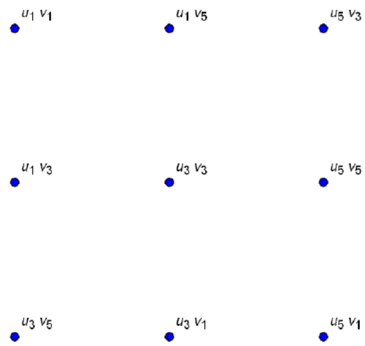


Figure 2(b).

Now total vertices we have removed in Graph of $[G(s)H]$ to obtain annihilator graph is $\gamma_a[G(s)H] = D_s + \{|\langle V_1 - D_1 \rangle| \times |D_2|\} = 10 + 3 \cdot 2 = 16.$

- The following results are identified for Regular Graphs in Strong Product of Graphs:

Number of vertices in G and H $[G(s)H]$	Split Domination number $\gamma_s[G(S)H]$	Annihilator Domination number $\gamma_a[G(S)H]$
$[G(S)H]$ with 4×4 Vertices	$\gamma_s[G(S)H] = 8$ $\gamma'_s[G(S)H] = 8$	$\gamma_a[G(S)H] = 12$ $\gamma'_a[G(S)H] = 12$
$[G(S)H]$ with 4×5 Vertices	$\gamma_s[G(S)H] = 10$ $\gamma'_s[G(S)H] = 8$	$\gamma_a[G(S)H] = 16$ $\gamma'_a[G(S)H] = 16$
$[G(S)H]$ with 4×6 Vertices	$\gamma_s[G(S)H] = 12$	$\gamma_a[G(S)H] = 18$

	$\gamma'_s[G(S)H] = 12$	$\gamma'_a[G(S)H] = 18$
$[G(S)H]$ with 4×7 Vertices	$\gamma_s[G(S)H] = 14$ $\gamma'_s[G(S)H] = 12$	$\gamma_a[G(S)H] = 22$ $\gamma'_a[G(S)H] = 22$
$[G(S)H]$ with 5×6 Vertices	$\gamma_s[G(S)H] = 12$ $\gamma'_s[G(S)H] = 15$	$\gamma_a[G(S)H] = 24$ $\gamma'_a[G(S)H] = 24$

And so on…

With the above results we have investigated some significant results on Split and Annihilator Domination of Regular graphs using Strong Product Graph.

Theorem 2. *If G and H are 2-regular graphs then the Strong product of G and H i.e., $G(S)H$ satisfies the following results.*

(i) $\gamma'_s[G(S)H] \leq (a)\gamma_s(G) \cdot |V_2|$ *or* $|V_1| \cdot \gamma_s(H)$, *if both* $|V_1|$ *and* $|V_2|$ *are same* $(b)\, Min\{\gamma_s(G) \cdot |V_2|, |V_1| \cdot \gamma_s(H)\}$, *if* $|V_1|$ *is odd and* $|V_2|$ *is even and Vice-versa.*

(ii) $\gamma'_s[G(S)H] \leq (c)D_s + \{\langle V_1 - D_1\rangle \times \langle V_2 - D_2\rangle\}$, *if both* $|V_1|$ *and* $|V_2|$ *are even* $(d)D_s + \{\langle V_1 - D_1\rangle \times \langle V_2 - D_2\rangle\} +$ *Cartesian product of one of the adjacent vertex in* $\langle V_1 - D_1\rangle$ *(or)* $\langle V_2 - D_2\rangle$ *with* D_2 *(or)* D_1 *respectively, if* $|V_1|$ *is odd and* $|V_2|$ *is even and Vice-versa also both* $|V_1|$ *and* $|V_2|$ *are odd.*

Proof. Let G be a 2-regular graph with p-vertices that is $V(G) = V_1 = \{x_1, x_2, \ldots, x_p\}$, here $|V_1| = p$ and H be another 2-regular graph with q-vertices that is $V(H) = V_2 = \{y_1, y_2, \ldots, y_q\}$, Here $|V_1| = q$ Suppose $D_1 = \gamma_s(G) = \{x_{d_1}, x_{d_2}, \ldots, x_{d_r}\}$ and $D_2 = \gamma_s(G) = \{y_{d_1}, y_{d_2}, \ldots, y_{d_r}\}$ be a split dominating sets of minimum cardinality of G and H respectively. Since every vertex in $V_1 - D_1$ and $V_2 - D_2$ is adjacent to a vertex in D_1 and D_2 respectively. Also the removal of D_1 in G, we get the induced sub graph $\langle V_1 - D_1\rangle$ to be disconnected graph and similarly the removal of D_2 in H, we get another induced sub graph $\langle V_2 - D_2\rangle$ to be disconnected graph.

(i) To prove Split domination.

Case (a). When $|V_1| = |V_2|$ (Number of vertices in both graphs are equal i.e., $p = q$).

Now $D_s = \{(x_{d_1}, y_1), (x_{d_1}, y_2), \ldots, (x_{d_1}, y_q), (x_{d_2}, y_1), (x_{d_2}, y_2), \ldots, (x_{d_2}, y_q), (x_{d_3}, y_1), (x_{d_3}, y_2), \ldots, (x_{d_3}, y_q), \ldots, (x_{d_r}, y_1), (x_{d_r}, y_2), \ldots, (x_{d_r}, y_q)$ is a dominating set of $G(S)H$ Suppose (u, v) be any vertex of $\langle V - D_s \rangle$ in $G(S)H$. Here u is adjacent with at least one vertex in $\{x_{d_1}, x_{d_2}, \ldots, x_{d_r}\}$ as this is dominating set of G being its.

Split dominating set and suppose v is adjacent with some vertex y_j in H.

Thus (u, v) is adjacent with (x_{d_i}, y_j) in D_s.

Therefore D_s is a dominating set.

Further we have to prove D_s is split dominating set as follows.

Let, x_i, x_j be any two vertices in $\langle V_1 - D_1 \rangle$. If, y_k, y_l are two non-adjacent vertices in H then the two vertices (x_i, y_k) and (x_j, y_l) will be two different components in the induced subgraph $\langle V - D_s \rangle$ in $G(S)H$.

Therefore D_s is a Split dominating set of $G(S)H$ as shown in the figure 3(a).

In this case we can also choose

$D_s = \{(x_1, y_{d_1}), (x_1, y_{d_2}), \ldots, (x_1, y_{d_s}), (x_2, y_{d_1}), (x_2, y_{d_2}), \ldots, (x_2, y_{d_s}), (x_3, y_{d_1}), (x_3, y_{d_2}), \ldots, (x_3, y_{d_s}), \ldots, (x_p, y_{d_1}), (x_p, y_{d_2}), \ldots, (x_p, y_{d_s})\}$.

Similarly with the same argument as we have proved D_s is Split dominating set here also it can easily proved that D'_s is split dominating set. Thus choose either D_s (or) D'_s in this case.

Hence $\gamma'_s[G(S)H] \leq (a)\gamma_s(G) \cdot |V_2|$ or $|V_1| \cdot \gamma_s(H)$, if $|V_1| = |V_2|$ as illustrated in figure 3(a).

Case (b). When $|V_1|$ is odd and $|V_2|$ is even and vice-versa

$D_s = \gamma_s(G) \times V_2 = D_1 \times V_2 = \{(x_{d_1}, y_1), (x_{d_1}, y_2), \ldots, (x_{d_1}, y_q), (x_{d_2}, y_1),$ $(x_{d_2}, y_2), \ldots, (x_{d_2}, y_q), (x_{d_3}, y_1), (x_{d_3}, y_2), \ldots, (x_{d_3}, y_q), \ldots, (x_{d_r}, y_1),$ $(x_{d_r}, y_2), \ldots, (x_{d_r}, y_q)\}.$

Similarly by the same argument like in case (a), it can be easily shown that D_s is a Split dominating set of $G(S)H$ as illustrated in figure 4(a).

Also we have

$D'_s = |V_1| \times \gamma_s(H) = |V_1| \times D_2 = \{(x_1, y_{d_1}), (x_1, y_{d_2}), \ldots, (x_1, y_{d_s}),$ $(x_2, y_{d_1}), (x_2, y_{d_2}), \ldots, (x_2, y_{d_s}), (x_3, y_{d_1}), (x_3, y_{d_2}), \ldots, (x_3, y_{d_s}), \ldots,$ $(x_p, y_{d_1}), (x_p, y_{d_2}), \ldots, (x_p, y_{d_s})\}.$

Similarly by the same argument like in case (a), it can be easily shown that D'_s is a Split dominating set of $G(S)H$ as illustrated in figure 4(a).

Hence $\gamma_s[G(S)H] \leq Min\{D_s, D'_s\}$

$\therefore \gamma_s[G(S)H] \leq Min\{\gamma_s(G) \cdot |V_2|, |V_1| \cdot \gamma_s(H)\}$, if $|V_1|$ is odd and $|V_2|$ is even and Vice-versa.

(ii) To prove Annihilator domination.

Case (c). When $|V_1|$ and $|V_2|$ are even (i.e., p and q are even)

We have $D_a = \{\langle V_1 - D_1 \rangle \times \langle V_2 - D_2 \rangle\}$

$$= \{(x_{a_1}, x_{a_2}, \ldots, x_{a_m}) \times (y_{a_1}, y_{a_2}, \ldots, y_{a_n})\}$$

$$\{(x_{a_1}, y_{a_1}), (x_{a_1}, y_{a_2}), \ldots, (x_{a_1}, y_{a_n}), (x_{a_2}, y_{a_1}), (x_{a_2}, y_{a_2}), \ldots, (x_{a_2}, y_{a_n}),$$
$$\ldots, (x_{a_m}, y_{a_1}), (x_{a_m}, y_{a_2}), \ldots, (x_{a_m}, y_{a_n})\}$$

where D_a is called as annihilator dominating set of $\langle V - D_s \rangle$ as illustrated in figure 3(b).

Let (u, v) be any vertex in D_a. Here u is not adjacent with any vertex in $\{x_{a_1}, x_{a_2}, \ldots, x_{a_m}\} = \langle V_1 - D_1 \rangle$ as this is a annihilator dominating set of G

and v is not adjacent with any vertex in $\{y_{a_1}, y_{a_2}, \ldots, y_{a_m}\} = \langle V_2 - D_2 \rangle$ as this is a annihilator dominating set of H.

$\therefore (u, v)$ is not adjacent with (x_{a_i}, y_{a_j}) in D_a.

Thus D_a is Annihilator dominating set of $\langle V - D_s \rangle$.

The removal of D_a in $\langle V - D_s \rangle$, then we get the induced sub graph to be annihilated graph.

From the main vertex set of $G(S)H$, we have to delete D_s as well as a D then we obtain induced sub graph is annihilated graph as illustrated in figure 3(b).

Hence $\gamma'_s[G(S)H] \leq D_s + \{\langle V_1 - D_1 \rangle \times \langle V_2 - D_2 \rangle\}$, if $|V_1| = |V_2| = even$

Case (d). When $|V_1| = odd, |V_2| = even$ and Vice-versa also $|V_1| = |V_2| = odd$.

Take $D'_a = D_{a_1} + D_{a_2}$, now we have to prove that D'_a is annihilator dominating set of $G < V - D_s > H$,

where $D_{a_1} + \{\langle V_1 - D_1 \rangle \times \langle V_2 - D_2 \rangle\}$ and $D_{a_2} =$ Cartesian product of one of the adjacent vertex in $\langle V_1 - D_1 \rangle$ (or) $\langle V_2 - D_2 \rangle$ with D_2 (or) D_1 respectively.

Similar argument as we discussed in case(c) is used to prove D_{a_1} is an annihilator dominating set, still we will not get the induced subgraph is annihilated graph because either $|V_1|$ or $|V_2|$ are odd or both $|V_1|$ and $|V_2|$ are odd that is G (or) H or both G and H contains odd number of vertices as illustrated in figure 4(b). In this situation, we have to take the cartesian product of one of the adjacent vertex in $\langle V_1 - D_1 \rangle$ or $\langle V_2 - D_2 \rangle$ with D_2 or D_1 respectively.

Now we prove that D_{a_2} is annihilator dominating set.

We have $D_a = \{(x_{a_1}) \times (y_{d_1}, y_{d_2}, y_{d_3}, \ldots, y_{d_s})\}$

$$= \{(x_{a_i}, y_{d_1}), (x_{a_i}, y_{d_2}), (x_{a_i}, y_{d_s}), \ldots, (x_{a_i}, y_{d_s})\}$$

Suppose (u, v) be any vertex in D_{a_2}. Here v is not adjacent with at least one of the vertex in $(y_{d_1}, y_{d_2}, y_{d_3}, \ldots, y_{d_s})$ as this is a split dominating set of H.

$\therefore (u, v)$ is not adjacent with (x_{a_i}, y_{a_j}) in D_{a_2}.

Therefore D_{a_2} is one of the annihilator dominating set of D'_a.

Now the annihilator dominating set of $V - D_s$ is $D'_a = D_{a_1} + D_{a_2}$.

The removal of D'_a in $\langle V - D_s \rangle$ then we obtain the induced sub graph to be annihilated graph.

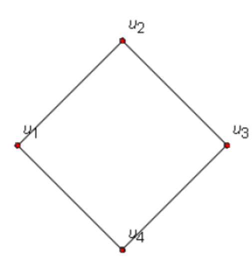
Also from the main vertex set of $G(S)H$ we delete D_s and D'_a then we get the induced sub graph of $G(S)H$ is annihilated graph as shown in figure 4(c).

Hence $\gamma_s[G(S)H] \leq D_s + \{\langle V_1 - D_1 \rangle \times \langle V_2 - D_2 \rangle\} +$ Cartesian product of one of the adjacent vertex in $\langle V_1 - D_1 \rangle$ (or) $\langle V_2 - D_2 \rangle$ with D_2 (or) D_1 respectively, if $| V_1 |$ is odd and $| V_2 |$ is even and Vice-versa also both $| V_1 | = | V_2 | = odd.$

Hence the theorem.

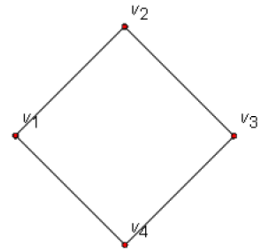
Illustrations.

$$G : | V_1 | = 4$$



Split dominating set of $G = D_1 = \{u_1, u_3\}$

$$H : | V_2 | = 4$$



Split dominating set of $H = D_1 = \{v_2, v_4\}$.

By the definition of Strong product of two graphs G and H we have

Graph of $[G(s)H]$

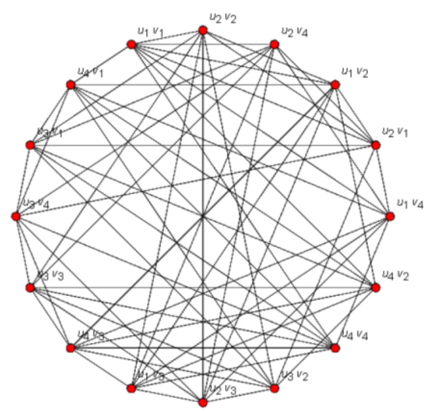


Figure 3.

The split dominating set is $D'_s = D_1 \times V_2 = \{u_1, u_3\} \times \{v_1, v_2, v_3, v_4\}$ $= \{(u_1v_1), (u_1v_2), (u_1v_3), (u_1v_4), (u_3v_1), (u_3v_2), (u_3v_3), (u_3v_4)\} = 8$ is the minimal split domination set.

Now removal of D_s from V we get the graph is splitted.

Graph of $\langle V[G(s)H] - D_s \rangle$

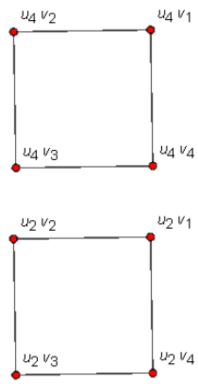


Figure 3(a).

D_s is a Split dominating set

$| D_s | = \gamma_s(G) \cdot | V_2 | = 2 \cdot 4 = 8.$

Hence $\gamma_s[G(S)H] \leq \gamma_s(G) \cdot | V_2 | = 2 \cdot 4 = 8.$

Now the annihilator dominating set is $D_a = D_S + \{\langle V_1 - D_1 \rangle\} \times \{V_2 - D_2\} = \{u_2, u_4\} \times \{v_1, v_3\} = \{(u_2 v_1), (u_2 v_3), (u_4 v_1), (u_4 v_3)\} = 4.$

Now removal of D_a from $\langle V[G(s)H] - D_s \rangle$ we get the graph with isolated vertices i.e. the graph is obtained with independent vertices is called as annihilator domination graph of $G(s)H$.

Graph of $\langle V[G(s)H] - D_a \rangle$

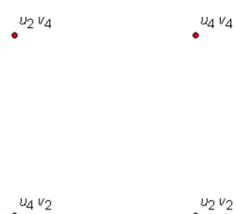
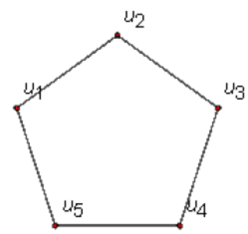


Figure 3(b).

Now total vertices we have removed in Graph of $G(s)H$ to obtain annihilator graph is

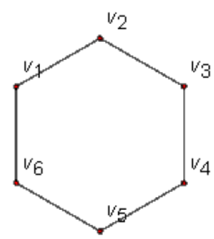
$\gamma_a[G(s)H] = D_s + \{| \langle V_1 - D_1 \rangle | \times | \langle V_2 - D_2 \rangle |\} = 8 + 2 \cdot 2 = 12$

$$G : | V_1 | = 5$$



Split dominating set of $G = D_1 = \{u_1, u_3\}$

$$H : | V_2 | = 6$$



Split dominating set of $H = D_2 = \{v_1, v_3, v_5\}$.

By the definition of Strong product of two graphs G and H we have

Graph of $[G(s)H]$

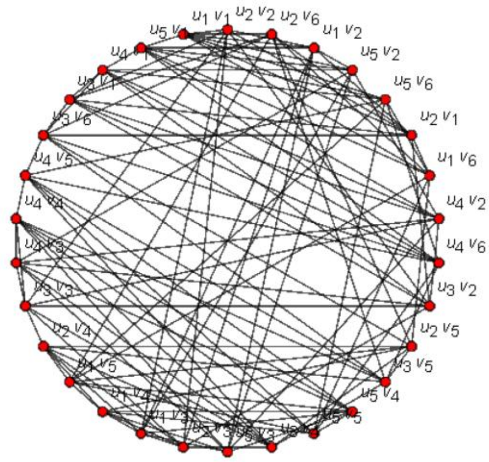


Figure 4.

The split dominating set is $D_s' = D_1 \times V_2 = \{u_1, u_3\} \times \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

That is $V - D_S = \{(u_1v_1), (u_1v_2), (u_1v_3), (u_1v_4), (u_1v_5), (u_3v_1), (u_3v_2), (u_3v_3), (u_3v_4), (u_3v_5), (u_3v_6)\} = 12$ is the minimal split domination set.

Now removal of D_s from V we get the graph is splitted.

Also we have $D_s' = | V_1 | \times D_2 = \{u_1, u_2, u_3, u_4, u_5\} \times \{v_1, v_3, v_5\} = 15$.

Now we have to choose $Min\{\gamma_s(G) \cdot | V_2 |, | V_1 | \cdot \gamma_s(H)\} = \{12, 15\} = 12$.

We consider $D_s = D_1 \times V_2 = \{u_1, u_3\} \times \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

Graph of $\langle V[G(s)H] - D_s \rangle$

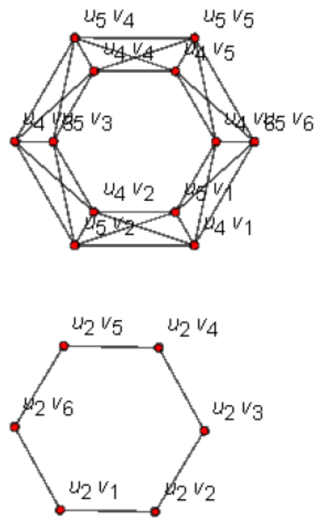


Figure 4(a).

D_s is a Split dominating set

$| D_s | = \gamma_s(G) \cdot | V_2 | = 2.6 = 12.$

Hence $\gamma_s[G(S)H] \leq Min\{\gamma_s(G) \cdot | V_2 |, | V_1 | \cdot \gamma_s(H)\}.$

Now the annihilator dominating set is

$D_a = D_S + [\{\langle V_1 - D_1 \rangle\} \times \{\langle V_2 - D_2 \rangle\} = D_S + \{u_2, u_4, u_5\} \times \{v_2, v_4, v_6\}$

$\Rightarrow D_a = D_S$

$+ \{(u_2 v_2), (u_2 v_4), (u_2 v_6), (u_4 v_2), (u_4 v_4), (u_4 v_6), (u_5 v_2), (u_5 v_4), (u_5 v_6)\}] = D_S + 9$

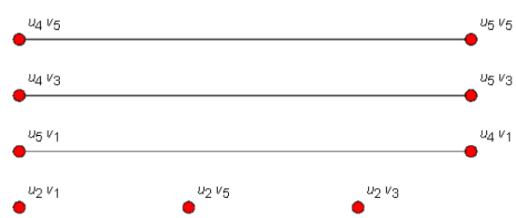


Figure 4(b).

Now removal of D_a from $\langle V[G(s)H] - D_s\rangle$, we will not obtain the graph with isolated vertices. Therefore we need to delete some other vertices in $\langle V[G(s)H] - D_s\rangle$

Hence the annihilator dominating set is

$D_a = D_S + [\{\langle V_1 - D_1\rangle\} \times \{\langle V_2 - D_2\rangle\}] + [\langle V_1 - D_1\rangle \times D_2] = D_S$

$+ [\{u_2, u_4, u_5\} \times \{v_2, v_4, v_6\}] + [u_4 \times \{v_1, v_3, v_5\}]$

$\Rightarrow D_a = D_S$

$+ \{(u_2v_2), (u_2v_4), (u_2v_6), (u_4v_2), (u_4v_4), (u_4v_6), (u_5v_2), (u_5v_4), (u_5v_6), (u_4v_1),$

$(u_4v_3), (u_4v_5)\} = D_S + 12$

Graph of $\langle V[G(s)H] - D_a\rangle$

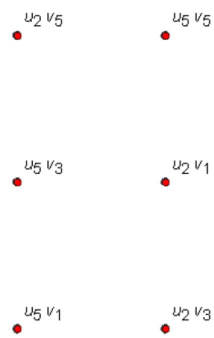


Figure 4(c).

i.e. the graph is obtained with independent vertices is called as annihilator domination graph of $G(S)H$.

Now total vertices we have removed in Graph of $G(S)H$ to obtain annihilator graph is

$D_a = D_S + [\{\langle V_1 - D_1\rangle\} \times \{\langle V_2 - D_2\rangle\}] + [\langle V_1 - D_1\rangle \times D_2] = 12 + 12 = 24.$

Hence

$\gamma_a[G(S)H] \leq D_s + \{\langle V_1 - D_1\rangle \times \langle V_2 - D_2\rangle\} +$ Cartesian product of one of the adjacent vertex in

$\langle V_1 - D_1 \rangle$ (or) $\langle V_2 - D_2 \rangle$ with D_2 (or) D_1 respectively,

If $| V_1 |$ is odd and $| V_2 |$ is even and Vice-versa.

Theorem 3. *If G be a regular graph and H be another regular graph then Strong product of G and H i.e.* $G(S)H$ *is also a regular graph.*

Proof. Let G be a k_1-regular graph and H be another k_2-regular graph then

$$\deg(u_i) = k_1, \forall u_i \in V_1 \text{ and } \deg(v_j) = k_2, \forall\, v_j \in V_2$$

Let (u_i, v_j) be any vertex in $G(S)H$ then $\deg(u_i, v_j) = 2[\deg(u_i) \cdot \deg(v_j)] = 2 \cdot k_1 \cdot k_2$.

Thus every vertex in $G(S)H$ is of degree $2k_1k_2$ i.e. $G(S)H$ is $2k_1k_2$-regular graph.

For example. Let G is a 2-regular graph and H is also a 2-regular graph.

Suppose the vertex set is $G = \{u_1, u_2, \ldots, u_p\}$ where $| G | = p$ and $H = \{v_1, v_2, \ldots, v_q\}$ where $| H | = q$

$$d(u_i) = 2, \forall\, u_i \in G \text{ where } i = 1, 2, 3, \ldots, p$$

$$d(v_j) = 2, \forall\, v_j \in H \text{ where } j = 1, 2, 3, \ldots, q.$$

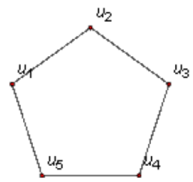
Let (u_i, v_j) be any vertex in $G(S)H$ then $\deg(u_i, v_j) = 2[\deg(u_i) \cdot \deg(v_j)] = 2[2 \cdot 2] = 8$.

Therefore every vertex in $G(S)H$ is having a degree 8 as shown in the figure 5.

Hence $G(S)H$ is 8-regular graph.

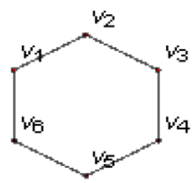
Illustration.

G: 2-Regular Graph with order $| G | = 5$



Here $\deg(u_i) = 2$ where $i = 1, 2, 3, 4, 5$

H: 2-Regular Graph with order $|H| = 6$



Here $\deg(v_j) = 2$ where $j = 1, 2, 3, 4, 5, 6$.

By the definition of Strong product of two graphs G and H we have

$[G(S)H]$ is a 8-Regular Graph with order $|G(S)H| = 30$

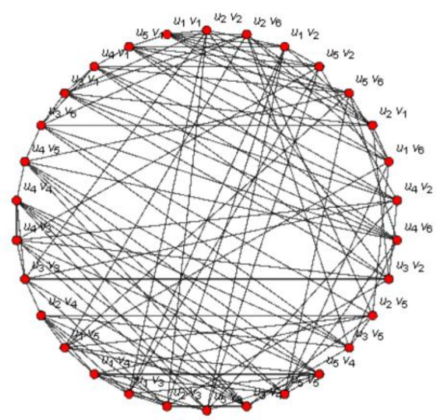


Figure 5.

Here $\deg(u_i, v_j) = 8$ where $i = 1, 2, 3, 4, 5$ and $j = 1, 2, 3, 4, 5, 6$.

Hence $\deg(u_i, v_j) = 2[\deg(u_i) \cdot \deg(v_j)] = 2[2 \cdot 2] = 8$.

Thus $G(S)H$ is 8-regular graph.

Theorem 4. *If G and H are two regular graphs then* $G(S)H$ *satisfies the following results*

(i) $\deg(x_i, y_j) = 2[\deg(x_i) \cdot \deg(y_j)]$

(ii) $| V_{G(S)H} | = | V_G | \times | V_H |$.

Proof. (i) Suppose $\deg(x_i) = p$ and $\deg(y_j) = q$ this means that x_i is adjacent with vertices $x_1, x_2, x_3, \ldots, x_p$ in G and y_j is adjacent with vertices $y_1, y_2, y_3, \ldots, y_q$ in H.

Let the vertex in strong product of G and H be (x_i, y_j)

(x_i, y_j) is adjacent with the vertex set

$\{(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_q); (x_2, y_1), (x_2, y_2), \ldots, (x_2, y_q), \ldots, (x_p, y_1),$

$(x_p, y_2), \ldots, (x_p, y_q)\}$.

If any vertex (x_k, y_l) not adjacent with (x_i, y_j) in $G(S)H$. Since x_i is not adjacent with x_k if $k > i$.

Similarly y_j is not adjacent with y_l if $l > j$.

Hence $\deg(x_i, y_j) = 2[\deg(x_i) \cdot \deg(y_j)]$

(ii) Suppose G, H be two graphs

Let $G = \{x_1, x_2, \ldots, x_p\}$ and $H = \{y_1, y_2, \ldots, y_p\}$.

So that $| V_G | = p$ and $| V_H | = q$.

Now the Cartesian product of G and H of strong product is

$V_{G(S)H} = G \times H = \{(x_1, y_1), (x_1, y_2), \ldots, (x_1, y_q); (x_2, y_1), (x_2, y_2), \ldots,$

$(x_2, y_q), \ldots, (x_p, y_1), (x_p, y_2), \ldots, (x_p, y_q)\}$ contains 'pq' number of vertices as illustrated in Figure 5

Hence $| V_{G(S)H} | = | V_G | \times | V_H | (\because p \cdot q = pq)$

Conclusion

The study of product graphs has provided us with enough stimulus to get some in-depth understanding of the graphs' different characteristics. It is anticipated that the inspiration gained from this examination of product graphs will serve as a starting point for further study. With the use of graph theory, we can easily create a simple technique for generating a graph with a specified cardinality of the split and annihilator dominating set. It's also wonderful to see how a graph with a certain dominance number may be extended in a systematic, straightforward way without altering the domination number. This may be used for a variety of purposes, including eradicating pests in agriculture, controlling viruses that cause epidemic diseases, and maintaining confidentiality while sharing information. This may be attributed, in part, to the rising significance of computer science and its relationship with the graph theory.

References

[1] F. Harary, Graph Theory, Addison-Wesley, Massachusetts, (1969).

[2] Bondy and Murty, Graph Theory with Applications, Macmillan, (1976).

[3] E. Sampathkumar, On tensor product graphs, J. Austral. Math. Soc. 20 (Series A) (1975), 268-273.

[4] V. R. Kulli and B. Janakiram, The split domination number of a graph, Graph theory notes of New York, Academy of Sciences XXXII (1997), 16-19.

[5] K. V. Suryanarayana Rao and V. Sreenivasan, The split domination in arithmetic graphs, International Journal of Computer Applications (0975-8887) 29(3) (2011).

[6] Dr P. Bhaskarudu, Some results on kronecker product of two graphs, International Journal of Mathematics trends and Technology- 3(1) (2012).

[7] K. V. Suryanarayana Rao and V. Sreeenivasan, The split domination in product graphs, I. J. Information Engineering and Electronic Business 4 (2013), 51-57.

[8] Venkata Suryanarayana Rao Kavaturi and Srinivasan Vangipuram, The annihilator domination in some standard graphs and arithmetic graphs, International Journal of Pure and Applied Mathematics 106(8) (2016), 123-135.

[9] P. Aparna, K. V. Suryanarayana Rao and E. Keshava Reddy, Generalized arithmetic graphs with equal and unequal powers of annihilator domination number, Communications in Mathematics and Applications 12(1) (2021), 171-187. ISSN0975-8607 (online);0976-5905.