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TOTALLY MAGIC CORDIAL LABELING ON ZERO DIVISOR GRAPHS

A. LOURDUSAMY, E. VERONISHA and F. JOY BEAULA

[1]Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai, Tamil Nadu, India
E-mail: lourdusamy15@gmail.com

[2]Research Scholar, Center Reg. No. 19211282092009
[3]PG and Research Reg. No: 20211282092004
Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai-627002
Manonmaniam Sundaranar University
Abisekapatti-627012, Tamilnadu, India
E-mail: nishaedwin1705@gmail.com
joybeaula@gmail.com

Abstract

A function $g : V \cup E \to \{0, 1\}$ such that $g(a) + g(b) + g(ab) \equiv c(\text{mod } 2)$ for all $ab \in E(G)$ and $| m_g(0) - m_g(1) | \leq 1$, where $m_g(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label i and c is the constant is called totally magic cordial labeling (TMC). Here, we prove that $\Omega(Z_{2p}), \Omega(Z_{3p}), \Omega(Z_{4p}), \Omega(Z_{pq})$, join of two zero-divisor graphs and product of zero-divisor graphs are TMC.

1. Introduction

Let G be a simple, finite and undirected graph. Since then many different types of graph labeling techniques have been investigated and over 2000 papers have been published in this area. The reader can refer to Gallian [4] for a survey of labeling.

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Definition 1.1. A function $g : V \cup E \to \{0, 1\}$ such that $g(a) + g(b) + g(ab) \equiv c(\text{mod } 2)$ for all $ab \in E(G)$ and $| m_g(0) - m_g(1) | \leq 1,$ where $m_g(i)(i = 0, 1)$ is the sum of the number of vertices and edges with label i and c is the constant is called totally magic cordial labelling (TMC). A graph with a totally magic cordial labeling is called a totally magic cordial graph.

Beck [3] introduced zero-divisor graph. Then Anderson and Livingston [1] modified it.

Definition 1.2. Take C to be a commutative ring with non-zero identity. Let $D(C)$ be set of all zero-divisors in C. Let $D^*(C) = D(C) \setminus \{0\}$. Now $\Omega(C)$ the zero-divisor graph of C, is the simple undirected graph with vertex set $D^*(C)$ and the edges are defined as xy where x and y are distinct vertices such that $xy = 0$.

Many researchers are interested in the structure of zero-divisor graph. More specifically many authors studied about connectedness, diameter, girth, domination, Eulerian and Hamiltonian nature of it. Tamizh Chelvam et al. [13] proposed graph labeling related to zero-divisors in a commutative ring. More specifically, they obtained certain labeling for certain class of zero-divisor graphs corresponding to finite rings.

Based on these concepts, we give several results on TMC for zero-divisor graphs. Here we study the TMC of $\Omega(Z_{2p}), \Omega(Z_{3p}), \Omega(Z_{4p}), \Omega(Z_{pq}),$ join of zero-divisor graphs and product of zero-divisor graphs.

Definition 1.3. The join $G_1 + G_2$, of two graphs G_1 and G_2 is defined as follows: $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}$.

2. TMC on Zero-Divisor Graphs

Theorem 2.1. *If p is a prime number with $p > 2$, then $\Omega(Z_{2p})$ is TMC.*

Proof. Usual notation $\Omega(Z_{2p})$ is the zero divisor graph of Z_{2p}, where the prime number p satisfies $p > 2$. Clearly $D^*(Z_{2p}) = \{2, 4, \ldots, 2(p-1), p\}$.

Let $V(\Omega(Z_{2p})) = \{v_1, \ldots, v_{p-1}, v_p\}$ and $E(\Omega(Z_{2p})) = \{v_iv_p : 1 \le i \le p-1\}$. Note that, $|V(\Omega(Z_{2p}))| = p$ and $|E(\Omega(Z_{2p}))| = p-1$.

We define $g : V(\Omega(Z_{2p})) \cup E(\Omega(Z_{2p})) \to \{0, 1\}$ as follows:

$g(v_i) = 1$ where i is odd with $1 \le i \le p-1$,

$g(v_i) = 0$ where i is even with $1 \le i \le p-1$,

$g(v_p) = 1$,

$g(v_iv_p) = 1$ where i is odd with $1 \le i \le p-1$,

$g(v_iv_p) = 0$ where i is even and $1 \le i \le p-1$.

Clearly, $m_g(0) = p-1$. Also $m_g(1) = p$. Thus, $\Omega(Z_{2p})$ for $p > 2$ is TMC with $c = 1$.

Example 2.1. $D^*(Z_{14}) = \{2, 4, 6, 7, 8, 10, 12\}$. The elements are represented by the vertices $v_1 = 2, v_2 = 4, v_3 = 6, v_4 = 8, v_5 = 10, v_6 = 12$ and $v_7 = 7$. A TMC labeling of $\Omega(Z_{14})$ is given in Figure 1.

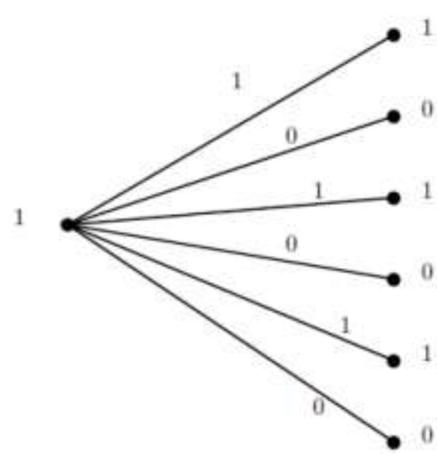

Figure 1.

Theorem 2.2. *If p is a prime number with* $p \ge 2$, *then* $\Omega(Z_{3p})$ *is TMC.*

Proof. Here $\Omega(Z_{3p})$ is the zero divisor graph of Z_{3p}, where p is prime.

Case 1. $p = 2$

$D^*(Z_6) = \{2, 3, 4\}$. Then the graph $\Omega(Z_6)$ is a path on three vertices. Obviously, it is TMC.

Case 2. If $p = 3$, $D^*(Z_9) = \{3, 6\}$. Then the graph $\Omega(Z_9)$ is a path on two vertices. Obviously, it is TMC.

Case 3. $p > 3$

Then $D^*(Z_{3p}) = \{p, 2p\} \cup \{3, 6, \ldots, 3(p-1)\} = \{u_1, u_2\} \cup \{v_1, \ldots, v_{p-1}\}$. Also, the edge set is $E(\Omega(Z_{3p})) = \{u_1 v_i, u_2 v_i : 1 \le i \le p-1\}$. Therefore, $|V(\Omega(Z_{3p}))| = p+1$ and $|E(\Omega(Z_{3p}))| = 2p-2$.

We define the labeling $g : V(\Omega(Z_{3p})) \cup E(\Omega(Z_{3p})) \to \{0, 1\}$ as follows:

$g(u_1) = 1$, $g(u_2) = 0$,

$g(v_i) = 1$ where $1 \le i \le p-1$ with i be odd;

$g(v_i) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_1 v_i) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_2 v_i) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_2 v_i) = 0$ where i is odd with $1 \le i \le p-1$,

$g(u_2 v_i) = 1$ where i is even with $1 \le i \le p-1$.

Clearly, $m_g(0) = m_g(1) = (3p-1)/2$. Hence, $\Omega(Z_{3p})$ for $p \ge 2$ is TMC with $c = 1$.

Theorem 2.3. *If p is a prime number with $p \ge 2$, then $\Omega(Z_{4p})$ is TMC.*

Proof. Here $\Omega(Z_{4p})$ be a zero divisor graph of Z_{4p}, where p is prime.

Case 1. $p = 2$

$D^*(Z_8) = \{2, 4, 6\}$. The graph $\Omega(Z_8)$ is a path on three vertices. So it is TMC.

Case 2. $p \geq 3$

We partition the vertex set of $\Omega(Z_{4p})$ as $V_1 = \{p, 2p, 3p\} = \{u_1, u_2, u_3\}$ and $V_2 = \{2, 4, \ldots, 2(p-1), 2(p+1), \ldots, 2(2p-1)\} = \{v_1, v_2, \ldots, v_{p-1}, v_{p+1}, \ldots, v_{2p-1}\}$. The edge set is $E(\Omega(Z_{4p})) = \{u_1v_2, u_1v_4, \ldots, u_1v_{p-1}, u_1v_{p+1}, \ldots, u_1v_{2p-2}, u_2v_1, u_2v_2, u_2v_3, \ldots, u_2v_{p-1}, u_2v_{p+1}, \ldots, u_2v_{2p-1}, u_3v_2, u_3v_2, u_3v_4, \ldots, u_3v_{p-1}, u_3v_{p+1}, \ldots, u_3v_{2p-2}\}$. Therefore, $|V(\Omega(Z_{4p}))| = 2p+1$ and $|E(\Omega(Z_{4p}))| = 4p-4$.

Define $g : V(\Omega(Z_{4p})) \cup E(\Omega(Z_{4p})) \to \{0, 1\}$ as:

$g(u_1) = 1,$

$g(u_2) = 1,$

$g(u_3) = 0,$

$g(v_i) = 0$ where i is odd with $1 \leq i \leq p-1,$

$g(v_i) = 1$ where i is even with $1 \leq i \leq p-1,$

$g(v_i) = 1$ where i is even with $p+1 \leq i \leq 2p-1,$

$g(v_i) = 0$ where i is odd with $p+1 \leq i \leq 2p-1,$

$g(u_1v_i) = 1,$ where i is even with $1 \leq i \leq 2p-2,$

$g(u_3v_i) = 0,$ where i is even with $1 \leq i \leq 2p-2,$

$g(u_2v_i) = 0,$ where i is odd with $1 \leq i \leq p-1,$

$g(u_2v_i) = 0,$ where i is even with $1 \leq i \leq p-1,$

$g(u_2v_i) = 0$ where i is even with $p+1 \leq i \leq 2p-1,$

$g(u_2v_i) = 0$ where i is odd with $p+1 \leq i \leq 2p-1.$

Clearly, $m_g(0) = 3p-2$ and $m_g(1) = 3p-1$. Hence, $\Omega(Z_{4p})$ for $p \geq 2$ is TMC with $c = 1$.

Theorem 2.4. *Let p and q be two distinct primes with* $p > 2$ *and* $p < q$. *Then the zero divisor graph* $\Omega(Z_{pq})$ *is TMC.*

Proof. $V(\Omega(Z_{pq}))$ is partitioned into V_1 and V_2 where

$V_1 = \{p, 2p, 3p, \ldots, (q-1)p\} = \{u_1, u_2, \ldots, u_{q-1}\}$ and

$V_2 = \{q, 2q, 3q, \ldots, (p-1)q\} = \{v_1, v_2, \ldots, v_{p-1}\}$.

The edge set is $E((Z_{pq})) = \{v_j u_i : u_i \in V_1$ and $v_j \in V_2, 1 \leq j \leq p-1, 1 \leq q-1\}$.

Therefore, $|V(\Omega(Z_{qp}))| = (q-1) + (p-1) = p + q - 2$ and $|E(\Omega(Z_{4p}))| = (q-1)(p-1)$.

Define $g : V(\Omega(Z_{pq})) \cup E(\Omega(Z_{pq})) \to \{0, 1\}$ as follows:

$g(v_j) = 1$ where j is odd with $1 \leq j \leq p-1$,

$g(v_j) = 0$ where j is even with $1 \leq j \leq p-1$,

$g(u_i) = 1$ where i is odd with $1 \leq i \leq q-1$,

$g(u_i) = 0$ where i is even with $1 \leq i \leq q-1$,

for $1 \leq j \leq p-1$ and $1 \leq i \leq q-1$,

$g(v_j u_i) = 1$ where i and j are odd;

$g(v_j u_i) = 0$ where i is odd with j is even;

$g(v_j u_i) = 0$ where i is even with j is odd;

$g(v_j u_i) = 1$ where i and j are even.

Clearly $m_g(0) = m_g(1) = (pq-1)/2$. So $\Omega(Z_{pq})$ for $p < q$ is TMC with $c = 1$.

3. TMC on Join of Zero-Divisor Graphs

Theorem 3.1. *If p is a prime number with* $p > 2$, *then* $\Omega(Z_{2p}) + \Omega(Z_4)$ *is TMC.*

Proof. Take $G = \Omega(Z_{2p}) + \Omega(Z_4)$. Then $V(G) = \{2, 4, \ldots, 2(p-1), p\} \cup \{x : x = 2 \in Z_4\} = \{u_1, \ldots, u_p, x\}$ and $E(G) = \{u_i u_p, u_i x, u_p x : 1 \leq i \leq p-1\}$. Note that, $|V(G)| = p+1$ and $|E(G)| = 2p-1$.

The function $g : V(G) \cup E(G) \to \{0, 1\}$ is:

$g(u_p) = 0,\ g(x) = 1,$

$g(u_i) = 1$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i) = 0$ where i is even with $1 \leq i \leq p-1$,

$g(u_i u_p) = 0$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i u_p) = 1$ where i is even with $1 \leq i \leq p-1$,

$g(u_i x) = 1$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i x) = 0$ where i is even with $1 \leq i \leq p-1$,

$g(u_p x) = 0.$

Note that, $m_g(0) = (3p+1)/2$ and $m_g(1) = (3p+1)/2$. Hence $\Omega(Z_{2p}) + \Omega(Z_4)$ for $p > 2$ is TMC with $c = 1$.

Theorem 3.2. *If p is a prime number with $p > 2$, then $\Omega(Z_{2p}) + \Omega(Z_9)$ is TMC.*

Proof. Take $G = \Omega(Z_{2p}) + \Omega(Z_9)$. The vertex set $V(G) = \{2, 4, \ldots, 2(p-1), p\} \cup \{x,\ y : x = 3,\ y = 6 \in Z_9\} = \{u_1, \ldots, u_p, x, y\}$. The edge set is $E(G) = \{u_i u_p, u_i x, u_i y, u_p x, u_p y, xy : 1 \leq i \leq p-1\}$. Therefore, $|V(G)| = p+2$ and $|E(G)| = 3p$.

The mapping $g : V(G) \cup E(G) \to \{0, 1\}$ is:

$g(x) = 1,\ g(y) = 1,\ g(u_p) = 0,$

$g(u_i) = 1$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i) = 0$ where i is even with $1 \leq i \leq p-1$,

$g(u_i u_p) = 0$ where i is odd with $1 \le i \le p-1$,

$g(u_i u_p) = 1$ where i is even with $1 \le i \le p-1$,

$g(u_i x) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i x) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_i y) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i y) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_p x) = 0,\ f(u_p y) = 0,\ f(xy) = 1.$

From the above labeling, we get $m_g(0) = m_g(1) = 2p+1$. Hence $\Omega(Z_{2p}) + \Omega(Z_9)$ for $p > 2$ is TMC with $c = 1$.

Theorem 3.3. *If p is a prime number with* $p > 2$, *then* $\Omega(Z_{2p}) + \Omega(Z_6)$ *is TMC.*

Proof. Take $G = \Omega(Z_{2p}) + \Omega(Z_6)$. Let $V(G) = \{2, 4, \ldots, 2(p-1), p\} \cup \{x, y, z : x = 2, y = 3, z = 4 \in Z_6\} = \{u_1, \ldots, u_p, x, y, z\}$ and $E(G) = \{u_i u_p, u_i x, u_p x, u_i y, u_p y, u_i z, u_p z, xy, yz : 1 \le i \le p-1\}$. Therefore, $|V(G)| = p+3$ and $|E(G)| = 4p+1$.

Define $g : V(G) \cup E(G) \to \{0, 1\}$ as follows:

$g(u_p) = 1,\ g(x) = 1,\ g(y) = 0,\ g(z) = 0,$

$g(u_i) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_i u_p) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i u_p) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_i x) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i x) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_i y) = 0$ where i is odd with $1 \le i \le p-1$,

$g(u_i y) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i z) = 0$ where i is odd with $1 \le i \le p-1$,

$g(u_i z) = 1$ where i is even with $1 \le i \le p-1$,

$g(u\,x) = 1,\ g(u\,y) = 0,\ g(u\,z) = 0,\ g(yz) = 1,\ g(xy) = 0.$

Observe that, $m_g(0) = (5p+5)/2$ and $m_g(1) = (5p+3)/2$. Hence $\Omega(Z_{2p}) + \Omega(Z_6)$ for $p > 2$ is TMC with $c = 1$.

Corollary 3.1. *If p is a prime number with $p > 2$, then $\overline{\Omega(Z_{2p})} + \Omega(Z_4)$ is TMC.*

Proof. Take $G = \overline{\Omega(Zp^2)} + \Omega(Z_4)$. Clearly G is isomorphic to the graph $\Omega(Z_{2p})$. Hence $|V(G)| = 2n+4$ and $|E(G)| = 6n$. By using the labeling given in Theorem 2.1, it is easy to verify that $|m_g(0) - m_g(1)| \le 1$. Hence $\overline{\Omega(Zp^2)} + \Omega(Z_4)$ for $p > 2$ is TMC with $c = 1$.

Theorem 3.4. *If p is a prime number with $p > 2$, then $\overline{\Omega(Zp^2)} + \Omega(Z_6)$ is TMC.*

Proof. Take $G = \overline{\Omega(Zp^2)} + \Omega(Z_6)$. The vertex set is $V(G) = \{u_1, \ldots, u_{p-1}\} \cup \{x, y, z : x = 2, y = 3, z = 4 \in Z_6\}$ $= \{u_1, \ldots, u_{p-1}, x, y, z\}$. The edge set is $E(G) = \{u_i x, u_i y, u_i z, xy, yz : 1 \le i \le p-1\}$. Therefore, $|V(G)| = p+2$ and $|E(G)| = 3p-1$.

The map $g : V(G) \cup E(G) \to \{0, 1\}$ is:

$g(x) = 1,\ g(y) = 1,\ g(z) = 0,$

$g(u_i) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_i x) = 1$ where i is odd with $1 \le i \le p-1$,

$g(u_i x) = 0$ where i is even with $1 \le i \le p-1$,

$g(u_i y) = 1$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i y) = 0$ where i is even with $1 \leq i \leq p-1$,

$g(u_i z) = 0$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i z) = 1$ where i is even with $1 \leq i \leq p-1$,

$g(xy) = 1,\ g(yz) = 0.$

Clearly, $m_g(0) = 2p$ and $m_g(1) = 2p+1$. Hence $\overline{\Omega(Zp^2)} + \Omega(Z_6)$ for $p > 2$ is TMC with $c = 1$.

Theorem 3.5. *If p is a prime number with $p > 2$, then $\overline{\Omega(Zp^2)} + \Omega(Z_9)$ is TMC.*

Proof. Take $G = \overline{\Omega(Zp^2)} + \Omega(Z_9)$. Let $V(G) = \{u_1, \ldots, u_{p-1}\} \cup \{x, y : x = 3, y = 6 \in Z_9\} = \{u_1, \ldots, u_{p-1}, x, y\}$ and $E(G) = \{u_i x, u_i y, xy : 1 \leq i \leq p-1\}$. Therefore, $|V(G)| = p+1$ and $|E(G)| = 2p-1$.

Define $g : V(G) \cup E(G) \to \{0, 1\}$ as follows:

$g(x) = 1,\ g(y) = 0,$

$g(u_i) = 1$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i) = 0$ where i is even with $1 \leq i \leq p-1$,

$g(u_i x) = 0$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i x) = 1$ where i is even with $1 \leq i \leq p-1$,

$g(u_i y) = 0$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i y) = 1$ where i is even with $1 \leq i \leq p-1$,

$g(xy) = 0.$

Note that, $m_g(0) = (3p+1)/2$ and $m_g(1) = (3p+1)/2$. Hence $\overline{\Omega(Zp^2)} + \Omega(Z_9)$ for $p > 2$ is TMC with $c = 1$.

Theorem 3.6. *If p is a prime number with* $p > 2$, *then* $\overline{\Omega(Zp^2)} + \overline{\Omega(Z_6)}$ *is TMC.*

Proof. Take $G = \overline{\Omega(Zp^2)} + \overline{\Omega(Z_6)}$. Then G is a graph obtained from the graph $\overline{\Omega(Zp^2)} + \Omega(Z_6)$ by deleting the edges xy and yz and adding an edge xz. By using the labeling given in Theorem 3.4 and $g(xy) = 0$, we get $m_g(0) = 2p$ and $m_g(1) = 2p$. Hence $\overline{\Omega(Zp^2)} + \overline{\Omega(Z_6)}$ for $p > 2$ is TMC with $c = 1$.

Theorem 3.7. *If p is a prime number with* $p > 2$, *then* $\overline{\Omega(Zp^2)} + \overline{\Omega(Z_9)}$ *is TMC.*

Proof. Let $G = \overline{\Omega(Zp^2)} + \overline{\Omega(Z_9)}$. Then G is isomorphic to a graph obtained from the graph $\overline{\Omega(Zp^2)} + \Omega(Z_9)$ by deleting an edge xy. By using the labeling given in Theorem 3.5, we get $m_g(0) = (3p-1)/2$ and $m_g(1) = (3p-1)/2$. Hence $\overline{\Omega(Zp^2)} + \overline{\Omega(Z_9)}$ for $p > 2$ is TMC with $c = 1$.

Theorem 3.8. *If p is a prime number with* $p \geq 3$ *and* $m \geq 1$ *is an integer, the join graph* $\overline{\Omega(Zp^2)} + m\Omega(Z_4)$ *is TMC.*

Proof. Take $\overline{G = \Omega(Zp^2)} + m\Omega(Z_4)$. Take $V(G) = \{u_1, u_2, \ldots, u_{p-1}, v_1, v_2, \ldots, v_m\}$ and $E(G) = \{u_i v_j : 1 \leq i \leq p-1, 1 \leq j \leq m\}$. Therefore, $|V(G)| = p + m - 1$ and $|E(G)| = m(p-1)$.

Define $g : V(G) \cup E(G) \to \{0, 1\}$ as follows:

$g(u_i) = 1$ where i is odd with $1 \leq i \leq p-1$,

$g(u_i) = 0$ where i is even with $1 \leq i \leq p-1$,

$g(v_j) = 1$ where j is odd with $1 \leq j \leq m-1$,

$g(v_j) = 0$ where j is even with $1 \leq j \leq m-1$, for $1 \leq i \leq p-1$ and $1 \leq j \leq m-1$,

$g(v_i v_j) = 1$ where i and j are odd;

$g(u_i v_j) = 0$ where i odd with j even;

$g(u_i v_j) = 0$ where i even with j odd;

$g(u_i v_j) = 1$ where i and j are even.

Table 1. $p \equiv 0, 1, 2 \pmod 3$.

Nature of m	$m_g(0)$	$m_g(1)$
$m \equiv 1, 5, 3 \pmod 6$	$(p + mp - 2)/2$	$(p + mp)/2$
$m \equiv 2, 4, 0 \pmod 6$	$(p + mp - 1)/2$	$(p + mp - 1)/2$

Hence, Table 1 shows that G is TMC with $c = 1$.

Corollary 3.2. *If p is a prime number with $p > 2$, then $\overline{\Omega(Zp^2)} + \overline{\Omega(Z_4)}$ is TMC.*

Proof. Note that, $\overline{\Omega(Z_4)} + \Omega(Z_4)$. By Theorem 3.1, $\overline{\Omega(Zp^2)} + \overline{\Omega(Z_4)}$ is TMC with $c = 1$.

Corollary 3.3. *If p is a prime number with $p > 2$, then $\Omega(Z_{2p}) + \overline{\Omega(Z_4)}$* is TMC.

Proof. Clearly $\overline{\Omega(Z_4)} + \Omega(Z_4)$. By Theorem 3.1, $\Omega(Z_{2p}) + \overline{\Omega(Z_4)}$ is TMC with $c = 1$.

Theorem 3.9. *If p is a prime number with $p > 2$, then $\Omega(Z_{2p}) + \overline{\Omega(Z_9)}$ is TMC.*

Proof. Take $G = \Omega(Z_{2p}) + \overline{\Omega(Z_9)}$. Clearly G is isomorphic to a graph obtained from the graph $\Omega(Z_{2p}) + \Omega(Z_9)$ by deleting an edge xy. By using the labeling given in Theorem 3.2, we get $m_g(0) = 2p + 1$ and $m_g(1) = 2p$. Hence $\Omega(Z_{2p}) + \overline{\Omega(Z_9)}$ for $p > 2$ is TMC with $c = 1$.

4. TMC on Product of two Zero-Divisor Graphs

Theorem 4.1. *If p is a prime number with* $p > 2$, *then the product graph* $\Omega(Z_{2p}) + \Omega(Z_4)$ *is TMC.*

Proof. Take $G = \Omega(Z_{2p}) \times \Omega(Z_4)$. Then G is isomorphic to the graph $\Omega(Z_{2p})$. By using the labeling given in Theorem 2.1, it is easy to verify that $| m_g(0) - m_g(1) | \leq 1$. Hence $\Omega(Z_{2p}) \times \Omega(Z_4)$ is TMC with $c = 1$.

Theorem 4.2. *If p is a prime number with* $p > 2$, *then* $\Omega(Z_{2p}) \times \Omega(Z_9)$ *is TMC.*

Proof. Take $G = \Omega(Z_{2p})$ and $H = \Omega(Z_9)$. Let $V(G) = \{2, 4, \ldots, 2(p-1), p\} = \{u_1, u_2, \ldots, u_{p-1}, u_p\}$ and $V(H) = \{3, 6\} = \{x, y\}$. Now $V(G \times H) = \{(u_i, x), (u_i, y) : 1 \leq i \leq p\}$ and $E(G \times H) = \{(u_i, x)(u_i, y), (u_i, x)(u_p, x), (u_i, y), (u_p, y), (u_p, x)(u_p, y) : 1 \leq i \leq p-1\}$. Then the number of vertices is $2p$ and that of edges is $3p - 2$.

Define $g : V(G) \cup E(G) \to \{0, 1\}$ as:

$g(u_p, y) = 0, f(u_p, x) = 0,$

$g(u_i, x) = 1$ where i odd with $1 \leq i \leq p-1$,

$g(u_i, x) = 0$ where i even with $1 \leq i \leq p-1$,

$g(u_i, y) = 1$ where $i = 1, 2$,

$g(u_i, y) = 0$ where $3 \leq i \leq p-1$,

$g((u_i, x)(u_i, y)) = 1$ where $i = 1$,

$g((u_i, x)(u_i, y)) = 0$ where $i = 2$,

$g((u_i, x)(u_i, y)) = 0$ where i odd with $3 \leq i \leq p-1$,

$g((u_i, x)(u_i, y)) = 1$ where i even with $3 \leq i \leq p-1$,

$g((u_p, x)(u_i, x)) = 0$ where i odd with $1 \leq i \leq p-1$,

$g((u_p, x)(u_i, x)) = 1$ where i even with $1 \leq i \leq p-1$,

$g((u_p, y)(u_i, y)) = 0$ where $i = 1, 2$,

$g((u_p, y)(u_i, y)) = 1$ where $3 \leq i \leq p-1$,

$g((u_p, x)(u_p, y)) = 1$.

Clearly $m_g(0) = (5p-1)/2$ and $m_g(1) = (5p-1)/2$. Hence $\Omega(Z_{2p}) \times \Omega(Z_9)$ for $p > 2$ is TMC with $c = 1$.

Theorem 4.3. *If p is a prime number with $p > 2$, then the product graph $\Omega(Z_{2p}) \times \Omega(Z_6)$ is TMC.*

Proof. Let $G = \Omega(Z_{2p})$ and $H = \Omega(Z_6)$. Let $V(G) = \{2, 4, \ldots, 2(p-1), p\} = \{u_1, u_2, \ldots, u_{p-1}, u_p\}$ and $V(H) = \{2, 3, 4\}$ $\{x, y, z\}$. Then $V(G \times H) = \{(u_i, x), (u_i, y), (u_i, z) : 1 \leq i \leq p\}$ and $E(G \times H)$ $= \{(u_i, x)(u_i, y), (u_i, y)(u_i, z), (u_i, x)(u_p, x), (u_i, y)(u_p, y), (u_i, z)(u_p, z),$ $(u_i, x)(u_p, y), (u_i, y), (u_p, z) : 1 \leq i \leq p-1\}$. Therefore, $|V(G \times H)| = 3p$ and $|E(G \times H)| = 5p - 3$. Define $g : V(G) \cup E(G) \to \{0, 1\}$ as follows:

$g(u_p, y) = 0$, $g(u_p, x) = 0$, $(u_p, z) = 1$,

$g(u_i, x) = 1$ where i odd with $1 \leq i \leq p-1$,

$g(u_i, x) = 0$ where i even with $1 \leq i \leq p-1$,

$g(u_i, y) = 1$ if $1 \leq i \leq p-1$,

$g(u_i, z) = 1$ where i odd with $1 \leq i \leq p-1$,

$g(u_i, z) = 0$ where i even with $1 \leq i \leq p-1$,

$g((u_p, x)(u_p, y)) = 1$ where i odd with $1 \leq i \leq p-1$,

$g((u_i, x)(u_i, y)) = 0$ where i even with $1 \leq i \leq p-1$,

$g((u_i, y)(u_i, z)) = 1$ where i odd with $1 \leq i \leq p-1$,

$g((u_i, y)(u_i, z)) = 0$ where i even with $1 \leq i \leq p-1$,

$g((u_i, x)(u_i, x)) = 0$ where i odd with $1 \leq i \leq p-1$,

$g((u_p, x)(u_i, x)) = 1$ where i even with $1 \leq i \leq p-1$,

$g((u_p, y)(u_i, y)) = 0$ if $1 \leq i \leq p-1$,

$g((u_p, z)(u_i, z)) = 1$ where i odd with $1 \leq i \leq p-1$,

$g((u_p, z)(u_i, z)) = 0$ where i even with $1 \leq i \leq p-1$,

$g((u_p, x)(u_p, y)) = 1$, $g((u_p, y)(u_p, z)) = 0$.

Note that, $m_g(0) = 4p-1$ and $m_g(1) = 4p-2$. Hence $\Omega(Z_{2p}) \times \Omega(Z_6)$ for $p > 2$ is TMC with $c = 1$.

5. Conclusion

We have proved that $\Omega(Z_{2p}), \Omega(Z_{3p}), \Omega(Z_{4p})$ and $\Omega(Z_{pq})$ are TMC graphs. We have proved that the join graphs $\Omega(Z_{2p}) + \Omega(Z_4)$, $\Omega(Z_{2p}) + \Omega(Z_9)$, $\Omega(Z_{2p}) + \Omega(Z_6)$, $\overline{\Omega(Z_{p^2})} + \Omega(Z_4)$, $\overline{\Omega(Z_{p^2})} + \Omega(Z_6)$, $\overline{\Omega(Z_{p^2})} + \Omega(Z_9)$, $\overline{\Omega(Z_{p^2})} + m\Omega(Z_4)$, $\overline{\Omega(Z_{p^2})} + \overline{\Omega(Z_4)}$, $\overline{\Omega(Z_{p^2})} + \overline{\Omega(Z_6)}$, $\overline{\Omega(Z_{p^2})} + \overline{\Omega(Z_9)}$, $\Omega(Z_{2p}) + \overline{\Omega(Z_4)}$, $\Omega(Z_{2p}) + \overline{\Omega(Z_6)}$ and $\Omega(Z_{2p}) + \overline{\Omega(Z_9)}$ are TMC graphs. Also, we have proved that $\Omega(Z_{2p}) \times \Omega(Z_4)$, $\Omega(Z_{2p}) \times \Omega(Z_6)$ and $\Omega(Z_{2p}) \times \Omega(Z_9)$ are TMC graphs. We further investigating total 3 sum cordial labeling on zero divisor graphs.

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