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ON PELL'S EQUATION $x^2 - py^2 - 1 = 0, \forall\ 1 < p < 100$

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Abstract

For the prime $p > 1$ and positive integers x and y, we consider Pell's equation $D : x^2 - py^2 - 1 = 0$. We discussed the solvability of the Diophantine equation D for every prime p between 1 and 100.

1. Introduction

Pell's equation is of the form $x^2 - ny^2 = 1$, where $n > 0$ is a non-perfect square integer and x and y both are integers and it is also known as Pell-Fermat equation in Diophantine equation and the negative Pell's equation can be written as $x^2 - ny^2 = -1$. There are many authors, they discussed the solvability of Pell's equation as well as the negative of Pell's equation [1].

S. Gupta and S. Kumar (2020), discussed the polynomial solution of the hyperbolic Diophantine equation $F^2(u) - RG^2(u) = 1$, where $R > 0$ is any polynomial. After the discussed they obtained some polynomial solutions and construct a solution table for the different values of the parameters [2]. S. Gupta and S. Kumar (2021), discussed the negative Pell's equation $F^2(u) - RG^2(u) = -1$, where $R > 0$ is any polynomial. After the discussed they obtained some polynomial solutions and construct a solution table for the different values of the parameters [3]. S. Gupta, S. Kumar, and H. Kishan

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(2018, 2020), discussed on some non-linear Diophantine equation $61^x + 67^y = z^2$, $67^x + 73^y = z^2$, $p^x + (p+2)^y = z^2$, $p^x + (p+6)^y = z^2$ and $n^x + (n+4r)^y = z^m$, where $x, y,$ and z are non-negative integers and $p, p+2, p+6$ all are primes and n, m, r are positive integers and obtained that all equations have no integer solution [4]-[7]. J. Kannan, Manju Somanath, and K. Raja [8] discussed the solution of the negative Pell's equation $x^2 = 41y^2 - 43^t, t \in N.$ They discussed on the solvability for some particular values of t and gave recurrence relation on the solution [8]. Ahmet Tekcan [9], discussed Pell's equation $x^2 - Dy^2 = \pm 4,$ where $D \neq 1$ be any positive non-square integer. He obtained some formulas for the integer solution of the equation as considered [9].

In this paper, for prime $p, 1 < p < 100,$ we consider the Diophantine equation

$$x^2 - py^2 - 1 = 0$$

where x and y both are positive integers. We discussed on the solvability of this equation and the only integer solution considered as the solution of the equation. We obtained fundamental solutions for this equation by using the CFE of $\sqrt{p}$ We find the penultimate convergent of the CFE of $\sqrt{p}$. for the values of $p = 3, 3, 11, 19, 47, 67, 79, 83.$

1. Notations

The meaning of used symbols in this paper are given below:

D : This will be used for Pell's equation $x^2 - py^2 - 1 = 0.$

N : Set of all positive integers.

$x, y:$ Positive integers.

p : Prime number.

$\in:$ Belongs to, (i.e. this is used to represent one or more elements in the set).

$\forall$: For all.

n, R : Non-perfect square integer.

CF : Continued fraction.

CFE : Continued fraction expansion.

3. Preliminaries

Definition: Continued Fraction (CF) [2], [10]: Let a_1 be an arbitrary integer and $a_2, a_3, \ldots, a_n$ be a positive integer then the expression

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots \atop {\cdots \atop {\cdots \atop \cdots + \frac{1}{a_n}}}}}}$$

is called a CF.

It is also written as $[a_1, a_2, a_3, \ldots, a_n] = a_1 \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \frac{1}{a_n}$

Where $a_1 =$ First partial quotient, $a_2 =$ Second partial quotient $, \ldots, a_n =$ nth partial quotient.

Definition. Penultimate Convergent [2], [10]: Let $x = [a_1, a_2, a_3, \ldots, a_n]$ and let $0 < n \leq N$.

Then $[a_1], [a_1, a_2], [a_1, a_2, a_3], \ldots, [a_1, a_2, a_3, \ldots, a_n]$ are called the first, second, third, $\ldots$, and nth convergent respectively to the CF. Let $C_n \geq 0$ be a sequence of n^{th} convergent of continued function expression $[a_1, a_2, a_3, \ldots, a_n]$ of $x \cdot C_n$ is a fraction which is of the form $\frac{p_n}{q_n}$, where p_n and q_n are integer and $q_n \neq 0$.

In a CF the first convergent is $\frac{p_1}{q_1} = a_1$ and the second convergent is $\frac{p_1}{q_1} = \frac{a_1 a_2 + 1}{a_2}$.

Let $n \geq 2$, then the nth convergent $\frac{p_2}{q_1}$ of a CF is given by the formulating

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$p_n = a_n q_{n-1} + p_{n-2}$$

where $p_1 = a_1$, $p_2 = a_1 a_2 + 1$, $q_1 = 1$, $q_2 = a_2$.

Theorem 3.1 [10]. *Let* $R > 0$ *then* $(x_0, y_0) = (0, 1)$ *is trivial solution of Pell's equation* $x^2 - Ry^2 = 1$.

Theorem 3.2 [10]. *Let* $R > 0$ *then Pell's equation* $x^2 - Ry^2 = 1$. *has infinite integral solutions*.

Theorem 3.3 [8]. *If* (x_1, y_1) *is the fundamental solution the Diophantine equation* (*Pell's Equation*) $x^2 - Ry^2 = 1$. *Then others solutions are obtained by the relation* $x_r + y_r\sqrt{R} = (x_1 + y_1\sqrt{R})$ *where* $r = 1, 2, 3, \ldots,$ *and* $x_r + y_r\sqrt{R}$ *is the fundamental solution*.

4. Results

Pell's equation $D : x^2 - py^2 - 1 = 0$ (i)

Here, we consider the Diophantine equation D, namely Pell's equation, where x and y both are positive integers and $p > 1$ is the prime.

We know that if the fundamental solution of the Pell equation exists then we can obtain infinite solutions by theorem 3.3. Here we shall discuss the fundamental solution of the equation D, for the particular values of the prime in various cases as given below:

Case I. If choice of prime p is 3

If $p = 3$ then Pell's equation becomes

$$x^2 - 3y^2 - 1 = 0 \qquad \text{(ii)}$$

Let $\alpha = \sqrt{3}$, then $[\sqrt{3}] = 1$.

Now we expand $\sqrt{3}$ into CFE as given below:

We have

$$\alpha = 1 + (\sqrt{3} - 1) \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{3}+1}{2}\right)} = a_1 + \frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{\sqrt{3}+1}{2} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{3}+1}{1}\right)} = a_2 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{\sqrt{3}+1}{1} \Rightarrow 2 + \frac{1}{\left(\frac{\sqrt{3}+1}{2}\right)} = a_3 + \frac{1}{\alpha_3}$$

Therefore the CFE of $\sqrt{3}$ is $[a_1, \overline{a_2, a_3}] = [1, \overline{1, 2}]$.

Let $\frac{u}{v} = \frac{p_2}{q_2}$ is the penultimate convergent of the CFE $[1, \overline{1, 2}]$ of $\sqrt{3}$.

We construct a table for the values of p_n and q_n, where $n = 1, 2, 3$.

Table (I)

n	1	2	3
a_n	1	1	2
p_n	1	2	-
q_n	1	1	-

Therefore the penultimate convergent of $\sqrt{3}$ is

$$\frac{u}{v} = 1 + \frac{1}{1+}\frac{1}{2} = \frac{2}{1}$$

Here we find that $(u + v\sqrt{3})$ is the fundamental solution of (2), where $u = 2$ and $v = 1$. Then others solutions (x_r, y_r) are obtained by the relation $x_r + y_r\sqrt{3} = (2 + \sqrt{3})^r$ where $r = 1, 2, 3, \ldots,$ and so on.

Case II. If choice of prime p is 11

If $p = 11$ then Pell's equation becomes

$$x^2 - 11y^2 - 1 = 0 \tag{iii}$$

Let $\alpha = \sqrt{11}$, then $[\sqrt{11}] = 3$.

Now we expand $\sqrt{11}$ into CFE as given below:

We have

$$\alpha = 3 + (\sqrt{11} - 3) \Rightarrow 3 + \frac{1}{\left(\frac{\sqrt{11} + 3}{2}\right)} = a_1 + \frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{\sqrt{11} + 3}{2} \Rightarrow 3 + \frac{1}{\left(\frac{\sqrt{11} + 3}{1}\right)} = a_2 + \frac{1}{\alpha_2}$$

$$\alpha_3 = \frac{\sqrt{11} + 3}{2} \Rightarrow 6 + \frac{1}{\left(\frac{\sqrt{11} + 3}{1}\right)} = a_3 + \frac{1}{\alpha_3}$$

Therefore the CFE of $\sqrt{11}$ is $[a_1, \overline{a_2, a_3}] = [3, \overline{3, 6}]$.

Let $\frac{u}{v} = \frac{p_2}{q_2}$ is the penultimate convergent of the CFE $[3, \overline{3, 6}]$ of $\sqrt{11}$.

We construct a table for the values of p_n and q_n, where $n = 1\ 2, 3$.

Table (II).

n	1	2	3
a_n	3	3	6
p_n	3	10	-
q_n	1	3	-

Therefore the penultimate convergent of $\sqrt{3}$ is

$$\frac{u}{v} = 3 + \frac{1}{3+}\frac{1}{6} = \frac{10}{3}$$

Here we find that $(u + v\sqrt{11})$ is the fundamental solution of (iii), where $u = 10$ and $v = 3$. Then others solutions (x_r, y_r) are obtained by the relation $x_r + y_r\sqrt{11} = (10 + 3\sqrt{11})^r$ where $r = 1, 2, 3, \ldots,$ and so on.

Case III. If choice of prime p is 19

If $p = 19$ then Pell's equation becomes

$$x^2 - 19y^2 - 1 = 0 \tag{iv}$$

Let $\alpha = \sqrt{19}$, then $[\sqrt{19}] = 4$.

Now we expand $\sqrt{19}$ into CFE as given below:

We have

$$\alpha = 4 + (\sqrt{19} - 4) \Rightarrow 4 + \frac{1}{\left(\frac{\sqrt{19}+4}{3}\right)} = a_1 + \frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{\sqrt{19}+4}{3} \Rightarrow 2 + \frac{1}{\left(\frac{\sqrt{19}+2}{5}\right)} = a_2 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{\sqrt{19}+2}{5} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{19}+3}{2}\right)} = a_3 + \frac{1}{\alpha_3}$$

$$\alpha_3 = \frac{\sqrt{19}+3}{2} \Rightarrow 3 + \frac{1}{\left(\frac{\sqrt{19}+3}{5}\right)} = a_4 + \frac{1}{\alpha_4}$$

$$\alpha_4 = \frac{\sqrt{19}+3}{2} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{19}+2}{3}\right)} = a_5 + \frac{1}{\alpha_5}$$

$$\alpha_5 = \frac{\sqrt{19}+2}{3} \Rightarrow 2 + \frac{1}{\left(\frac{\sqrt{19}+4}{1}\right)} = a_6 + \frac{1}{\alpha_6}$$

$$\alpha_6 = \frac{\sqrt{19}+4}{3} \Rightarrow 2 + \frac{1}{\left(\frac{\sqrt{19}+4}{3}\right)} = a_7 + \frac{1}{\alpha_7}$$

Therefore the CFE of $\sqrt{19}$ is $[a_1, \overline{a_2, a_3, a_4, a_5, a_6, a_7}]$ $= [4, \overline{2, 1, 3, 1, 2, 8}]$.

Let $\frac{u}{v} = \frac{p_6}{q_6}$ is the penultimate convergent of the CFE $= [4, \overline{2, 1, 3, 1, 2, 8}]$ of $\sqrt{19}$.

We construct a table for the values of p_n and q_n, where $n = 1, 2, 3, 7$.

Table (III).

n	1	2	3	4	5	6	7
a_n	4	2	1	3	1	2	8
p_n	4	9	13	48	61	170	-
q_n	1	2	3	11	14	39	-

Therefore the penultimate convergent of $\sqrt{19}$ is

$$\frac{u}{v} = 4 + \frac{1}{2+}\frac{1}{1+}\frac{1}{3+}\frac{1}{1+}\frac{1}{2+}\frac{1}{8} = \frac{170}{39}$$

Here we find that $(u + v\sqrt{19})$ is the fundamental solution of (iv), where $u = 170$ and $v = 39$. Then others solutions (x_r, y_r) are obtained by the relation $x_r + y_r\sqrt{19} = (170 + 39\sqrt{19})^r$ where $r = 1, 2, 3, \ldots,$ and so on.

Case IV. If choice of prime p is 47

If $p = 47$ then Pell's equation becomes

$$x^2 - 47y^2 - 1 = 0 \tag{v}$$

Let $\alpha = \sqrt{47}$, then $\alpha = [\sqrt{47}] = 1$.

Now we expand $\sqrt{47}$ into CFE as given below:

We have

$$\alpha = 6 + (\sqrt{19} - 6) \Rightarrow 6 + \frac{1}{\left(\frac{\sqrt{47}+6}{11}\right)} = a_1 + \frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{\sqrt{47}+6}{11} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{47}+5}{2}\right)} = a_2 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{\sqrt{47}+5}{2} \Rightarrow 5 + \frac{1}{\left(\frac{\sqrt{47}+5}{11}\right)} = a_3 + \frac{1}{\alpha_3}$$

$$\alpha_3 = \frac{\sqrt{47}+5}{11} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{47}+6}{1}\right)} = a_4 + \frac{1}{\alpha_4}$$

$$\alpha_4 = \frac{\sqrt{47}+6}{1} \Rightarrow 12 + \frac{1}{\left(\frac{\sqrt{47}+6}{11}\right)} = a_5 + \frac{1}{\alpha_5}$$

Therefore the CFE of $\sqrt{47}$ is $[a_1, \overline{a_2, a_3, a_4, a_5}] = [6, \overline{1, 5, 1, 12}]$.

Let $\frac{u}{v} = \frac{p_4}{q_4}$ is the penultimate convergent of the CFE $[6, \overline{1, 5, 1, 12}]$ of $\sqrt{47}$.

We construct a table for the values of p_n and q_n, where $n = 1, 2, 3, 4, 5$.

Table (IV).

n	1	2	3	4	5
a_n	6	1	5	1	12
p_n	6	7	41	48	-
q_n	1	1	6	7	-

Therefore the penultimate convergent of $\sqrt{47}$ is

$$\frac{u}{v} = 6 + \frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{12} = \frac{48}{7}$$

Here we find that $(u + v\sqrt{47})$ is the fundamental solution of (v), where $u = 48$ and $v = 7$. Then others solutions (x_r, y_r) are obtained by the relation $x_r + y_r\sqrt{47} = (48 + 7\sqrt{47})^r$ where $r = 1, 2, 3, \ldots,$ and so on.

Case V. If choice of prime p is 67

If $p = 67$ then Pell's equation becomes

$$x^2 - 67y^2 - 1 = 0 \tag{vi}$$

Let $\alpha = \sqrt{67}$, then $[\sqrt{67}] = 8$.

Now we expand $\sqrt{67}$ into CFE as given below:

We have

$$\alpha = 8 + (\sqrt{67} - 8) \Rightarrow 6 + \frac{1}{\left(\frac{\sqrt{67}+8}{3}\right)} = a_1 + \frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{\sqrt{67}+8}{3} \Rightarrow 5 + \frac{1}{\left(\frac{\sqrt{67}+7}{6}\right)} = a_2 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{\sqrt{67}+5}{6} \Rightarrow 2 + \frac{1}{\left(\frac{\sqrt{67}+5}{7}\right)} = a_3 + \frac{1}{\alpha_3}$$

$$\alpha_3 = \frac{\sqrt{67}+5}{7} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{67}+2}{9}\right)} = a_4 + \frac{1}{\alpha_4}$$

$$\alpha_4 = \frac{\sqrt{67}+2}{9} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{67}+7}{2}\right)} = a_5 + \frac{1}{\alpha_5}$$

$$\alpha_5 = \frac{\sqrt{67}+7}{2} \Rightarrow 7 + \frac{1}{\left(\frac{\sqrt{67}+7}{9}\right)} = \alpha_6 + \frac{1}{\alpha_6}$$

$$\alpha_6 = \frac{\sqrt{67}+7}{9} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{67}+2}{7}\right)} = \alpha_7 + \frac{1}{\alpha_7}$$

$$\alpha_7 = \frac{\sqrt{67}+2}{7} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{67}+5}{6}\right)} = \alpha_8 + \frac{1}{\alpha_8}$$

$$\alpha_8 = \frac{\sqrt{67}+5}{6} \Rightarrow 2 + \frac{1}{\left(\frac{\sqrt{67}+7}{3}\right)} = \alpha_9 + \frac{1}{\alpha_9}$$

$$\alpha_9 = \frac{\sqrt{67}+7}{3} \Rightarrow 5 + \frac{1}{\left(\frac{\sqrt{67}+8}{1}\right)} = \alpha_{10} + \frac{1}{\alpha_{10}}$$

$$\alpha_{10} = \frac{\sqrt{67}+8}{1} \Rightarrow 16 + \frac{1}{\left(\frac{\sqrt{67}+8}{3}\right)} = \alpha_{11} + \frac{1}{\alpha_{11}}$$

Therefore the CFE of $\sqrt{67}$ is $[a_1, \overline{a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}}]$ $= [8, \overline{5, 2, 1, 1, 7, 1, 1, 2, 5, 16}]$.

Let $\frac{u}{v} = \frac{p_9}{q_9}$ is the penultimate convergent of the CFE $= [8, \overline{5, 2, 1, 1, 7, 1, 1, 2, 5, 16}]$. of $\sqrt{67}$.

We construct a table for the values of p_n and q_n where $n = 1, 2, 3, \ldots, 11$.

Table (V).

n	1	2	3	4	5	6	7	8	9	10	11
a_n	8	5	2	1	1	7	1	1	2	5	16
p_n	8	41	90	131	221	1678	1899	3577	9053	48842	-
q_n	1	5	11	16	27	205	232	437	1106	5967	-

Therefore the penultimate convergent of $\sqrt{67}$ is

$$\frac{u}{v} = 8 + \frac{1}{5+}\frac{1}{2+}\frac{1}{1+}\frac{1}{1+}\frac{1}{7+}\frac{1}{1+}\frac{1}{1+}\frac{1}{2+}\frac{1}{5+}\frac{1}{16} = \frac{48842}{5967}$$

Here we find that $(u + v\sqrt{67})$ is the fundamental solution of (vi), where $u = 48842$ and $v = 5967$. Then others solutions (x_r, y_r) are obtained by the relation $x_r + y_r\sqrt{67} = (48842 + 5967\sqrt{67})^r$ where $r = 1, 2, 3, \ldots,$ and so on.

Case VI. If choice of prime p is 79

If $p = 79$ then Pell's equation becomes

$$x^2 - 79y^2 - 1 = 0 \tag{vii}$$

Let $\alpha = \sqrt{79}$, then $[\sqrt{79}] = 8$.

Now we expand $\sqrt{79}$ into CFE as given below:

We have

$$\alpha = 8 + (\sqrt{79} - 8) \Rightarrow 8 + \frac{1}{\left(\frac{\sqrt{79}+8}{15}\right)} = a_1 + \frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{\sqrt{79}+8}{15} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{79}+7}{2}\right)} = a_2 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{\sqrt{79}+7}{2} \Rightarrow 7 + \frac{1}{\left(\frac{\sqrt{79}+8}{1}\right)} = a_3 + \frac{1}{\alpha_3}$$

$$\alpha_3 = \frac{\sqrt{79}+7}{15} \Rightarrow 1 + \frac{1}{\left(\frac{\sqrt{79}+8}{1}\right)} = a_4 + \frac{1}{\alpha_4}$$

$$\alpha_4 = \frac{\sqrt{79}+8}{1} \Rightarrow 16 + \frac{1}{\left(\frac{\sqrt{79}+8}{15}\right)} = a_5 + \frac{1}{\alpha_5}$$

Therefore the CFE of $\sqrt{79}$ is $[a_1, \overline{a_2, a_3, a_4, a_5}] = [8, \overline{1, 7, 1, 16}]$.

Let $\frac{u}{v} = \frac{p_4}{q_4}$ is the penultimate convergent of the CFE $[8, \overline{1, 7, 1, 16}]$ of $\sqrt{47}$.

We construct a table for the values of p_n and q_n, where $n = 1, 2, 3, 4, 5$.

Table (VI).

n	1	2	3	4	5
a_n	8	1	7	1	16
p_n	8	9	71	80	-
q_n	1	1	8	9	-

Therefore the penultimate convergent of $\sqrt{79}$ is

$$\frac{u}{v} = 8 + \frac{1}{1+}\frac{1}{7+}\frac{1}{1+}\frac{1}{16} = \frac{80}{9}$$

Here we find that $(u + v\sqrt{79})$ is the fundamental solution of (vii), where $u = 80$ and $v = 9$. Then others solutions (x_r, y_r) are obtained by the relation $x_r + y_r\sqrt{79} = (80 + 9\sqrt{67})^r$ where $r = 1, 2, 3, \ldots,$ and so on.

Case VII. If choice of prime p is 83

If $p = 83$ then Pell's equation becomes

$$x^2 - 83y^2 - 1 = 0 \qquad \text{(viii)}$$

Let $\alpha = \sqrt{83}$, then $\alpha = [\sqrt{83}] = 9$.

Now we expand $\sqrt{83}$ into CFE as given below:

We have

$$\alpha = 9 + (\sqrt{83} - 9) \Rightarrow 9 + \frac{1}{\left(\frac{\sqrt{83} + 9}{2}\right)} = a_1 + \frac{1}{\alpha_1}$$

$$\alpha_1 = \frac{\sqrt{83} + 9}{2} \Rightarrow 9 + \frac{1}{\left(\frac{\sqrt{83} + 9}{1}\right)} = a_2 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{\sqrt{83} + 9}{1} \Rightarrow 18 + \frac{1}{\left(\frac{\sqrt{83} + 9}{2}\right)} = a_3 + \frac{1}{\alpha_3}$$

Therefore the CFE of $\sqrt{83}$ is $[a_1, \overline{a_2, a_3}] = [9, \overline{9, 18}]$.

Let $\frac{u}{v} = \frac{p_2}{q_2}$ is the penultimate convergent of the CFE $[9, \overline{9, 18}]$ of $\sqrt{3}$.

We construct a table for the values of p_n and q_n, where $n = 1, 2, 3$.

Table (VII).

n	1	2	3
a_n	9	9	18
p_n	9	82	-
q_n	1	9	-

Therefore the penultimate convergent of 3 is

$$\frac{u}{v} = 9 + \frac{1}{9 +} \frac{1}{18} = \frac{82}{9}$$

Here we find that $(u + v\sqrt{83})$ is the fundamental solution of (viii), where $u = 82$ and $v = 9$. Then others solutions (x_r, y_r) are obtained by the relation $x_r + y_r\sqrt{83} = (82 + 9\sqrt{83})^r$ where $r = 1, 2, 3, \ldots,$ and so on.

5. Solution Table for Choices of prime *p*

We construct a table of solutions of equation (i) for giving particular values of prime p.

Table (IV).

Sr. No.	Choice of Prime p	Fundamental Solution (x_0, y_0)	Other Solutions (x_n, y_n), $n = 2, 3, 4, 5 \ldots$
1	3	(2, 1)	(7, 4), (26, 15), (97, 56), (362, 209), (1351, 780), (5042, 2911)
2	11	(10, 3)	(199, 60), (3970, 1197), (79201, 23880), (1580050, 476403), (31521799, 9504180)
3	19	(170, 39)	(57799, 13260), (10342970, 4508361), (1934130979, 1169797200),
4	47	(48, 7)	(4607, 672), (442224, 64505), (42448897, 6191808)
5	67	(48842, 5967)	(4771081927, 582880428), (466058366908226, 56938091722785)
6	79	(80, 9)	(12799, 1440), (2047760, 230391), (327628801, 36861120), (52418560400, 3243755520)
7	83	(82, 9)	(13447, 1476), (2205226, 242055), (361643617, 39695544), (59307347962, 6509827161)

6. Conclusion

In this paper, for the various prime p, we discussed Pell's equation $x^2 - py^2 - 1 = 0$. We discussed the solvability of this Diophantine equation for every prime $p, 1 < p < 100$ and observed that this equation is solvable only if prime $p \in S$ where $S = \{3, 11, 19, 47, 67, 79, 83\}$.

We obtained that (2, 1), (10, 3), (170, 39), (48, 7), (48842, 5967), (80, 9), (82, 9) are the fundamental solutions for the prime 3, 11, 19, 47, 67, 79, 83 respectively and also obtained some other solutions which are given in the table (4)

If we take 5, 7, 13, 17, 23, 29, 31, 37, 41, 43, 53, 59, 61, 71, 73, 89, 97 as the value of prime p, the equation (i) is not solvable.

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