Advances and Applications in Mathematical Sciences
Volume 24, Issue 7-8, May-June 2025, Pages 43-55

NEW REFINEMENTS OF MINKOWSKI'S INEQUALITY

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Abstract

In this paper, we define two mappings, investigate their properties and obtain some new refinements of Minkowski's inequality.

1. Introduction

Throughout this paper, for any given positive $n\,(n \geq 2)$, let $a_i > 0$, $b_i > 0\,(i = 1, 2, \cdots, n)$, p and q be two non-zero real numbers such that $p^{-1} + q^{-1} = 1$. If $p > 1$, then

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}. \tag{1.1}$$

The inequality (1.1) is equivalent to the following:

$$\sum_{i=1}^{n}(a_i + b_i)^p \leq \left(\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{q}}. \tag{1.2}$$

2020 Mathematics Subject Classification: 26D15.

Keywords: Minkowski's inequality, Hölder's inequality, Finite, Infinite, Refinement, Jensen's inequality.

*Fund Project: Science and Technology Research Project of Chongqing Education Commission (KJQN201902001). Fund Project: Scientific Research Project of Chongqing College of International Business and Economics (KYZK2024053).

Received March 11, 2025; Accepted March 25, 2025

If $p < 1\,(p \neq 0)$, then the inequalities in (1.1) and (1.2) are reversed.

The inequality (1.1) is called the Minkowshi's inequality (see [2, 3, 6-8]). For some recent results which generalize, improve and extend this classical inequality, see [1, 5, 10].

To go further into (1.2), we define two mappings M_1 and M_2 by

$$M_1 : \{(n, k) \mid n \geq 2, k = 1, 2, \cdots, n; n, k \in \mathbb{N}\} \to \mathbb{R},$$

$$M_1(n, k) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n-1}^{k-1}} \left(\frac{\left(\sum_{j=1}^{k} a_{i_j}{}^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{k} b_{i_j}{}^p\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p\right)^{\frac{1}{p}}} \right)$$

$$M_2 : \{(n, k) \mid n \geq 2, k = 1, 2, \cdots; n, k \in \mathbb{N}\} \to \mathbb{R},$$

$$M_2(n, k) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n+k-1}^{k-1}} \left(\frac{\left(\sum_{j=1}^{k} a_{i_j}{}^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{k} b_{i_j}{}^p\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p\right)^{\frac{1}{p}}} \right).$$

The aim of this paper is to study the properties of M_1 and M_2, thus obtaining some new refinements of (1.1).

2. Main Results

Theorem 2.1. *Let* $a_i > 0, b_i > 0\,(i = 1, 2, \cdots, n; n \geq 2)$, p *and* q *be two non-zero real numbers such that* $p^{-1} + q^{-1} = 1$, *and* M_1 *be defined as in the first section. We write*

$$F_1(n, k) = \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{-\frac{1}{q}} M_1(n, k), \; (1 \leq k \leq n).$$

When $p > 1$. *We obtain the following results:*

(1) *The following finite refinements of* (1.1) *are hold*

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} = F_1(n, 1) \le F_1(n, 2) \le \cdots$$

$$\le F_1(n, n) = \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}. \tag{2.1}$$

(2) *For any two positive integers* $k, r\,(1 \le k, r \le n)$ *and any two real numbers* $a \ge 0, b \ge 0,$ *such that* $a + b = 1,$ *we get the following refinements of* (1.1)

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \le aF_1(n, k) + bF_1(n, r) \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}. \tag{2.2}$$

When $p < 1\,(p \ne 0),$ *the inequalities in* (2.1)-(2.2) *are reversed.*

Remark 1. Theorem 2.1 is the finite refinements of Minkowski's inequality with non-repetitive sample.

Theorem 2.2. *Let* a_i, b_i, p *and* q *be defined as in the Theorem* 2.1, *and* M_2 *be defined as in the first section. We write*

$$F_2(n, k) = \left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{-\frac{1}{q}} M_2(n, k).$$

When $p > 1$. *We obtain the following results*:

(1) *The following infinite refinements of* (1.1) *are hold*

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} = F_2(n, 1) \le F_2(n, 2) \le \cdots$$

$$\le F_2(n, n) \le \cdots \le \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}. \tag{2.3}$$

(2) *For any two positive integers* $k, r\,(k, r \geq 1)$ *and any two real numbers* $a \geq 0, b \geq 0,$ *such that* $a + b = 1,$ *we get the following refinements of* (1.1)

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \leq aF_2(n, k) + bF_2(n, r) \leq \left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p\right)^{\frac{1}{p}}. \qquad (2.4)$$

When $p < 1\,(p \neq 0),$ *the inequalities in* (2.3)-(2.4) *are reversed.*

Remark 2. Theorem 2.2 is the infinite refinements of Minkowski's inequality with repetitive sample.

3. Several Lemmas

In order to prove the above theorems, we need the following two lemmas.

Lemma 3.1. *Let* $a_i > 0, b_i > 0\,(i = 1, 2, \cdots, n; n \geq 2),$ p *and* q *be two non-zero real numbers such that* $p^{-1} + q^{-1} = 1.$ *If* $p > 1,$ *then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} b_i{}^q\right)^{\frac{1}{q}}. \qquad (3.1)$$

We write

$$H_1(n, k; a_i, b_i) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\sum_{j=1}^{k} b_{i_j}{}^q}{C_{n-1}^{k-1}}\left(\frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} b_{i_j}{}^q}\right)^{\frac{1}{p}}.$$

We obtain the following finite refinements of (3.1)

$$\sum_{i=1}^{n} a_i b_i = H_1(n, 1; a_i, b_i) \leq H_1(n, 2; a_i, b_i)$$

$$\leq \cdots \leq H_1(n, n; a_i, b_i) = \left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} b_i{}^q\right)^{\frac{1}{q}}. \qquad (3.2)$$

The inequality (3.1) *is called the Hölder's inequality* (*see* [2-4, 6, 8]). *If* $p < 1\,(p \neq 0)$, *then the inequalities in* (3.1) *and* (3.2) *are reversed.*

Proof of Lemma 3.1. From the definition of H_1, since a simple calculation shows that

$$H_1(n, 1; a_i, b_i) = \sum_{i=1}^{n} a_i b_i, \tag{3.3}$$

$$H_1(n, n; a_i, b_i) = \left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i{}^q\right)^{\frac{1}{q}}. \tag{3.4}$$

For $k = 2, 3, \cdots, n$, using some elementary identity involved combinatorial numbers, from (3.1) and (3.6) in [9], we can get

$$\sum_{j=1}^{k} a_{i_j}{}^p = \sum_{\{r_1, \cdots, r_{k-1}\} \subset \{i_1, \cdots, i_k\}} \frac{\sum_{l=1}^{k-1} a_{r_l}{}^p}{k-1} \tag{3.5}$$

and

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{\{r_1, \cdots, r_{k-1}\} \subset \{i_1, \cdots, i_k\}} \sum_{l=1}^{k-1} a_{r_l}{}^p$$

$$= (n - k + 1) \sum_{1 \leqslant j_1 < \cdots < j_{k-1} \leq n} \sum_{m=1}^{k-1} a_{j_m}{}^p, \tag{3.6}$$

respectively.

(1) When $p > 1$, then $x^{1/p}$ is a concave function on $(0, +\infty)$ with x. Using Jensen's inequality of concave function (see [3, 8]), from (3.5) and (3.6), we get

$$H_1(n, k; a_i, b_i) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\sum_{j=1}^{k} b_{i_j}{}^q}{C_{n-1}^{k-1}} \left(\frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} b_{i_j}{}^q}\right)^{\frac{1}{p}}$$

$$= \sum_{1\le i_1<\cdots<i_k\le n} \frac{\sum_{j=1}^{k} b_{i_j}{}^q}{C_{n-1}^{k-1}} \left(\sum_{\{r_1,\,\cdots,\,r_{k-1}\}\subset\{i_1,\,\cdots,\,i_k\}} \frac{\sum_{l=1}^{k-1} b_{r_l}{}^q}{(k-1)\sum_{j=1}^{k} b_{i_j}{}^q} \cdot \frac{\sum_{l=1}^{k-1} a_{r_l}{}^p}{\sum_{l=1}^{k-1} b_{r_l}{}^q} \right)^{\frac{1}{p}}$$

$$\ge \sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{\{r_1,\,\cdots,\,r_{k-1}\}\subset\{i_1,\,\cdots,\,i_k\}} \frac{\sum_{l=1}^{k-1} b_{r_l}{}^q}{(k-1)C_{n-1}^{k-1}} \left(\frac{\sum_{l=1}^{k-1} a_{r_l}{}^p}{\sum_{l=1}^{k-1} b_{r_l}{}^q} \right)^{\frac{1}{p}}$$

$$= \sum_{1\le j_1<\cdots<j_{k-1}\le n} \frac{(n-k+1)\sum_{m=1}^{k-1} b_{j_m}{}^q}{(k-1)C_{n-1}^{k-1}} \left(\frac{\sum_{m=1}^{k-1} a_{j_m}{}^p}{\sum_{m=1}^{k-1} b_{j_m}{}^q} \right)^{\frac{1}{p}}$$

$$= \sum_{1\le j_1<\cdots<j_{k-1}\le n} \frac{\sum_{m=1}^{k-1} b_{j_m}{}^q}{C_{n-1}^{k-2}} \left(\frac{\sum_{m=1}^{k-1} a_{j_m}{}^p}{\sum_{m=1}^{k-1} b_{j_m}{}^q} \right)^{\frac{1}{p}} = H_1(n,\, k-1;\, a_i,\, b_i). \tag{3.7}$$

Combination of (3.3)-(3.4) and (3.7) yields (3.2).

(2) When $p < 1\,(p \neq 0)$, then $x^{1/p}$ is a convex function on $(0, +\infty)$ with x. Using Jensen's inequality of convex function, we get the reverse of (3.7), which implies the reverse of (3.2).

This completes the proof of Lemma 3.1.

Lemma 3.2. *Let a_i, b_i, p and q be defined as in the Lemma 3.1. We write*

$$H_2(n, k; a_i, b_i) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} \frac{\sum_{j=1}^{k} b_{i_j}{}^q}{C_{n+k-1}^{k-1}} \left(\frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} b_{i_j}{}^q} \right)^{\frac{1}{p}}.$$

If $p > 1$, *we obtain the following infinite refinements of* (3.1)

$$\sum_{i=1}^{n} a_i b_i = H_2(n, 1; a_i, b_i) \le H_2(n, 2; a_i, b_i)$$

$$\le \cdots \le H_2(n, n; a_i, b_i) \le \cdots \le \left(\sum_{i=1}^{n} a_i{}^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i{}^q \right)^{\frac{1}{q}}. \tag{3.8}$$

If $p < 1\,(p \ne 0)$, *then the inequalities in* (3.8) *are reversed.*

Proof of Lemma 3.2. From the definition of H_2, since a simple calculation shows that

$$H_2(n, 1; a_i, b_i) = \sum_{i=1}^{n} a_i b_i. \tag{3.9}$$

For $k = 2, 3, \ldots,$ using some elementary identity involved combinatorial numbers, we have

$$\sum_{i=1}^{n} a_i{}^p = \frac{1}{C_{n+k-1}^{k-1}} \sum_{1 \le i_1 \le \cdots \le i_k \le n} \sum_{j=1}^{k} a_{i_j}{}^p \tag{3.10}$$

and

$$\sum_{1 \le i_1 \le \cdots \le i_k \le n} \sum_{\{r_1, \cdots, r_{k-1}\} \subset \{i_1, \cdots, i_k\}} \left(\sum_{l=1}^{k-1} a_{r_l}{}^p \right)$$

$$= (n + k - 1) \sum_{1 \le j_1 \le \cdots \le j_{k-1} \le n} \left(\sum_{m=1}^{k-1} a_{j_m}{}^p \right). \tag{3.11}$$

(1) When $p > 1$, then $x^{1/p}$ is a concave function on $(0, +\infty)$ with x. Using Jensen's inequality of concave function and $p^{-1} + q^{-1} = 1$, from (3.10), we get

$$\left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} b_i{}^q\right)^{\frac{1}{q}} = \left(\sum_{i=1}^{n} b_i{}^q\right)\left(\frac{\sum_{i=1}^{n} a_i{}^p}{\sum_{i=1}^{n} b_i{}^q}\right)^{\frac{1}{p}}$$

$$= \left(\sum_{i=1}^{n} b_i{}^q\right)\left(\sum_{1 \le i_1 \le \cdots \le i_k \le n} \frac{\sum_{j=1}^{k} b_{i_j}{}^q}{C_{n+k-1}^{k-1}\sum_{i=1}^{n} b_i{}^q} \cdot \frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} b_{i_j}{}^q}\right)^{\frac{1}{p}} \tag{3.12}$$

$$\ge \sum_{1 \le i_1 \le \cdots \le i_k \le n} \frac{\sum_{j=1}^{k} b_{i_j}{}^q}{C_{n+k-1}^{k-1}}\left(\frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} b_{i_j}{}^q}\right)^{\frac{1}{p}} = H_2(n,\ k; a_i,\ b_i).$$

From the definition of H_2 and (3.5), (3.11), we get

$$H_2(n,\ k; a_i,\ b_i) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} \frac{\sum_{j=1}^{k} b_{i_j}{}^q}{C_{n+k-1}^{k-1}}\left(\frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} b_{i_j}{}^q}\right)^{\frac{1}{p}}$$

$$\ge \sum_{1 \le i_1 \le \cdots \le i_k \le n}\ \sum_{\{r_1,\ \cdots,\ r_{k-1}\} \subset \{i_1,\ \cdots,\ i_k\}} \frac{\sum_{l=1}^{k-1} b_{r_l}{}^q}{(k-1)C_{n+k-1}^{k-1}}\left(\frac{\sum_{l=1}^{k-1} a_{r_l}{}^p}{\sum_{l=1}^{k-1} b_{r_l}{}^q}\right)^{\frac{1}{p}}$$

$$= \sum_{1 \le j_1 \le \cdots \le j_{k-1} \le n} \frac{(n+k-1)\sum_{m=1}^{k-1} b_{j_m}{}^q}{(k-1)C_{n+k-1}^{k-1}} \left(\frac{\sum_{m=1}^{k-1} a_{j_m}{}^p}{\sum_{m=1}^{k-1} b_{j_m}{}^q} \right)^{\frac{1}{p}}$$

$$= \sum_{1 \le j_1 \le \cdots \le j_{k-1} \le n} \frac{\sum_{m=1}^{k-1} b_{j_m}{}^q}{C_{n+k-2}^{k-2}} \left(\frac{\sum_{m=1}^{k-1} a_{j_m}{}^p}{\sum_{m=1}^{k-1} b_{j_m}{}^q} \right)^{\frac{1}{p}} = H_2(n,\, k-1; a_i,\, b_i). \qquad (3.13)$$

Combination of (3.9) and (3.12)-(3.13) yields (3.8).

(2) When $p < 1\,(p \neq 0)$, then $x^{1/p}$ is a convex function on $(0, +\infty)$ with x. Using Jensen's inequality of convex function, we get the reverse of (3.12) and (3.13), which implies the reverse of (3.8).

This completes the proof of Lemma 3.2.

4. Proof of Theorems

Proof of Theorem 2.1. From the definitions of M_1 and F_1 since a simple calculation shows that

$$F_1(n,\, n) = \left(\sum_{i=1}^{n} a_i{}^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p \right)^{\frac{1}{p}}, \qquad (4.1)$$

$$F_1(n,\, 1) = \left(\sum_{i=1}^{n} (a_i + b_i)^p \right)^{\frac{1}{p}}. \qquad (4.2)$$

For $k = 1, 2, \cdots, n,$ from the definitions of M_1 and $H_1,$ we have

$$M_1(n,\ k) = \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n-1}^{k-1}} \left(\frac{\left(\sum_{j=1}^{k} a_{i_j}{}^p \right)^{\frac{1}{p}} + \left(\sum_{j=1}^{k} b_{i_j}{}^p \right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p \right)^{\frac{1}{p}}} \right)$$

$$= \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n-1}^{k-1}} \left(\frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p} \right)^{\frac{1}{p}} \tag{4.3}$$

$$+ \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n-1}^{k-1}} \left(\frac{\sum_{j=1}^{k} b_{i_j}{}^p}{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p} \right)^{\frac{1}{p}}$$

$$= H_1(n,\ k;\ a_i,\ (a_i + b_i)^{p-1}) + H_1(n,\ k;\ b_i,\ (a_i + b_i)^{p-1}).$$

(1) When $p > 1$, then $x^{1/p}$ is a concave function on $(0, +\infty)$ with x. For $k = 2, 3, \cdots, n$, from (4.3) and (3.2), with a simple calculation, we get

$$M_1(n,\ k) \ge M_1(n,\ k-1). \tag{4.4}$$

From (4.4) and the definition of F_1, since a simple calculation shows that

$$F_1(n,\ k) \ge F_1(n,\ k-1). \tag{4.5}$$

Combination of (4.1)-(4.2) and (4.5) yields (2.1)

For any two positive integers $k, r\,(1 \le k,\ r \le n)$ and any two real numbers $a \ge 0, b \ge 0$, such that $a + b = 1$. From (2.1), with a simple calculation, we obtain (2.2).

(2) When $p < 1\,(p \neq 0)$, then $x^{1/p}$ is a convex function on $(0, +\infty)$ with x. Using Jensen's inequality of convex function, we get the reverse of (4.4) and (4.5), which implies the reverse of (2.1)-(2.2).

The proof of Theorem 2.1 is completed.

Proof of Theorem 2.2. From the definitions of M_2 and F_2, we have

$$F_2(n, 1) = \left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{p}}. \tag{4.6}$$

For $k = 1, 2, \cdots$, from the definitions of M_2 and H_2, with a simple calculation, we get

$$M_2(n, k) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n+k-1}^{k-1}} \left(\frac{\left(\sum_{j=1}^{k} a_{i_j}{}^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^{k} b_{i_j}{}^p\right)^{\frac{1}{p}}}{\left(\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p\right)^{\frac{1}{p}}} \right)$$

$$= \sum_{1 \le i_1 \le \cdots \le i_k \le n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n+k-1}^{k-1}} \left(\frac{\sum_{j=1}^{k} a_{i_j}{}^p}{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p} \right)^{\frac{1}{p}} \tag{4.7}$$

$$+ \sum_{1 \le i_1 \le \cdots \le i_k \le n} \frac{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p}{C_{n+k-1}^{k-1}} \left(\frac{\sum_{j=1}^{k} b_{i_j}{}^p}{\sum_{j=1}^{k} (a_{i_j} + b_{i_j})^p} \right)^{\frac{1}{p}}$$

$$= H_2(n, k;\, a_i,\, (a_i + b_i)^{p-1}) + H_2(n, k;\, b_i,\, (a_i + b_i)^{p-1}).$$

(1) When $p > 1$, then $x^{1/p}$ is a concave function on $(0, +\infty)$ with x. For $k = 2, 3, \ldots,$ from (4.7) and (3.8), with a simple calculation, we get

$$M_2(n, k-1) \le M_2(n, k) \le \cdots$$

$$\le \left(\left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{\frac{1}{q}}. \tag{4.8}$$

From (4.8) and the definition of F_2, since a simple calculation shows that

$$F_2(n, k-1) \le F_2(n, k) \le \cdots \le \left(\sum_{i=1}^{n} a_i{}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i{}^p\right)^{\frac{1}{p}}. \tag{4.9}$$

Combination of (4.6) and (4.9) yields (2.3).

For any two positive integers $k, r\,(k, r \ge 1)$ and any two real numbers $a \ge 0, b \ge 0,$ such that $a + b = 1.$ From (2.3), with a simple calculation, we obtain (2.4).

(2) When $p < 1\,(p \ne 0)$, then $x^{1/p}$ is a convex function on $(0, +\infty)$ with x. Using Jensen's inequality of convex function, we get the reverse of (4.8) and (4.9), which implies the reverse of (2.3)-(2.4).

The proof of Theorem 2.2 is completed.

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