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FIXED POINT THEOREMS FOR MAPPINGS INVOLVING RATIONAL TYPE EXPRESSIONS IN DUALISTIC PARTIAL METRIC SPACES

ANIL BAKHRU, MANOJ UGHADE and RICHA GUPTA

Department of Mathematics
Sardar Vallabhbhai Polytechnic College
College of University of Technology of
Madhya Pradesh, Bhopal 462002, India

Department of Post Graduate
Studies and Research in Mathematics
Jaywanti Haksar Government
Post Graduate College
College of Chhindawara University
Betul, 460001, India

Department of Engineering
Mathematics and Research Center
Sarvepalli Radhakrishanan University
Bhopal 462026, India
E-mail: anilbakhru@gmail.com

Abstract

The aim of this paper is to establish some fixed point theorems for mappings involving rational expressions in a complete dualistic partial metric space using a class of pairs of functions satisfying certain assumptions. Our result extends and generalizes some well-known results of [8], [9], [26] and [33]. We also provide examples which show the usefulness of these results.

1. Introduction

Matthews [17] introduced a new generalized metric space called partial metric space. He established the precise relationship between partial metric

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spaces and the so-called weightable quasi-metric spaces. After this contribution, many researchers focused on partial metric spaces (see [1], [11], [12], [13], [14], [15], [22], [28]).

The concept of dualistic partial metric, which is more general than partial metric, was studied by O'Neill [29] and established a robust relationship between dualistic partial metric and quasi metric. For the more details of fixed point results on dualistic partial metric spaces, the readers may refer to [4], [16], [19] [20], [23], [25], [27], [28].

Das and Gupta [8] established first fixed point theorem for rational contractive type conditions in metric space.

Theorem 1.1 (see [8]). *Let* (X, d) *be a complete metric space, and let* $\mathcal{T} : X \to X$ *be a self-mapping. If there exist* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$ *such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \alpha d(x, y) + \beta \frac{[1 + d(x, \mathcal{T}x)]d(y, \mathcal{T}y)}{1 + d(x, y)} \tag{1.1}$$

for all $x, y \in X$, *then* $\mathcal{T}$ *has a unique fixed point* $x^* \in X$.

Nazam et al. [26] proved a real generalization of Das-Gupta fixed point theorem in the frame work of dualistic partial metric spaces. The main purpose of this paper is to present some fixed point theorems for mappings involving rational expressions in the context of complete dualistic partial metric spaces using a class of pairs of functions satisfying certain assumptions. Our result extends and generalizes some well-known results of [8], [9], [26] and [33]. We also provide examples to show significance of the obtained results involving rational type dualistic contractive conditions.

2. Preliminaries

We recall some mathematical basics and definitions to make this paper self-sufficient.

Definition 2.1 (see [17]). Let X be a non-empty set. A partial metric on X is a function $p : X \times X \to [0, \infty)$ complying with following axioms, for all $x, y, z \in X$

$(p_1)\, x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$;

$(p_2)\, p(x, x) \leq p(x, y)$;

$(p_3)\, p(x, y) = p(y, x)$;

$(p_4)\, p(x, y) \leq p(x, z) + p(v, y) - p(z, z)$.

The pair (X, p) is called a partial metric space.

Definition 2.2 (see [29])**.** Let X be a non-empty set. A dualistic partial metric on X is a function $p^* : X \times X \to (-\infty, \infty)$ satisfying the following axioms, for all $x, y, z \in X$

$(p_1^*)\, x = y \Leftrightarrow p^*(x, y) = p^*(x, x) = p^*(y, y)$;

$(p_2^*)\, p^*(x, x) \leq p^*(x, y)$;

$(p_3^*)\, p^*(x, y) = p^*(y, x)$;

$(p_4^*)\, p^*(x, z) + p^*(y, y) \leq p^*(x, y) + p^*(y, z)$.

The pair (X, p^*) is called a dualistic partial metric space.

Remark 2.3. Noting that each partial metric is a dualistic partial metric but the converse is false. Indeed, define a function p^* on $(-\infty, \infty)$ as $p^*(x, y) = \max\{x, y\}, \forall x, y \in (-\infty, \infty)$. Obviously, p^* is a dualistic partial metric on $(-\infty, \infty)$. Since $p^*(x, y) < 0 \notin [0, \infty), \forall x, y \in (-\infty, 0)$ and then p^* is not a partial metric on $(-\infty, \infty)$. This confirms our remark. Unlike other metrics, in dualistic partial metric $p^*(x, y) = 0$ does not imply $x = y$. Indeed, for all $k > 0$, $p^*(-k, 0) = 0$ and $-k \neq 0$. The self-distance $p^*(x, x)$ is a feature utilized to describe the amount of information contained in x. The restriction of p^* to $[0, \infty)$ is a partial metric. This situation creates a problem in obtaining a fixed point of a self-mapping in dualistic partial metric space. For the solution of this problem, Nazam et al. [21] introduced concept of convergence comparison property (CCP) and established some fixed point by using (CCP) along with axioms (p_1^*) and (p_2^*).

Definition 2.4 (see [21]). Let (X, p^*) be a dualistic partial metric space and $\mathcal{T}$ be a self-mapping on X. We say that $\mathcal{T}$ has a convergence comparison property (CCP) if for each sequence $\{x_n\}$ in X such that $x_n \to x$, $\mathcal{T}$ satisfies

$$p^*(x, x) \leq p^*(\mathcal{T}x, \mathcal{T}x) \tag{2.1}$$

Example 2.5. Let $X = (-\infty, 0]$, $X_1 = (-4, 0]$, $X_2 = (-\infty, -4]$. Define a mapping $p^* : X \times X \to (-\infty, \infty)$ by $p^*(x, y) = |x - y|$ if $x \neq y$ and $p^*(x, y) = x \vee y$ if $x = y$. Clearly, (X, p^*) is a complete dualistic partial metric space. Consider $\left\{x_n = \frac{1}{n} - 2, n \geq 1\right\}_{n \in \mathbb{N}} \subset X$. Here $\lim_{n\to\infty} p^*(x_n, -2) = p^*(-2, -2) \Rightarrow \lim_{n\to\infty} x_n = -2$ in (X, p^*). Define $\mathcal{T} : X \to X$ by $\mathcal{T}x = -1$ if $x \in X_2$ and $\mathcal{T}x = 0$ if $x \in X_1$. For such $x = -2$, observe that $p^*(x, x) = x \vee x = x = -2 \leq 0 = p^*(0, 0) = p^*(\mathcal{T}(-2), \mathcal{T}(-2))$. So $\mathcal{T}$ has the (CCP).

Example 2.6 (see [21], [29]). (1) Define $p_d^* : X \times X \to (-\infty, \infty)$ by $p_d^*(x, y) = d(x, y) + b$, where d is a metric on a nonempty set X and $b \in (-\infty, \infty)$ is arbitrary constant, then it is easy to check that p_d^* verifies axioms $(p_1^*) - (p_4^*)$ and hence (X, p^*) is a dualistic partial metric space.

(2) Let p be a partial metric defined on a non empty set X. The function $p^* : X \times X \to (-\infty, \infty)$ defined by $p^*(x, y) = p(x, y) - p(x, x) - p(y, y)$ satisfies the axioms $(p_1^*) - (p_4^*)$ and so it defines a dualistic partial metric on X. Note that $p^*(x, y)$ may have negative values.

(3) Let $X = (-\infty, \infty)$. Define $p^* : X \times X \to (-\infty, \infty)$ by $p^*(x, y) = |x - y|$ if $x \neq y$ and $p^*(x, y) = -\beta$ if $x = y$ and $\beta > 0$. We can easily see that p^* is a dualistic partial metric on X.

O'Neill [29] established that each dualistic partial metric p^* on X generates a T_0 topology $\tau(p^*)$ on X having a base, the family of p^*-balls

$\{\mathcal{B}_{p^*}(x, \epsilon) | x \in X, \epsilon > 0\}$, where

$$\{\mathcal{B}_{p^*}(x, \epsilon) = \{y \in X | p^*(x, y) < p^*(x, x) + \epsilon\}.$$

If (X, p^*) is a dualistic partial metric space, then the function $d_{p^*} : X \times X \to [0, \infty)$ defined by

$$d_{p^*}(x, y) = p^*(x, y) - p^*(x, x) \tag{2.2}$$

defines a quasi-metric on X such that $\tau(p^*) = \tau(dp^*)$ and

$$d^s_{p^*}(x, y) = \max\{d_{p^*}(x, y), d_{p^*}(y, x)\} \tag{2.3}$$

defines a metric on X.

Definition 2.7 (see [28]). Let (X, p^*) be a dualistic partial metric space.

1. A sequence $\{x_n\}$ in X is said to converge or to be convergent if there is a $x \in X$ such that $\lim_{n\to\infty} p^*(x_n, x) = p^*(x, x)$. x is called the limit of $\{x_n\}$ and we write $x_n \to x$.

2. A sequence $\{x_n\}$ in X is said to be Cauchy sequence if $\lim_{n, m\to\infty} p^*(x_n, x_m)$ exists and is finite.

3. A dualistic partial metric space $X = (X, p^*)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to $\tau(p^*)$, to a point $x \in X$ such that $p^*(x, x) = \lim_{n, m\to\infty} p^*(x_n, x_m)$.

Remark 2.8. For a sequence, convergence with respect to metric space may not imply convergence with respect to dualistic partial metric space. Indeed, if we take $\beta = 1$ and $\{x_n = \frac{1-n}{n} : n \geq 1\}_{n\in\mathbb{N}} \subset X$ as in Example 2.6 (3). Mention that $\lim_{n\to\infty} d(x_n, -1) = -1$ and therefore, $x_n \to -1$ with respect to d. On the other hand, we make a conclusion that $x_n \nrightarrow -1$ with respect to p^* because $\lim_{n\to\infty} p^*(x_n, -1) = \lim_{n\to\infty} p^*| x_n - (-1) |$ $= \lim_{n\to\infty}| \frac{1-n}{n} + 1 | = 0$ and $p^*(-1, -1) = -1$.

Lemma 2.9 (see [28]). *Let (X, p^*) be a dualistic partial metric space.*

(1) *Every Cauchy sequence in $(X, d^s_{p^*})$ is also a Cauchy sequence in (X, p^*).*

(2) *A dualistic partial metric (X, p^*) is complete if and only if the induced metric space $(X, d^s_{p^*})$ is complete.*

(3) *A sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to $\tau(d^s_{p^*})$ if and only if $p^*(x, x) = \lim_{n\to\infty} p^*(x_n, x) = \lim_{n\to\infty} p^*(x_n, x_m)$.*

Definition 2.9 (see [26]). Let (X, p^*) be a dualistic partial metric space. A mapping $\mathcal{T} : X \to X$ is said to be a dualistic Dass-Gupta contraction if there exist $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$ such that

$$| p^*(\mathcal{T}x, \mathcal{T}y) | \leq \alpha \left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \right| + \beta | p^*(x, y) | \tag{2.4}$$

$$\forall x, y \in \Delta = \{(x, y) \in X \times X \mid p^*(x, y) \neq -1\}.$$

Nazam et al. [26] studied the following fixed point theorems on dualistic contraction of rational type.

Theorem 2.10. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a dualistic Das-Gupta contraction. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}_n(x_0)\}$ with initial point x_0, converges to the fixed point.*

One of the most important ingredients of a contractive condition is to study the kind of involved functions, like altering distance functions introduced by Khan et al. [16] as follows.

Definition 2.11 (see [16]). A function $\varphi: [0, \infty) \to [0, \infty)$ is said to be altering distance function if (a1) φ is monotone increasing and continuous,

(a2) $\varphi(t) = 0 \Leftrightarrow t = 0, \forall t \in [0, \infty)$.

Definition 2.12 (see [5]). The pair (φ, ϕ), where $\varphi, \phi : [0, \infty) \to [0, \infty)$ is called a pair of generalized altering distance functions if

(b1) φ is continuous;

(b2) φ is non-decreasing;

(b3) $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$.

The condition (b3) was introduced by Moradi and Farajzadeh [18]. The above conditions do not determine the values of $\varphi(0)$ and $\phi(0)$.

Definition 2.13 (see [2]). We will denote by $\mathcal{F}$ the family of all pairs (φ, ϕ), where $\varphi, \phi : [0, \infty) \to [0, \infty)$ are functions satisfying the following conditions.

(F1) φ is non-decreasing;

(F2) if $\exists t_0 \in [0, \infty)$ such that $\phi(t_0) = 0$, then $t_0 = 0$ and $\varphi^{-1}(0) = \{0\}$.

(F3). if $\{\alpha_n\}, \{\beta_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lambda$ satisfying $\lambda < \{\beta_n\}$ and $\varphi(\beta_n) \le (\varphi - \phi)(\alpha_n), \forall n \in \mathbb{N}$, then $\lambda = 0$.

Definition 2.14 (see [33]). A pair of functions (φ, ϕ) is said to belong to the class $\mathfrak{F}$ if they satisfy the following conditions:

(c1) $\varphi, \phi : [0, \infty) \to [0, \infty)$;

(c2) if $t, s \in [0, \infty), \varphi(t) \le \phi(s)$ then $t \le s$;

(c3). if $\{t_n\}, \{s_n\} \subset [0, \infty), \lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta$ and $\varphi(t_n) \le \phi(s_n), \forall n \in \mathbb{N}$, then $\delta = 0$.

If (φ, ϕ) satisfies (F1) and (F2), then $(\varphi, \phi = \varphi - \phi)$ satisfies (c1) and (c2). Furthermore, if $(\varphi, \phi = \varphi - \phi)$ satisfies (c3), then (φ, ϕ) satisfies (F3).

Remark 2.15 (see [33]). If $(\varphi, \phi) \in \mathfrak{F}$ and $\varphi(t) \le \phi(t)$, then $t = 0$, since we can take $t_n = s_n, t, \forall n \in \mathbb{N}$ and by (c3), we deduce $t = 0$.

Example 2.16. The conditions (c1)-(c3) of Definition 2.14 are fulfilled for the functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ defined by

(1) $\varphi(t) = \ln\left(\frac{5t+1}{12}\right)$ and $\phi(t) = \ln\left(\frac{3t+1}{12}\right), \forall t \in [0, \infty)$.

(2) $\varphi(t) = \ln\left(\frac{2t+1}{2}\right)$ and $\phi(t) = \ln\left(\frac{t+1}{2}\right), \forall t \in [0, \infty)$.

Example 2.17 (see [33])**.** Let $\mathcal{S} = \{\ell : [0, \infty) \to [0, \infty) | \ell(t_n) \to 1 \Rightarrow t_n \to 0\}$. Consider the pairs of functions $(1_{[0, \infty)}, \ell(1_{[0, \infty)}))$, where $\ell \in \mathcal{S}$ and $\ell(1_{[0, \infty)})$ is defined as

$$(\ell(1_{[0, \infty)}))(t) = \ell(t)t, \forall t \in [0, \infty).$$

It is easy to check that $(1_{[0, \infty)}, \ell(1_{[0, \infty)})) \in \mathfrak{F}$.

Example 2.18 (see [33])**.** Let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous and increasing function such that $\varphi(t) = 0 \Leftrightarrow t = 0, \forall t \in [0, \infty)$. Let $\phi : [0, \infty) \to [0, \infty)$ be a non-decreasing function such that $\phi(t) = 0 \Leftrightarrow t = 0, \forall t \in [0, \infty)$ and $\phi \le \varphi$. We make a conclusion that $(\varphi, \varphi - \phi) \in \mathfrak{F}$.

An interesting particular case is when φ is the identity mapping, $\varphi = 1_{[0, \infty)}$ and $\phi : [0, \infty) \to [0, \infty)$ is a non-decreasing function such that $\phi(t) = 0 \Leftrightarrow t = 0$ and $\phi(t) \le t, \forall t \in [0, \infty)$.

Remark 2.19 (see [33])**.** Let $g : [0, \infty) \to [0, \infty)$ be an increasing function and $(\varphi, \phi) \in \mathfrak{F}$. Then $(g \circ \varphi, g \circ \phi) \in \mathfrak{F}$.

3. Main Results

In this section, using the class $\mathfrak{F}$ functions, we give generalizations of some fixed point theorems from the literature.

Theorem 3.1. *Let* (X, p^*) *be a complete dualistic partial metric space.*

Let $\mathcal{T} : X \times X$ *be a mapping such that there exists a pair of functions* $(\varphi, \phi) \in \mathfrak{F}$ *satisfying*

$$\varphi(|\ p^*(\mathcal{T}x,\ \mathcal{T}y)\ |) \leq \max\{\phi(|\ p^*(x,\ y)\ |),\ \phi(|\ \frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\ |)\} \quad (3.1)$$

$\forall x,\ y \in \Delta$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{T_n(x_0)\}$ with initial point x_0, converges to the fixed point.

Proof. Let $x_0 \in X$ be an initial element and define Picard iterative sequence $\{x_n\}$ by $\mathcal{T}x_{n-1} = x_n$, $\forall n \in \mathbb{N}$. If there is a positive integer n_0 such that $x_{n_0} = x_{n_0+1}$, then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$. So x_{n_0} is a fixed point of $\mathcal{T}$. In this case, the proof is finished. Now, we suppose that $x_n \neq x_{n+1}$, $\forall n \in \mathbb{N}$, applying (3.1), we have

$$\varphi(|\ p^*(x_{n+1},\ x_n)\ |) = \varphi(|\ p^*(\mathcal{T}x_n,\ \mathcal{T}x_{n-1})\ |)$$

$$\leq \max\{\varphi(|\ p^*(x_n,\ x_{n-1})\ |),\ \varphi(|\ \frac{p^*(x_{n-1},\ \mathcal{T}x_{n-1})(1 + p^*(x_n,\ \mathcal{T}x_n))}{1 + p^*(x_n,\ x_{n-1})}\ |)\}$$

$$= \max\{\phi(|\ p^*(x_n,\ x_{n-1})\ |),\ \phi(|\ \frac{p^*(x_{n-1},\ x_n)(1 + p^*(x_n,\ x_{n+1}))}{1 + p^*(x_n,\ x_{n-1})}\ |)\}. \quad (3.2)$$

Now, we can distinguish two cases.

Case 1. Consider

$$\max\{\phi(|\ p^*(x_n,\ x_{n-1})\ |),\ \phi(|\ \frac{p^*(x_{n-1},\ x_n)(1 + p^*(x_n,\ x_{n+1}))}{1 + p^*(x_n,\ x_{n-1})}\ |)\}$$

$$= \phi(|\ p^*(x_n,\ x_{n-1})\ |). \quad (3.3)$$

Due to inequality (3.2), we have

$$\phi(|\ p^*(x_{n+1},\ x_n)\ |) \leq \phi(|\ p^*(x_n,\ x_{n-1})\ |). \quad (3.4)$$

Since $(\varphi, \phi) \in \mathfrak{F}$, we deduce that

$$|\, p^*(x_{n+1},\, x_n)\,| \leq |\, p^*(x_n,\, x_{n-1})\,|.$$

Case 2. If

$$\max\{\phi(|\, p^*(x_n,\, x_{n-1})\,|),\, \phi(|\, \frac{p^*(x_{n-1},\, x_n)(1 + p^*(x_n,\, x_{n+1}))}{1 + p^*(x_n,\, x_{n-1})}\,|)\}$$

$$= \phi(|\, \frac{p^*(x_{n-1},\, x_n)(1 + p^*(x_n,\, x_{n+1}))}{1 + p^*(x_{n-1},\, x_n)}\,|). \tag{3.5}$$

Then from (3.3), we have

$$\varphi(|\, p^*(x_{n+1},\, x_n)\,|) \leq \phi(|\, \frac{p^*(x_{n-1},\, x_n)(1 + p^*(x_n,\, x_{n+1}))}{1 + p^*(x_{n-1},\, x_n)}\,|). \tag{3.6}$$

Since $(\varphi,\, \phi) \in \mathfrak{F}$ we get

$$|\, p^*(x_n,\, x_{n+1})\,| \leq |\, \frac{p^*(x_{n-1},\, x_n)(1 + p^*(x_n,\, x_{n+1}))}{1 + p^*(x_{n-1},\, x_n)}\,|$$

which implies that

$$|\, p^*(x_{n+1},\, x_n)\,| \leq |\, p^*(x_n,\, x_{n-1})\,|.$$

From both cases, we conclude that the sequence $\{|\, p^*(x_{n+1},\, x_n)\,|\}$ is a monotone and bounded below sequence of non-negative real numbers, it is convergent and converges to a point r, i.e. $\lim_{n\to\infty}|\, p^*(x_{n+1},\, x_n)\,| = r \geq 0$. If $r = 0$. Then we have done. Let $r > 0$ and denote $A = \{n \in \mathbb{N} \,|\, n$ satisfies (3.3)$\}$ and $B = \{n \in \mathbb{N} \,|\, n$ satisfies (3.5)$\}$. Now, we make the following remark.

(1) If Card $A = \infty$, then from (3.2), we can find infinitely natural numbers n satisfying inequality (3.4) and since $\lim_{n\to\infty}|\, p^*(x_{n+1},\, x_n)\,| = \lim_{n\to\infty}|\, p^*(x_n,\, x_{n-1})\,| = r$ and $(\varphi,\, \phi) \in \mathfrak{F}$, we deduce that $r = 0$.

(2) If Card $B = \infty$, then from (3.2), we can find infinitely many $n \in \mathbb{N}$ satisfying inequality (3.6). Since $(\varphi,\, \phi) \in \mathfrak{F}$ and using the similar argument

to the one used in case 2, we obtain

$$| \, p^*(x_n, x_{n+1}) \, | \leq | \, \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_n, x_{n+1}))}{1 + p^*(x_{n-1}, x_n)} \, |$$

(3) for infinitely many $n \in \mathbb{N}$. On letting the limit as $n \to \infty$ and taking into account that $\lim_{n\to\infty} | \, p^*(x_{n+1}, x_n) \, | = r,$ we deduce that $r \leq \frac{r(1+r)}{1+r}$ and consequently, we obtain $r = 0.$

Therefore, in both cases we have

$$\lim_{n\to\infty} | \, p^*(x_{n+1}, x_n) \, | = 0 \text{ and then } \lim_{n\to\infty} | \, p^*(x_{n+1}, x_n) \, | = 0. \tag{3.7}$$

We use (3.1) to find the self-distance $p^*(x_n, x_{n-1}),$ as follows:

$$\begin{aligned} \varphi(| \, p^*(x_n, x_n) \, |) &= \varphi(| \, p^*(\mathcal{T}x_{n-1}, \mathcal{T}x_{n-1}) \, |) \\ &\leq \max\{\varphi(| \, p^*(x_{n-1}, x_{n-1}) \, |), \\ &\quad \varphi(| \, \frac{p^*(x_{n-1}, \mathcal{T}x_{n-1})(1 + p^*(x_{n-1}, \mathcal{T}x_{n-1}))}{1 + p^*(x_{n-1}, x_{n-1})} \, |)\} \\ &= \max\{\phi(| \, p^*(x_{n-1}, x_{n-1}) \, |), \\ &\quad \phi(| \, \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} \, |)\}. \end{aligned} \tag{3.8}$$

Put

$$C = \{n \in \mathbb{N} \,|\, \varphi(| \, p^*(x_n, x_n) \, |) \leq \phi(| \, p^*(x_{n-1}, x_{n-1}) \, |)\}$$

$$D = \{n \in \mathbb{N} \,|\, \varphi(| \, p^*(x_n, x_n) \, |) \leq \phi(| \, \frac{p^*(x_{n-1}, x_{n-1})(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} \, |)\}.$$

By (3.8), we have Card $C = \infty$ or Card $D = \infty.$ If Card $C = \infty,$ then there exists infinitely many $n \in \mathbb{N}$ satisfying

$$\varphi(| p^*(x_n, x_n) |) \le \phi(| p^*(x_{n-1}, x_{n-1}) |) \tag{3.9}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, we have

$$| p^*(x_n, x_n) | \le | p^*(x_{n-1}, x_{n-1}) |.$$

Thus, $\{| p^*(x_{n+1}, x_n) |\}$ is a non increasing sequence of positive real numbers and arguing like case of (3.7), we have $\lim_{n\to\infty}| p^*(x_n, x_n) | = 0$. On the other hand, if Card $D = \infty$, then we can find infinitely many $n \in \mathbb{N}$ satisfying

$$\varphi(| p^*(x_n, x_n) |) \le \phi\left(\left| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} \right|\right) \tag{3.10}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, we infer

$$| p^*(x_n, x_n) | \le \left| \frac{p^*(x_{n-1}, x_n)(1 + p^*(x_{n-1}, x_n))}{1 + p^*(x_{n-1}, x_{n-1})} \right| \tag{3.11}$$

taking the $\lim_{n\to\infty}$ on (3.11) and using (3.7), we obtain that $\lim_{n\to\infty}| p^*(x_n, x_n) | \le 0$ and then $\lim_{n\to\infty}| p^*(x_n, x_n) | = 0$. Thus, in both cases, we infer that $\lim_{n\to\infty}| p^*(x_n, x_n) | = 0$ and then

$$\lim_{n\to\infty} p^*(x_n, x_n) = 0. \tag{3.12}$$

We deduce from (2.2) that

$$d_{p^*}(x_n, x_{n+1}) = p^*(x_n, x_{n+1}) - p^*(x_n, x_n).$$

So using (3.7) and (3.12), we get

$$\lim_{n\to\infty} d_{p^*}(x_n, x_{n+1}) = 0. \tag{3.13}$$

Next step is to show that $\{x_n\}$ is a Cauchy sequence in $(X, d^s_{p^*})$. For this we have to show that

$$\lim_{m,\, n \to \infty} d^{s}_{p^*}(x_n, x_m) = \lim_{m,\, n \to \infty} \max\{d_{p^*}(x_n, x_m), d_{p^*}(x_m, x_n)\} = 0.$$

Suppose on contrary that $\{x_n\}$ is not a Cauchy sequence, that is $\lim_{n, m \to \infty} d_{p^*}(x_n, x_m) \neq 0$. Then given $\epsilon > 0$, we will construct a pair of subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is smallest index for which for all $n_k > m_k > k$, where $k \in \mathbb{N}$

$$d_{p^*}(x_{n_k}, x_{m_k}) \geq \epsilon. \tag{3.14}$$

It follows directly that

$$d_{p^*}(x_{n_k - 1}, x_{m_k}) < \epsilon. \tag{3.15}$$

By (3.14) and (3.15), we have

$$\begin{aligned} \epsilon &\leq d_{p^*}(x_{n_k}, x_{m_k}) \\ &\leq d_{p^*}(x_{n_k}, x_{n_k - 1}) + d_{p^*}(x_{n_k - 1}, x_{m_k}) \\ &< d_{p^*}(x_{n_k}, x_{n_k - 1}) + \epsilon. \end{aligned}$$

Taking $\lim_{k \to \infty}$ on both sides in above inequality and from (3.13), we obtain

$$\lim_{k \to \infty} d_{p^*}(x_{n_k}, x_{m_k}) = \epsilon. \tag{3.16}$$

Using triangle inequality, we have

$$\begin{aligned} d_{p^*}(x_{n_k}, x_{m_k}) &\leq d_{p^*}(x_{n_k}, x_{n_k - 1}) + d_{p^*}(x_{n_k - 1}, x_{m_k}) \\ &\leq d_{p^*}(x_{n_k}, x_{n_k - 1}) + d_{p^*}(x_{n_k - 1}, x_{m_k - 1}) + d_{p^*}(x_{m_k - 1}, x_{m_k}) \end{aligned}$$

and

$$\begin{aligned} d_{p^*}(x_{n_k - 1}, x_{m_k - 1}) &\leq d_{p^*}(x_{n_k - 1}, x_{n_k}) + d_{p^*}(x_{n_k}, x_{m_k - 1}) \\ &\leq d_{p^*}(x_{n_k - 1}, x_{n_k}) + d_{p^*}(x_{n_k}, x_{m_k}) + d_{p^*}(x_{m_k}, x_{m_k - 1}). \end{aligned}$$

Taking the limit as $k \to \infty$ in the above two inequalities and using (3.13) and (3.16), we get

$$\lim_{k\to\infty} d_{p^*}(x_{n_k-1}, x_{m_k-1}) = \epsilon. \tag{3.17}$$

Now applying contractive condition (3.1), for $x_{n_k} \neq x_{m_k}$, we have

$$\begin{aligned}
\varphi(|\, p^*(x_{n_k}, x_{m_k})\,|) &= \varphi(|\, p^*(\mathcal{T}x_{n_k-1}, \mathcal{T}x_{m_k-1})\,|) \\
&\leq \max\{\phi(|\, p^*(x_{n_k-1}, x_{m_k-1})\,|), \\
&\phi(|\, \frac{p^*(x_{n_k-1}, \mathcal{T}x_{n_k-1})(1 + p^*(x_{m_k-1}, \mathcal{T}x_{m_k-1}))}{1 + p^*(x_{m_k-1}, \mathcal{T}x_{m_k-1})}\,|)\} \\
&= \max\{\phi(|\, p^*(x_{n_k-1}, x_{m_k-1})\,|), \\
&\phi(|\, \frac{p^*(x_{n_k-1}, x_{n_k})(1 + p^*(x_{m_k-1}, x_{m_k}))}{1 + p^*(x_{m_k-1}, x_{m_k} - 1)}\,|)\}
\end{aligned} \tag{3.18}$$

Let us put

$$E = \{n \in \mathbb{N} \,|\, \varphi(|\, p^*(x_{n_k}, x_{m_k})\,|) \leq \phi(|\, p^*(x_{n_k-1}, x_{m_k-1})\,|)\}$$

$$F = \{n \in \mathbb{N} \,|\, \varphi(|\, p^*(x_{n_k}, x_{m_k})\,|) \leq \phi(|\, \frac{p^*(x_{n_k-1}, x_{n_k})(1 + p^*(x_{m_k-1}, x_{m_k}))}{1 + p^*(x_{n_k-1}, x_{m_k-1})}\,|)\}.$$

By (3.18), we have Card $E = \infty$ or Card $F = \infty$. Let us suppose that Card $E = \infty$, then there exists infinitely many $k \in \mathbb{N}$ satisfying

$$\varphi(|\, p^*(x_{n_k}, x_{m_k})\,|) \leq \phi(|\, p^*(x_{n_k-1}, x_{m_k-1})\,|) \tag{3.19}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, by letting the limit as $k \to \infty$, we have

$$\lim_{k\to\infty} |\, p^*(x_{n_k}, x_{m_k})\,| \leq \lim_{k\to\infty} |\, p^*(x_{n_k-1}, x_{m_k-1})\,|.$$

In the view of (3.16) and (3.17), we get $\epsilon = 0$ a contradiction. On the other hand, if Card $F = \infty$, then we can find infinitely many $k \in \mathbb{N}$ satisfying

$$\varphi(|\, p^*(x_{n_k}, x_{m_k})\,|) \leq \phi\left(|\, \frac{p^*(x_{n_k-1}, x_{n_k})(1 + p^*(x_{m_k-1}, x_{m_k}))}{1 + p^*(x_{n_k-1}, x_{m_k-1})}\,|\right) \tag{3.20}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, we infer

$$|\, p^*(x_{n_k}, x_{m_k})\,| \leq |\, \frac{p^*(x_{n_k-1}, x_{n_k})(1 + p^*(x_{m_k-1}, x_{m_k}))}{1 + p^*(x_{n_k-1}, x_{m_k-1})}\,|.$$

Taking the limit as $k \to \infty$ and in the view of (3.13) and (3.16), it follows that $\epsilon \leq 0$ and we reach a contradiction. Therefore, in both the possibilities, we reach a contradiction and therefore $\lim_{m,n\to\infty} d_{p^*}(x_n, x_m) = 0.$ Similarly we can prove that $\lim_{m,n\to\infty} d_{p^*}(x_m, x_n) = 0.$ Hence $\lim_{m,n\to\infty} d^s_{p^*}(x_n, x_m) = 0,$ which ensures that $\{x_n\}$ is a Cauchy sequence in $(X, d^s_{p^*}).$ Since (X, p^*) is a complete dualistic partial metric space, by Lemma 2.9(2), $(X, d^s_{p^*})$ is a complete metric space. Thus, there exists $v \in (X, d^s_{p^*})$ such that $x_n \to v$ as $n \to \infty,$ that is $\lim_{n\to\infty} d_{p^*}(x_n, v) = 0$ and by Lemma 2.9 (3), we know that

$$p^*(v, v) = \lim_{n\to\infty} p^*(x_n, v) = \lim_{n\to\infty} p^*(x_n, x_m). \tag{3.21}$$

Since, $\lim_{n\to\infty} d_{p^*}(x_n, v) = 0,$ by (2.2), (3.7) and (3.12), we have

$$p^*(v, v) = \lim_{n\to\infty} p^*(x_n, v) = \lim_{n\to\infty} p^*(x_n, x_m) = 0. \tag{3.22}$$

This shows that $\{x_n\}$ is a Cauchy sequence converging to $v \in (X, p^*).$ We are left to prove that v is a fixed point of $\mathcal{T}.$ Suppose that $\mathcal{T}v \neq v.$ Now applying contractive condition (3.1) and Lemma 2.9(3), we have

$$\varphi(|\, p^*(\mathcal{T}v, \mathcal{T}x_n)\,|) \leq \max\left\{\phi(|\, p^*(v, x_n)\,|), \phi(|\, \frac{p^*(x_n, \mathcal{T}x_n)(1 + p^*(v, \mathcal{T}v))}{1 + p^*(x_n, v)}\,|)\right\}. \tag{3.23}$$

Denote

$$G = \{n \in \mathbb{N} \mid \varphi(| p^*(\mathcal{T}v, \mathcal{T}x_n) |) \leq \phi(| p^*(v, x_n) |)\}$$

$$H = \{n \in \mathbb{N} \mid \varphi(| p^*(\mathcal{T}v, \mathcal{T}x_n) |) \leq \phi(| \frac{p^*(x_n, \mathcal{T}x_n)(1 + p^*(v, \mathcal{T}v))}{1 + p^*(x_n, v)} |)\}.$$

We have Card $G = \infty$ or Card $H = \infty$. If Card $G = \infty$, then there exists infinitely many $n \in \mathbb{N}$ such that

$$\varphi(| p^*(\mathcal{T}v, \mathcal{T}x_n) |) \leq \phi(| p^*(v, x_n) |) \tag{3.24}$$

and since $(\varphi, \phi) \in \mathfrak{F}$, by taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} | p^*(\mathcal{T}v, \mathcal{T}x_n) | \leq \lim_{n\to\infty} | p^*(v, x_n) |.$$

To simplify our consideration, we will denote the subsequence by the same symbol $\{\mathcal{T}x_n\}$ Since $\mathcal{T}x_n = x_{n+1}$ and $x_n \to v \in X$, this means that $\limsup p^*(v, x_n) \to 0$ and consequently $\lim_{n\to\infty} x_{n+1} = v$. We infer $| p^*(\mathcal{T}v, v) | \leq 0$ and then $| p^*(\mathcal{T}v, v) | = 0$. On the other hand, if Card $H = \infty$, then we can find infinitely many $n \in \mathbb{N}$, such that

$$\varphi(| p^*(\mathcal{T}v, \mathcal{T}x_n) | \leq \phi(| \frac{p^*(x_n, \mathcal{T}x_n)(1 + p^*(v, \mathcal{T}v))}{1 + p^*(x_n, v)} |). \tag{3.25}$$

Since $(\varphi, \phi) \in \mathfrak{F}$, $\mathcal{T}x_n = x_{n+1}$ and $\lim_{n\to\infty} x_n = v$, on letting limit as $n \to \infty$, we have

$$\lim_{n\to\infty} | p^*(\mathcal{T}v, x_{n+1}) | \leq \lim_{n\to\infty} | \frac{p^*(x_n, x_{n+1})(1 + p^*(v, \mathcal{T}v))}{1 + p^*(x_n, v)} |. \tag{3.26}$$

In the view of (3.7), arguing like above, we conclude that $| p^*(\mathcal{T}v, v) | = 0$. Therefore, in both the cases, we obtain $| p^*(\mathcal{T}v, v) | = 0$ and then $p^*(\mathcal{T}v, v) = 0$. Since $\mathcal{T}$ has (CCP), we get

$$0 = p^*(v, v) \leq kp^*(\mathcal{T}v, \mathcal{T}v) \tag{3.27}$$

On the other hand, by axiom (p_4^*) we have

$$p^*(v, v) \leq p^*(v, \mathcal{T}v) + p^*(\mathcal{T}v, v) - p^*(\mathcal{T}v, \mathcal{T}v)$$

which implies that

$$p^*(\mathcal{T}v, \mathcal{T}v) \leq 0. \tag{3.28}$$

The inequalities (3.27) and (3.28) imply that $p^*(\mathcal{T}v, \mathcal{T}v) = 0$. Thus

$$p^*(\mathcal{T}v, \mathcal{T}v) = p^*(v, v) = p^*(v, \mathcal{T}v). \tag{3.29}$$

By using axiom (p_1^*), we have $\mathcal{T}v = v$ and hence v is a fixed point of $\mathcal{T}$. Finally, we will prove the uniqueness of the fixed point. Suppose that $v^* \in X$ is another fixed point of $\mathcal{T}$ such that $v \neq v^*$. Now using contractive condition (3.1), we get

$$\begin{aligned}
\varphi(| p^*(v, v^*) |) &= \varphi(| p^*(\mathcal{T}v, \mathcal{T}v^*) |) \\
&\leq \max\left\{\phi(| p^*(v, v^*) |), \phi\left(\left| \frac{p^*(v, \mathcal{T}v^*)(1 + p^*(v, \mathcal{T}v^*))}{1 + p^*(v, v^*)} \right|\right)\right\} \\
&= \max\left\{\phi(| p^*(v, v^*) |), \phi\left(\left| \frac{p^*(v, v^*)(1 + p^*(v, v))}{1 + p^*(v, v^*)} \right|\right)\right\} \\
&= \max\{\phi(| p^*(v, v^*) |), \phi(0)\}.
\end{aligned} \tag{3.30}$$

If $\max\{\phi(| p^*(v, v^*) |), \phi(0)\} = \phi(| p^*(v, v^*) |)$, in this case from (3.30), $\varphi(| p^*(v, v^*) |) \leq \phi(| p^*(v, v^*) |)$. Since $(\varphi, \varphi) \in \mathfrak{F}$ and by Remark 2.18, we deduce that $| p^*(v, v^*) | = 0$. Similarly, if $\max\{\phi(| p^*(v, v^*) |), \phi(0)\} = \phi(0)$, then from (3.30), $\varphi(| p^*(v, v^*) |) \leq \phi(0)$. We infer that $| p^*(v, v^*) | \leq 0$ and then $| p^*(v, v^*) | = 0$. Hence in the both possibilities, $| p^*(v, v^*) | = 0$ and then $p^*(v, v^*) = 0$. Thus $p^*(v, v^*) = p^*(v, v) = p^*(v^*, v^*)$, by using axiom (p_1^*), we have $v = v^*$ and hence v is a unique fixed point of $\mathcal{T}$. This completes the proof.

From Theorem 3.1 we obtain the following corollaries.

Corollary 3.2. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying*

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) \leq \phi(| p^*(x, y) |) \tag{3.31}$$

$\forall x, y \in X$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Corollary 3.3. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying*

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) | \leq \phi(| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} |) \tag{3.32}$$

$\forall x, y \in \Delta$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with initial point x_0, converges to the fixed point.

Taking into account Example 2.21, we have the following corollary.

Corollary 3.4. *Let (X, p^*) be a complete dualistic partial metric space. Let $\mathcal{T} : X \to X$ be a mapping such that there exists two functions $\varphi, \phi : [0, \infty) \to [0, \infty)$ satisfying*

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) | \leq \max\{\varphi(| p^*(x, y) | - \phi(| p^*(x, y) |)),$$

$$\varphi(| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} | - \phi| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} |)\} \tag{3.33}$$

for all $x, y \in X$, where φ is an increasing function and ϕ is a non-decreasing function and they satisfy $\varphi(t) = \phi(t) = 0$ if and only if $t = 0$ and φ is continuous with $\phi \leq \varphi$. $\forall x, y \in X$. If $\mathcal{T}$ satisfies (CCP). Then $\mathcal{T}$ has a unique fixed point in X and the Picard iterative sequence $\{\mathcal{T}^n(x_0)\}$ with *initial point* x_0, *converges to the fixed point*.

Corollary 3.4 has the following consequences.

Corollary 3.5. *Let* (X, p^*) *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a mapping such that there exists two functions* $\varphi, \phi : [0, \infty) \to [0, \infty)$ *satisfying the same conditions as in Corollary* 3.4

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) \le \phi(| p^*(x, y) | - \phi| p^*(x, y) |) \tag{3.34}$$

for all $x, y \in X$. *If* $\mathcal{T}$ *satisfies* (*CCP*). *Then* $\mathcal{T}$ *has a unique fixed point in* X *and the Picard iterative sequence* $\{\mathcal{T}^n(x_0)\}$ *with initial point* x_0, *converges to the fixed point*.

Corollary 3.6. *Let* (X, p^*) *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a mapping such that there exists two functions* $\varphi, \phi : [0, \infty) \to [0, \infty)$ *satisfying the same conditions as in Corollary* 3.4.

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) \le \varphi\left(\left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \right|\right.$$
$$\left. - \phi\left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \right|\right) \tag{3.35}$$

for all $x, y \in X$. *If* $\mathcal{T}$ *satisfies* (*CCP*). *Then* $\mathcal{T}$ *has a unique fixed point in* X *and the Picard iterative sequence* $\{\mathcal{T}^n(x_0)\}$ *with initial point* x_0, *converges to the fixed point*.

Remark 3.8. The main result of [26] is Theorem 2.10. Notice that the rational contractive condition appearing in this theorem

$$| p^*(\mathcal{T}x, \mathcal{T}y) | \le \alpha\left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \right| + \beta | p^*(x, y) |$$

for any $x, y \in \Delta$, where $\alpha, \beta \ge 0$ and $\alpha + \beta < 1$ implies that

$$| p^*(\mathcal{T}x, \mathcal{T}y) | \le (\alpha + \beta) \max\left\{\phi\left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)} \right|, | p^*(x, y) |\right\}$$

$$\leq \max\left\{(\alpha+\beta)\left|\frac{p^*(y, \mathcal{T}y)(1+p^*(x, \mathcal{T}x))}{1+p^*(x, y)}\right|, (\alpha+\beta)|p^*(x, y)|\right\}.$$

This condition is a particular case of the contractive condition appearing in Theorem 3.1 with the pair of functions $(\varphi, \phi) \in \mathfrak{F}$ given by $\varphi = 1_{[0, \infty)}$ and $\phi = (\alpha + \beta)1_{[0, \infty)}$. Therefore, Theorem 2.10 is a particular case of the following corollary and considered as an extension and generalizations of Theorem 2.10 in the setting of complete dualistic partial metric spaces.

Corollary 3.9. *Let* (X, p^*) *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a mapping such that*

$$|p^*(\mathcal{T}x, \mathcal{T}y)| \leq \max\left\{(\alpha+\beta)\left|\frac{p^*(y, \mathcal{T}y)(1+p^*(x, \mathcal{T}x))}{1+p^*(x, y)}\right|, (\alpha+\beta)|p^*(x, y)|\right\} \tag{3.36}$$

for any $x, y \in X$, *where* $\alpha, \beta \geq 0$ *and* $\alpha + \beta < 1$. *If* $\mathcal{T}$ *satisfies* (*CCP*). *Then* $\mathcal{T}$ *has a unique fixed point in X and the Picard iterative sequence* $\{\mathcal{T}^n(x_0)\}$ *with initial point* x_0, *converges to the fixed point.*

Observations 3.10.

1. If in Corollary 3.9, we put $\alpha + \beta = c$ and $\max\left\{\left|\frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)}\right|, |p^*(x, y)|\right\} = |p^*(x, y)|$, then we get Theorem 2.3 of Oltra and Valero [28].

2. In Corollary 3.9, if we replace the range of p^* by $[0, \infty)$, put $\alpha + \beta = c$ and $\max\left\{\left|\frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x))}{1 + p^*(x, y)}\right|, |p^*(x, y)|\right\} = |p^*(x, y)|$, then we get fixed point theorem of Matthews [17].

3. If we set $p^*(x, x) = 0, \forall x \in X$ and replace the range of p^* by $[0, \infty)$, in Theorems 3.1, we retrieve corresponding theorems in metric spaces (see [8]).

4. If we set $p^*(x, x) \in [0, \infty), \forall x, y \in X$ in Theorems 3.1, we retrieve

corresponding theorems in partial metric spaces (see [33]).

Taking into account Example 2.20, we have the following corollary.

Corollary 3.11. *Let* (X, p^*) *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a mapping such that there exist* $\ell \in \mathcal{S}$ *(see Example 2.20) satisfying*

$$\varphi(|p^*(\mathcal{T}x, \mathcal{T}y)|) \leq \max\{\ell(|p^*(x, y)|)|p^*(x, y)|,$$
$$\ell\left(\left|\frac{p^*(y, \mathcal{T}y)(1+p^*(x, \mathcal{T}x))}{1+p^*(x, y)}\right|\right)\left|\frac{p^*(y, \mathcal{T}y)(1+p^*(x, \mathcal{T}x))}{1+p^*(x, y)}\right|\} \quad (3.37)$$

for all $x, y \in \Delta$. *If* $\mathcal{T}$ *satisfies (CCP). Then* $\mathcal{T}$ *has a unique fixed point in X and the Picard iterative sequence* $\{\mathcal{T}^n(x_0)\}$ *with initial point* x_0, *converges to the fixed point.*

Following Corollary is a generalization of main result of Geraghty [9].

Corollary 3.11. *Let* (X, p^*) *be a complete dualistic partial metric space. Let* $\mathcal{T} : X \to X$ *be a mapping such that there exist* $\ell \in \mathcal{S}$ *(see Example 2.20) satisfying*

$$\varphi(|p^*(\mathcal{T}x, \mathcal{T}y)|) \leq \ell(|p^*(x, y)|)|p^*(x, y)| \quad (3.38)$$

for all $x, y \in X$. *If* $\mathcal{T}$ *satisfies (CCP). Then* $\mathcal{T}$ *has a unique fixed point in X and the Picard iterative sequence* $\{\mathcal{T}^n(x_0)\}$ *with initial point* x_0, *converges to the fixed point.*

4. Examples

In this section, we give an example in support of our main result.

Example 4.1. Let $X = (-\infty, 0]^2$. Define $p^* : X \times X \to (-\infty, \infty)$ by $p^*(x, y) = \max\{x_1, y_1\}$ where $x = (x_1, y_1)$ and $y = (x_2, y_2)$. It is easy to check that $((-\infty, 0]^2, p^*)$ is a complete dualistic partial metric space. Define $\mathcal{T} : (-\infty, 0]^2 \to (-\infty, 0]^2$ by $\mathcal{T}x = x^2$, $\forall x \in (-\infty, 0]^2$. Since

$$\max\{x_1, y_1\} \le \max\left\{\frac{x_1}{2}, \frac{y_1}{2}\right\} \Rightarrow p^*(x, y) \le p^*(\mathcal{T}x, \mathcal{T}y), \forall x, y \in (-\infty, 0]^2.$$

Hence $\mathcal{T}$ satisfies (CCP). Define $\varphi, \phi : [0, \infty) \to [0, \infty)$ as follows: $\varphi(t) = \ln\left(\frac{5t+1}{2}\right)$ and $\varphi(t) = \ln\left(\frac{3t+1}{2}\right)$, $\forall t \in [0, \infty)$. Clearly, $(\varphi, \phi) \in \mathfrak{F}$. We shall show (3.1) is satisfied. Without loss of generality, assume that $x_1 \le y_1$. Then we have

$$\varphi(| p^*(\mathcal{T}x, \mathcal{T}y) |) = \ln\left(\frac{5| p^*(\mathcal{T}x, \mathcal{T}y) | + 1}{12}\right)$$

$$= \ln\left(\frac{5| p^*\left(\frac{x}{2}, \frac{y}{2}\right) | + 1}{12}\right) = \ln\left(\frac{5| \left(\frac{y_1}{2}\right) | + 1}{12}\right)$$

$$= \ln\left(\frac{5}{24}| y_1 | + \frac{1}{12}\right).$$

On the other hand,

$$\phi(| p^*(x, y) |) = \ln\left(\frac{3| p^*(x, y) | + 1}{12}\right) = \ln\left(\frac{3| y_1 | + 1}{12}\right) = \ln\left(\frac{3}{12}| y_1 | + \frac{1}{12}\right)$$

$$\phi\left(\left| \frac{p^*(y, \mathcal{T}y)(1 + p^*(x, \mathcal{T}x)}{1 + p^*(x, y)}\right)\right. = \phi\left(\left| \frac{\frac{y_1}{2}\left(1 + \frac{x_1}{2}\right)}{1 + y_1}\right)\right. = \left(\left| \frac{y_1(2 + x_1)}{4(1 + y_1)} \right|\right)$$

$$= \ln\left(\left| \frac{3\left| \frac{y_1(2 + x_1)}{4(1 + y_1)} \right| + 1}{12}\right)\right.$$

$$= \ln\left(\frac{3| y_1(2 + x_1) | + 4| 1 + y_1 |}{24} \left.\right|\right).$$

Combining the observations above, we get

$$\varphi(p^*(\mathcal{T}x,\ \mathcal{T}y)\,|) = \ln\left(\frac{5}{24}|\ y_1\ | + \frac{1}{12}\right) = \ln\left(\frac{3}{12}|\ y_1\ | + \frac{1}{12}\right)$$

$$\leq \max\left\{\ln\left(\frac{3}{12}|\ y_1\ | + \frac{1}{12}\right),\ \ln\left(\frac{3|\ y_1(2 + x_1)\,| + 4|\ 1 + y_1\ |}{24}\right)\right\}$$

$$= \max\left\{\phi(p^*(x,\ y)\,|),\ \phi\left(\frac{p^*(y,\ \mathcal{T}y)(1 + p^*(x,\ \mathcal{T}x))}{1 + p^*(x,\ y)}\ |\right)\right\}.$$

Thus all the conditions of Theorem 3.1 are satisfied. Hence $\mathcal{T}$ has a fixed point, indeed $v = (0,\ 0)$ is a fixed point.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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