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ENERGY AND WIENER INDEX OF INCOMPARABILITY GRAPHS OF SPECIAL LATTICES

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Abstract

In this paper, we study the energy of incomparability graphs of lattice L_n, where n is product of two distinct prime numbers. We discuss Wiener index of incomparability graph of such lattices. A formula is given for Wiener index of incomparability graph of such lattices.

1. Introduction

The concept of graphs associated with an algebraic structure like ring is introduced by Beck, [4]. He studied coloring of commutative ring R. In [5], Anderson and Livingston studied zero divisor graphs of ring R. Duffus and Rival discuss the path lengths in the covering graphs from lattices in [6]. This motivated the study of various type of graphs in lattices. In [2], Wasadikar and Survase defined the incomparability graphs of lattices. In 2015, Wasadikar and Dabhole [1], studied graphs of some special lattices L_n where n is positive integer.

The energy of graph G was first defined by Gutman, [7]. This term is originated from theoretical chemistry. Wiener introduced the concept of wiener index in 1947.

We study the concept Energy and Wiener index of incomparability graph of lattice L_n. First section is introduction and second section deals with

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preliminaries. In third section, we discuss energy of incomparability graph of lattices L_n where n is product of two distinct prime numbers. We obtain energy and Wiener index of the incomparability graph of L_n whenever $n = pq,\ p^2q,\ p^3q.$ In fourth section, formula for Wiener index is obtained for $n = p^m q$ whenever $m \geq 2$.

2. Preliminaries

We recall some basic concepts of graph theory from [8]. A graph G consists of nonempty set of vertices (points) and edges (lines) joining to unordered pair of vertices. A graph is a simple graph if it is free from loop and parallel edges. Complement of graph G is a graph G' in which the edges which are adjacent in graph G are non adjacent. The graph G is said to be self- complementary if it is isomorphic to itself. The adjacency matrix $A(G)$ of graph G is a square matrix in which if v_i is adjacent to v_j then $a_{ij} = 1$ otherwise $a_{ij} = 0$, $V(G) = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix of a simple graph is symmetric therefore the eigenvalues are real. The diameter $diam(G)$ of a graph G is $\sup\{d(x, y)/x, y \in G\}$. The energy of a graph G is absolute sum of the eigenvalues of the adjacency matrix of a graph G. The distance $d(u, v)$ between two vertices u and v is the minimum number of edges in a path of G joining to u and v. The Wiener index of graph G is defined as $W(G) = \sum d(u, v)$.

A poset $(L, \leq)$ is a lattice whenever in $f(u, v) = u \wedge v$ and $\sup(u, v) = u \vee v$ exists in L for every pair of elements $u, v \in L$. Any two elements $u, v \leq L$. are incomparable if $u \not\leq v$ and $v \not\leq u$ it is denoted by $u \| v$. Throughout this paper L_n denotes the lattice of divisors of positive integer n under partial order defined as divisibility. The incomparability graph of L_n is $G(L_n)$ in which $u - v$ is adjacent if and only if $\gcd(u, v) \neq u$ or $\gcd(u, v) \neq v$ for any $u, v \in L_n$.

Undefined terms of graph are from [8] and undefined terms from lattice theory are from [3].

3. Energy and Wiener index

In this section, we discuss energy of incomparability graph of lattice L_n where $n = pq,\ p^2q$.

Theorem 3.1. *Let L_n be a lattice such that $n = pq$ and p, q are distinct prime integers. The energy of incomparability graph $G(L_n)$ is $E(G(L_{pq}) = 2$ and the Wiener index $W(G(Lpq)) = 1$.*

Proof. Figure 1 represents the lattice Lpq for distinct prime integers p and q. We note that p and q are atoms and hence they are incomparable. Figure 2 represents incomparability graph $G(L_{pq})$ of lattice Lpq. Being incomparable $p - q$ is connected by an edge. This is a simple graph with two vertices p and q and an edge $e = (p, q)$. Therefore the Adjacency matrix $A[G(L_{pq})]$ of $G(L_{pq})$ is $A[G(L_{pq})] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

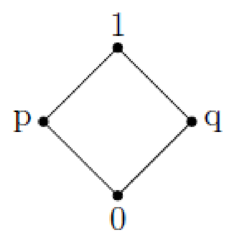

Figure 1.

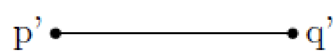

Figure 2.

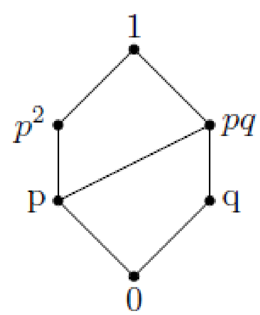

Figure 3.

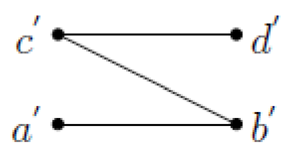

Figure 4.

This matrix is symmetric with diagonal elements zero. The eigenvalues of matrix $A[G(Lpq)]$ are 1 and -1. Hence Energy of a graph $G(Lpq)$ is $E(G(Lpq)) = 1 + 1 = 2$

Wiener index of a graph

$$W(G) = \frac{1}{2}\sum d(x,\ y)$$

$$W(G(L_n)) = \frac{1+1}{2}$$

$$= 1.$$

Theorem 3.2. *Let* L_n *be a lattice such that* $n = p^2 q$ *and* p, q *are distinct prime integers. The energy of incomparability graph* $G(Ln)$ *is* $E(G(L_{p^2q})) = 4.47213$ *and the Wiener index* $W(G(L_{p^2q})) = 10$.

Proof. Figure 3 represents the lattice L_{p^2q} for distinct prime integers p and q. We note that the pairs $(p,\ q), (p^2,\ pq)$ are incomparable in L_{p^2q}. Figure 4 represents incomparability graph $G(L_{p^2q})$ of lattice L_{p^2q}. It is a simple graph with four vertices and three edges. The Adjacency matrix $A[G(L_n)]$ of $G(L_{p^2q})$ is

$$A[G(L_n)] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This matrix is symmetric with diagonal elements zero. The eigenvalues of matrix $A[G(L_n)]$ are ± 1.61 and ± 0.61. Hence Energy of the graph is $E(G(L_n)) = 2(1.61 + 0.61) = 4.47213$. And, Wiener index of a graph $W(G(L_n)) = 10$.

Corollary 3.1. *The incomparability graph* $G(L_{p^2q})$ *is self complementary*.

Proof. The graph $G(L_{p^2q})$ is given in figure 4. We know that in a complement graph G' of a graph G, any two vertices are adjacent if they are non adjacent G. The complement of $G(L_{p^2q})$ is represented in figure 5

Figure 5.

The number of vertices and edges in both the graphs are same. There are two pendant vertices which are adjacent to vertex of degree two and both the vertices of degree two are adjacent to each other. Hence these two graphs are isomorphic. Therefore the graph $G(L_{p^2q})$ is self complementary.

Following remark follows from corollary 3.1.

Remark 3.1. Energy and Wiener index of $G(L_{p^2q})$ and $G'(L_{p^2q})$ is same.

4. Wiener Index of L_{p^mq}

In this section, a formula for Wiener index of incomparability graph L_{p^mq} is obtained.

Remark 4.1. The diameter of the incomparability graph of lattice L_{p^mq}, $m \geq 2$ is 3.

The incomparability graph of lattice L_{p^mq} is a simple connected graph. The distance between any two vertices $x, y \in G(L_{p^mq})$ is ≤ 3.

Hence the diameter of $G(L_{p^mq})$ is 3. This fact can be used to calculate Wiener index of L_{p^mq}.

Example 4.1. The lattice L_{p^3q} is given in Figure 6 and its incomparability graph $G(L_{p^3q})$ is given in the figure 7.

The graph has 6 vertices and 6 edges. Hence, the Wiener index

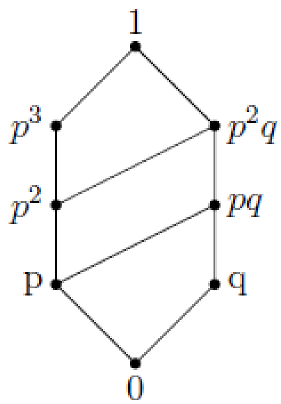

Figure 6.

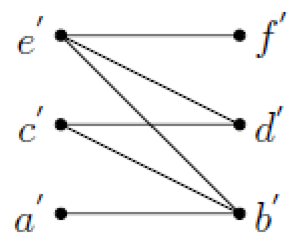

Figure 7.

$W(G(L_{p^3q})) =$

$((1+2+3+2+3)+(1+2+1+2)+(1+2+3)+(1+2)+1) = 27.$

Theorem 4.1. *Let* L_n *be a lattice such that* $n = p^m q$, $m \geq 2$ *and* p, q *are distinct prime integers. The Winner index of incomparability graph of lattice is* $W(L_{p^m q}) = m + 8\binom{m}{2}$, $m \geq 2$.

Proof. Let $L_{p^m q}$ be a lattice and $G(L_{p^m q})$ be the incomparability graph of $L_{p^m q}$. Figure 8 represents $L_{p^m q}$ and Figure 9 represents $G(L_{p^m q})$. The graph $G(L_{p^m q})$ is a simply connected graph on $2m$ vertices. The vertex set of incomparability graph $G(L_{p^m q})$ is $\{p, q, pq, p^2, p^2q, \ldots, p^{m-2}q, p^m\}$.

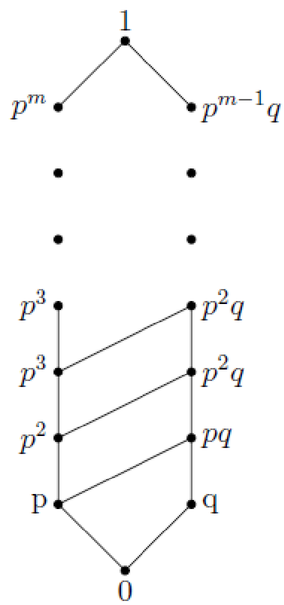

Figure 8.

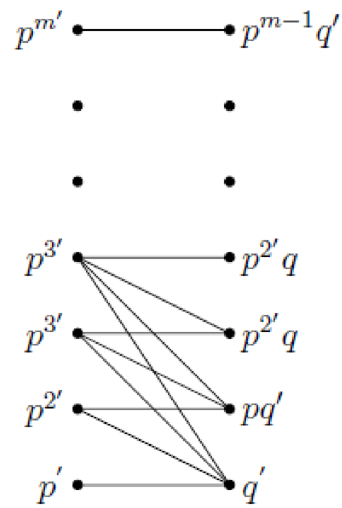

Figure 9.

To obtain the Wiener index, fix vertex p, the distance $d(p, x_i)$, $x_i \in V(G(L_{p^m q})) = (1 + (2+3) + (2+3) + \ldots + (2+3))$. We note that here $(2+3)$ occurred $m-1$ times. Now fix vertex q and find $d(q, x_i) = ((1+2) + (1+2) + (1+2) + \ldots + (1+2))$, here $(2+3)$ occurred $m-1$ times and $x_i \neq p$ and so on.

Since the diameter of incomparability graph is 3, the distances between any two vertices is ≥ 3. Therefore

$$W(G(L_{p^m q}))$$

$$= m + ((m-1)(2+3) + (m-2)(2+3) + \ldots + 1(2+3))$$

$$+ ((m-1)(1+2) + (m-2)(1+2) + \ldots + 1(1+2))$$

$$= m + \binom{m}{2}(2+3) + \binom{m}{2}(1+2)$$

$$= m + 8\binom{m}{2}$$

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