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# SURVEY ON HEXAGONAL PICTURE LANGUAGES

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## Abstract

Hexagonal pictures occur in several applicational areas especially in picture processing and image analysis. Motivated by the fact that hexagonal arrays on triangular grids can be treated as a two dimensional representation of three dimensional blocks. Siromoney and Siromoney proposed grammatical model for generation of hexagonal pictures, introducing a very natural notion of arrowhead for hexagonal arrays. Subsequently Meena Mahajan and Kamala Krithivasan studied hexagonal array languages and their properties. Later Dersanambika et al. introduced the notion of local and recognizable picture languages. Labeled Wang tiles are also used in recognition of picture languages.

## 1. Introduction

Hexagonal arrays and hexagonal patterns are found in literature on picture processing and scene analysis. In (2003) K.S. Dersanambika et al. have defined that hexagonal pictures are used particularly in picture processing and also in image analysis [3]. The hexagonal picture language

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over $\Sigma$ with all sides of equal length is recognizable but not local. Here they used different formalism for hexagonal picture languages. The formalism's are hexagonal tiling system, local hexagonal picture languages, recognizable hexagonal picture languages, labeled hexagonal wang tiles and finally about wang systems. Also they introduced $xyz$ domino systems and also obtained its equivalence with hexagonal tiling systems.

On the other hand, a lot of research relating DNA computing to many areas such as designing DNA sequences, DNA encoding and many other topics has been studied. Splicing systems on biological considerations make use of new operations called splicing on strings of symbols. This splicing operations on images of hexagonal arrays was introduced by M. H. Begum [2]. In chromosome analysis programme the circumscribing polygons associated with each image turn out to be hexagon.

Some of the formalism to generate hexagonal arrays are hexagonal kolam array grammars and its generalization. Hexagonal array grammars, sequential and parallel applications of rewriting rules and arrowhead catenations are features of this model. Hexagonal tile rewriting grammars and regional hexagonal tile rewriting grammars are the hexagonal tiling based isometric grammar models, which have more generating capacity than hexagonal array grammars. Hexagonal pursa grammars and pure 2D hexagonal context free grammars are non isometric formalism where parallel application of rewriting rules are attempted in the derivation of hexagonal pictures.

Another hexagonal picture generating mechanism is hexagonal array token petrinet structure. Petrinets have been used for analysing systems that are concurrent, asynchronous, distributed, parallel non deterministic or stochastic. Hexagonal array tokens are used here [13]. Here they introduced a generalization to this model, Adjunt hexagonal array token Petri Net structure in cooperating adjunction operation, a variation in the position of arrowhead catenations.

## 2. Hexagonal Arrays and Grammars

Hexagonal arrays and hexagonal patterns are found in the literature on picture processing and scene analysis. In [11] Siromoney and Siromoney defined a new kind of catenation called arrowhead catenation. It is very natural for hexagonal arrays. The main advantage is that an arrowhead catenated to a hexagon results in a hexagon and amounts to catenation on three dimensional block to another along a hidden face. Starting from a hexagon representing a block and allowing catenations along the three dimensions we can generate hexagons with desired properties.

**Definition 2.1.** A hexagon ABCDEF is called an arrowhead if $AB = EF$, $BC = DE$, $AB$ parallel to $EF$ and $BC$ parallel to $DE$. Opposite sides $CD$ and $FA$ are equal and parallel.

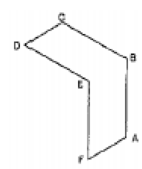


**Figure 1.** Arrowhead.

Hexagonal shape is maintained in every generation by applying catenations of arrowheads.

By using this catenation hexagonal kolam arrays were introduced. A hexagonal kolam arrays $G$ is a 5-tuple $G = (V, I, P, S, L)$ where

- $V$ is the union of non terminals and intermediates
- $I$ is the finite set of terminals.
- $P$ is the production rule.
- $S$ is the start symbol and
- $L$ is the set of intermediate languages.

The examples of hexagonal array kolam grammars can be seen in [11]. In the hexagonal array kolam models, the sequence of application of rules is parenthesized and the removal of parenthesis starts subject to conditions of catenations. The growth patterns in this model takes place along the edges by catenating arrowheads in any one of fixed directions.

A hexagonal kolam array grammar is called $(R : R)$, $(R : CF)$, $(R : CS)$ according to the intermediate language are regular or context sensitive or context free. Also it generates various hexagonal patterns. For example $(R : R)$ hexagonal kolam array grammar generates a hexagonal pattern found on the shell of tortoise. By constructing arrowheads in the lower left, lower right and up axes also corresponding upper left, upper right and down axes. We can obtain the corresponding perceptual twins as pictures of the given set of blocks. $(R : R)$ hexagonal kolam arrays and its dual given below generates the family of perceptual twins. A single operation rotation through 1800 on a single grammar corresponds to transformations on the set of pictures obtaining the perceptual twins of original set of blocks. In the kolam models arrowheads are catenated independent of the original hexagon to which they are catenated but dependent on a specific grammar generating that arrowhead.

In [12] by retaining the arrowhead catenations they modified the definitions of hexagonal kolam array grammar by allowing initial rules causing derivations from the start symbol to begin with an initial array. The rules are used for subsequent derivation until non terminals are replaced. The model is named as hexagonal array grammar. In this model the initial hexagonal array is kept in the leftmost position and from left to right the arrowheads are catenated. This model helps to increase the generative capacity resulting in interesting picture classes of hexagonal arrays.

**2.1. P systems**

K. S. Dersanambika et al. [4] have defined that a hexagonal picture is generated by evolving an initial *p* system into a stable *p* system. The graph of this *p* system is a triangular grid in which each inner node is connected with six neighbours each border node is connected with four neighbours each border node is connected with four neighbours and each corner with three neighbours. Always the hexagonal picture generation starts from the top left corner of hexagonal picture. The *p* systems generate only hexagonal pictures by distinguishing 13 types of nodes. They are represented by 13 symbols. $\{P_{000}, P_{001}, P_{002}, P_{100}, P_{200}, P_{111}, P_{012}, P_{022}, P_{210}, P_{122}, P_{220}, P_{221}, P_{222}\}$ Hexagonal picture over an alphabet and the hexagonal grid is shown in the figure below.

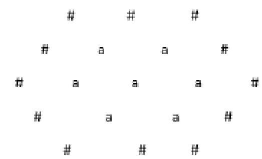


**Figure 2.** Hexagonal picture over one letter alphabet.

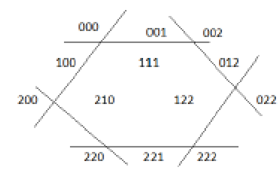


**Figure 3.** Hexagonal grid.

Generation of hexagonal arrays by *p* systems illustrates the technique to handle arrays in triangular grids. The *p* systems use rules that rewrite arrays in triangular grids.

### 3. Hexagonal Tiling System and Wang Tiles

K. S. Dersanambika et al. defined hexagonal pictures and three direction online tessilation automata used for recognizing hexagonal picture languages. In [5] they used different formalism for hexagonal picture languages. The formalism of hexagonal tiling systems, local hexagonal picture languages, recognizable hexagonal picture languages, labelled hexagonal wang tiles and finally about wang systems. They introduced hfbs domino systems and proved that this is equivalent to hexagonal tiling system. Hexagonal picture over the alphabet abcdef surrounded by a special boundary symbol is given below.

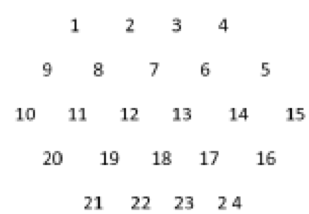


**Figure 4.** Bordered hexagonal picture.

The axis *xyz* and the coordinates of each element for the hexagonal pictures is given below.

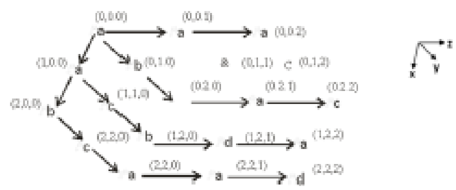

**Figure 5.** Hexagonal picture in terms of coordinates.

A hexagonal tile is of the form

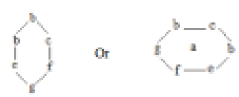

**Figure 6.** Hexagonal tile.

**Definition 3.1.** Let $\Gamma$ be a finite alphabet. A hexagonal picture $\mathrm{L} \subset \Gamma^{**\mathrm{H}}$ is local if there exist a finite set $\Delta$ of tiles over $\Gamma \cup \{\#\}$ such that $\mathrm{L} = \{p \in \Gamma^{**H} \mid B_{222}(\hat{p}) \subseteq \Delta\}$.

The family of hexagonal pictures is denoted by HLOC. The language of hexagons over one letter alphabet is a local language. But the language of hexagons over one letter alphabet with all sides of equal length is not local.

**Definition 3.2.** A hexagonal picture language $\mathrm{L} \subset \Sigma^{**\mathrm{H}}$ is recognizable if there exist a local language *L*' over the alphabet $\Gamma$ and a mapping $\prod : \Gamma \to \Sigma$ such that $\mathrm{L} = \prod(L')$.

**Definition 3.3.** A tiling system *T* is a 4-tuple $(\Gamma, \Sigma, \prod, \theta)$ where $\Sigma$ and $\Gamma$ are two finite set of symbols. $\prod : \Gamma \to \Sigma$ is a projection and $\theta$ is a set of tiles over the alphabet $\Gamma \cup \{\#\}$.

A hexagonal wang tile is a coloured tile on each of its edges. By using this hexagonal wang tiles a hexagonal wang tile system is defined and the equivalence between the hexagonal tiling system and hexagonal wang system is proved in [5].

### 3.1. Hexagonal contextual array *p* systems

In [6] contextual hexagonal array *p* systems in which the rules consist of attaching contents to hexagonal arrays depending upon a choice mapping. Contextual hexagonal array *p* systems with rules of the types corresponding to these variants are more powerful than the usual hexagonal array contextual grammars and their variants. They also studied about the external and internal hexagonal array contextual *p* systems with choice functions. Some properties like the relation between external hexagonal array contextual *p* systems with erased context and external hexagonal array contextual grammars with regular controls. These studies can be extended to tissue *p* systems which generate hexagonal arrays using contextual rules.

#### 3.1.1. Hexagonal Tile rewriting grammars

Tile rewriting grammars [14] is another formalism for defining hexagonal picture languages. It combines the rewriting rules with the tiling systems. Local hexagonal picture language is strictly included in the language generated by hexagonal tile rewriting grammars.

## 4. Recognizability of Hexagonal Wang Automata

A new version of automaton called hexagonal wang automata was introduced in [1]. To recognize hexagonal pictures a formalism called hexagonal wang systems were used. The hexagonal wang system can also be proved equivalent to hexagonal tiling system. A hexagonal wang system is defined as follows.

**Definition 4.1.** A hexagonal wang system is a triplet $W = \langle \Sigma, Q, \theta \rangle$, where $\Sigma$ is the finite alphabet, $Q$ is a set of colours, $\theta$ is a subset of $\Sigma$.

### 4.1. Scanning strategies

In a hexagonal picture a scanning strategy provides a method to visit all positions in it. There are different types of scanning strategies. One pass strategies means those visit each position in each domain exactly once. A strategy is said to be blind if it proceeds locally by scanning adjacent positions. Some one pass strategies for scanning hexagonal pictures are given below.


**Figure 7.** Boustrophedon.

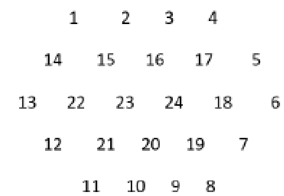


**Figure 8.** Spiral.

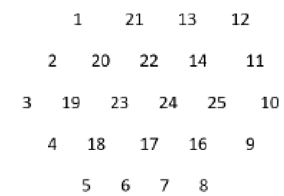


**Figure 9.** Nested Pentagon.

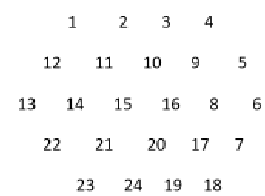


**Figure 10.** Slanted $L$ like path.

Some scanning strategies may be blind if it is included by a triplet $\langle \eta, S, d \rangle$, where $\eta$ is the next position, $S$ is the start symbol, $d$ is the direction. If a scanning strategy is blind and one pass it is called polite.

### 4.2. Hexagonal Wang automata

Hexagonal wang automata use the polite scanning strategies. It can be viewed as a pointer that visit a hexagonal picture by moving from a position to an adjacent one, colouring each edge of the position where it visits. The scanning strategies can be considered as an alternative for the online tessilation automata since in polite scanning strategies the size of the picture is not needed. Also hexagonal wang automata can be considered as a model defining HREC.

### 4.3. Hexagonal Array Splicing and Petri nets

The splicing system make use of a new operation called splicing on strings of symbols. A method of splicing operation on images of hexagonal arrays was introduced by M.H. Begum. Works on this led to introduce Parallel double splicing on images. In [10] they defined $Z$-double, $X$-double and $Y$-double hexagonal array splicing of a picture language $L$ and also defined the corresponding double hexagonal array splicing system.

Another new model to generate hexagonal arrays was by using Petri net structure. It is an abstract formal model. They are used for analysing systems that are concurrent, asynchronous, distributed, parallel, non-deterministic and stochastic. In order to stimulate concurrent activities, tokens are used in petri net. A language is associated with its execution. By defining a labeling function for transitions over an alphabet the set of all firing sequences, starting from a specific initial marking leading to a finite set of terminal markings, generates a language over the alphabet by the process of catenation. The resulting model on hexagonal picture language is known as hexagonal array token petri net structure.

If some adjunction are defined to join the arrowhead into the hexagonal array of the input place the resulting model is called Adjunct Hexagonal Array Token Petri Net structure. It has got application in the tilling patterns and generation of kolam patterns. It is also used in biomedical image processing. In [13] HATPN structure is defined along with a control feature called inhibitor arcs. The non empty intersection of the hexagonal picture languages generated by this models with other models clearly suggests that this model can generate a wide variety of digitized hexagonal pictures and patterns.

## 5. Hexagonal Tiling Recognizable Picture Series

A picture is used for understanding things in a better way while tiling systems are used to define the languages in REC. The *xyz* domino system were introduced earlier and proved the equivalence with tiling system. In [7] weights are assigned to these local and *xyz* local hexagonal picture using hexagonal tiles or hexagonal dominos. As in local languages here also the language is *xyz* local or simply local if there exist a set of hexagonal dominos or hexagonal tiles over the alphabet such that the whole picture is generated by a finite set of dominos or tiles. A weighted hexagonal tile system is defined in [7] and using it hexagonal picture series can be computed. In a similar way a weighted hexagonal domino system is also defined there. By using this a weighted hexapolic picture automaton (WHPA) is introduced.

The properties like projection is made clear for WHPA. For a hexagonal picture $p$ with successful computation, the product values coincide with weight. This is because for every position $p$ in a hexagonal picture there exist precisely one factor weight in the product. If $p$ has no successful computation then it reduces to zero. Every *xyz* local series is hexagonal tile local. By using the construction along arrowheads a weighted hexapolic picture automaton can be generated. This creates a successful run for every element. The run stimulates the distribution of elements along the canonical hexagonal tile covering of a hexagonal picture. The hexagonal picture series is the projection of hexagonal tile local series and also of *xyz* local series.

**5.1. Unambiguous hexagonal picture series**

A hexagonal tiling or a hexagonal domino system $(\sum, \Gamma, \theta', \prod')$ is called unambiguous hexagonal representation of language $L$. [8]. The unambiguous hexagonal picture (UHP) tile $\sum^{**H}$ represents the family of languages that are unambiguously hexagonal tiling recognizable over an alphabet $\Sigma$. The unambiguously hexagonal rational operations are 8 in numbers [8]. Among the families of subsets of $\sum^{**H}$, unambiguously hexagonal projection of rational languages is the smallest and is closed under injective projections. Every rational string language and every *xyz* local hexagonal picture languages are unambiguously rational. The family UHP tile is closed under the above mentioned unambiguously hexagonal rational operations and injective projections.

For every input hexagonal picture, there exist at most one successful run then the weighted hexapolic picture automaton is said to be unambiguous. If there is an injective mapping $\prod$ on the language $L$, generated by unambiguously hexagonal picture automaton computing $\prod(L)$. The languages generated by UHP tile are unambiguous.

## 6. Conclusion

The various properties and methods of representations of hexagonal picture languages has been discussed. The fundamental analysis of hexagonal picture languages, hexagonal arrays and grammars, P systems generating hexagonal pictures, hexagonal tiling system and wang tiles, hexagonal tile rewriting grammars, recognizability using wang automata, Petri nets generating hexagonal pictures, hexagonal splicing systems, hexagonal tiling recognizable picture series and unambiguous hexagonal picture series have been discussed.

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