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THE THREE-DIMENSIONAL DIAMOND FUZZY SETS AND THEIR CUT SETS

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Abstract

In this paper, we have attempted another sort of L-fuzzy set name it as diamond fuzzy set and their cut sets with the assistance of precious diamond fuzzy number. We have given meaning of α-lower and upper cut precious diamond fuzzy set, α-upper and lower Q-cut diamond fuzzy sets and a few properties with verification.

1. Introduction

Since the concept of fuzzy sets was introduced by Zadehin 1965 [1], the theories of fuzzy sets and fuzzy systems developed rapidly. As is well known, the cut set (or level set) of fuzzy set [2] is an important concept in theory of fuzzy sets and systems, which plays a significant role in fuzzy algebra [3, 4], fuzzy reasoning [5, 6], fuzzy measure [7-9] and soon. The cut set is the bridge connecting the fuzzy sets and classical sets. Based on the cut sets, the decomposition theorems and representation theorems can be established [2]. The cut sets on fuzzy sets are described in [10] by using the neighborhood relations between fuzzy point and fuzzy set. It is pointed out that there are four kinds of definitions of cutsets on fuzzy sets, each of which has similar properties. Also, the decomposition theorems and representation theorems can be established based on each kind of cut sets. With the development of the theory on fuzzy sets, Goguen introduced L-fuzzy sets as an extension of Zadeh fuzzy sets in 1967 [11]. Since then, many L-fuzzy sets are put forward, such as the interval-valued fuzzy sets [12], the intuitionistic fuzzy sets [13],

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the interval-valued intuitionistic fuzzy sets [14] and the type-2 fuzzy sets [15]. In [16], four new kinds of cutsets of intuitionistic fuzzy sets and interval-valued fuzzy sets were put forward, which are defined by the 3-valued fuzzy sets. The cut sets on intuitionistic fuzzy sets and interval-valued fuzzy sets have similar properties with the cutest of fuzzy sets. Furthermore, based on those cutsets [16], the decomposition theorems and representation theorems of the intuitionistic fuzzy sets were obtained. In practical applications, we can use triple value form such as (very good, good, more or less good) or (very young, young, more or less young) to describe with there an object belongs to a notion. From this point of view, a new kind of L-fuzzy set is introduced, which is called the three-dimensional fuzzy set in this paper. Furthermore, the cutsets of three-dimensional fuzzy sets are defined by the 4-valued fuzzy sets and their properties are discussed. Based on these kinds of cutsets, the decomposition theorems and representation theorems of three-dimensional fuzzy sets are obtained. Left interval-valued intuitionistic fuzzy sets and the right interval-valued intuitionistic fuzzy sets are introduced. We prove that the lattices constructed by these two special L-fuzzy sets are not equivalent to sublattices of lattice constructed by interval-valued intuitionistic fuzzy sets. Furthermore, we show that the three-dimensional fuzzy sets are equivalent to the left interval-valued intuitionistic fuzzy sets or the right interval-valued intuitionistic fuzzy sets. This paper is organized as follows. In Section 2, some definitions and theorems are given. In Section 3, cut sets of the three-dimensional diamond fuzzy sets are defined and their properties are given.

2. Preliminaries

Definition 2.1. ([1]). Let X be a set. Them aping $A : X \to [0, 1]$ is called a fuzzy set on X.

Definition 2.2 ([10]). Let A be a fuzzy set on X and $\alpha \in [0, 1]$. We call

$$A_{\alpha} = \{x \mid x \in X, A(x) \geq \alpha\}, A_{\alpha} = \{x \mid x \in X, A(x) \geq \alpha\}$$

α-upper cut set and α-strong upper cut set of A, respectively. We call

$$A^{\alpha} = \{x \mid x \in X, A(x) \leq \alpha\}, A^{\alpha} = \{x \mid x \in X, A(x) \geq \alpha\},$$

α-lower cut set and α-strong lower cut set of A, respectively.

$$A_{[\alpha]} = \{x \mid x \in X,\ \alpha + A(x) \geq 1\},\ A_{\underline{[\alpha]}} = \{x \mid x \in X,\ \alpha + A(x) > 1\}$$

α-upper Q-cut set and α-strong upper Q-cut set of A, respectively.

$$A^{[\alpha]} = \{x \mid x \in X,\ \alpha + A(x) \leq 1\},\ A^{\underline{[\alpha]}} = \{x \mid x \in X,\ \alpha + A(x) > 1\}$$

α-lower Q-cut set and α-strong lower Q-cut set off A, respectively.

Definition 2.3 ([16])**.** If L is a completely distributive lattice and there is a mapping $X : L \to L$ such that

(i) $a \leq b \Rightarrow b' \leq a'$;

(ii) $(a')' = a.$ Then L is called an F lattice.

Definition 2.4 ([13])**.** Let X be a set and $\mu_A : X \to [0, 1], \upsilon_A : X \to [0, 1]$ be two mappings. If $\mu_A(x) + \upsilon_A(x) \leq 1,\ \forall x \in X,$ then we call $A = (X,\ \mu_A,\ \upsilon_A)$ an intuitionistic fuzzy sub set (IFS) over X.

Definition 2.5 ([10])**.** Let X be a set 2^X represents the power set of X and $H : [0, 1] \to 2^X$ is a mapping.

(i) If $(a_1 < a_2 \Rightarrow H(a_1) \supset H(a_2))$, then we call H an inverse order nested set of X;

(ii) If $(a_1 < a_2 \Rightarrow H(a_1) \subset H(a_2))$, then we call H an order nested set of X.

Definition 2.6 ([12])**.** Let X be a set. If $A(x) = [A^-(x),\ A^+(x)],\ \forall x \in X,$ then we call A an interval-valued fuzzy set (IVFS) over X, where $0 \leq A^-(x) \leq A^+(x) \leq 1,\ \forall x \in X.$

Definition 2.7 ([16])**.** Let $A = (X,\ \mu_A,\ \upsilon_A)$ be an intuitionistic fuzzy set and $a \in [0, 1]$.

(i) We call

$$A_\alpha(x) = \begin{cases} 1 & \mu_A(x) \geq a \\ \frac{1}{2} & \mu_A(x) < a \leq 1 - \vartheta_A(x) \\ 0 & a > 1 - \mu_A(x) \end{cases} \text{ and } A_{\underline{\alpha}}(x) = \begin{cases} 1 & \mu_A(x) \geq a \\ \frac{1}{2} & \mu_A(x) < a \leq 1 - \vartheta_A(x). \\ 0 & \sigma > 1 - \mu_A(x) \end{cases}$$

α-upper cut set and α-strong upper cut set of A, respectively.

(ii) We call

$$A^{\alpha}(x)=\begin{cases}1 & \vartheta_A(x)\geq a\\ \frac{1}{2} & \vartheta_A(x)<a\leq 1-\mu_A(x)\\ 0 & a>1-\mu_A(x)\end{cases} \text{ and } A^{\underline{\alpha}}(x)=\begin{cases}1 & \vartheta_A(x)\geq a\\ \frac{1}{2} & \vartheta_A(x)<a\leq 1-\vartheta_A(x),\\ 0 & a>1-\mu_A(x)\end{cases}$$

α-lower cut set and α-strong lower cut set of A, respectively.

(iii) We call

$$A_{[\alpha]}(x)=\begin{cases}1 & \alpha+\mu_A(x)\geq 1\\ \frac{1}{2} & \vartheta_A(x)<a\leq 1-\mu_A(x)\\ 0 & \vartheta_A(x)>a\end{cases} \text{ and } A_{[\alpha]}(x)=\begin{cases}1 & \alpha+\mu_A(x)>1\\ \frac{1}{2} & \vartheta_A(x)<a\leq 1-\mu_A(x)\\ 0 & \vartheta_A(x)\geq a\end{cases}$$

α-upper Q-cut set and α-strong upper Q-cut set of A, respectively.

(iv) We call

$$A_{[\alpha]}(x)=\begin{cases}1 & \alpha+\vartheta_A(x)\geq 1\\ \frac{1}{2} & \mu_A(x)<a\leq 1-\vartheta_A(x)\\ 0 & \mu_A(x)>a\end{cases} \text{ and } A_{[\alpha]}(x)=\begin{cases}1 & \alpha+\vartheta_A(x)>1\\ \frac{1}{2} & \mu_A(x)<a\leq 1-\vartheta_A(x),\\ 0 & \mu_A(x)>a\end{cases}$$

α-lower Q-cut set and α-strong lower Q-cut set of A, respectively.

Definition 2.8 ([17])**.** A Diamond fuzzy number of a set A is defined as $A_D = \{a, b, c, (\alpha_b, \beta_b)\}$, and its membership function is given by,

$$\mu_{A_D}(x)=\begin{cases}0 & \text{for } x<a\\ \frac{(x-a)}{(b-a)} & \text{for } a\leq x\leq b\\ \frac{(c-x)}{(c-b)} & \text{for } b\leq x\leq c\ \alpha_b-\text{base}\\ \frac{(a-x)}{a-b} & \text{for } a\leq x\leq b\\ \frac{(x-c)}{(b-c)} & \text{for } b\leq x\leq c\\ 1 & x=\beta_b\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 2.1 ([16]). *Let* $3^X = \{A \mid A : X \to \left\{0, \frac{1}{2}, 1\right\}$ *is a mapping*$\}$. *For* $A \in 3^X$ *and* $\alpha \in [0, 1]$, αA *is defined as follows*:

$$(\alpha A)(x) = \begin{cases} (0, 1) & A(x) = 0 \\ (\alpha, 1 - \alpha) & A(x) = 1 \\ (0, 1 - \alpha) & A(x) = \frac{1}{2}. \end{cases}$$

then αA *is an intuitionistic fuzzy set over X,* $A = \bigcup_{\alpha \in [0, 1]} \alpha A_\alpha$ *and* $A = \bigcup_{\alpha \in [0, 1]} \alpha A_{\underline{\alpha}}$.

3. The Three-Dimensional Diamond Fuzzy Sets and their Cut Sets

As is well known, the Zadeh fuzzy set D over X is a mapping $D : X \to [0, 1]$, where $D(x)$ denotes the degree of membership of x in A. From the definition of Zadeh fuzzy set, we know that Zadeh fuzzy set characters the notion by using the numbers in $[0, 1]$, which is an approximation to the notion. By extension from $[0, 1]$ to $\bar{I} = \{[a^-, a^+] \mid 0 \leq a^- \leq a^+ \leq 1\}$, we can obtain the interval-valued fuzzy sets. That is the mapping $D : X \to \bar{I}, D(x) \equiv [D^-(x), D^+(x)]$, where $D^-(x)$ denotes the least degree of membership of x in D and $D^+(x)$ denotes the most degree of membership of x in D. Similarly, the mapping $D : X \to L = \{(d, e) \mid d, e \in [0, 1], d + e \leq 1\}, D(x) \equiv (\mu_D(x), \vartheta_D(x))$ is an intuitionistic fuzzy set, where $\vartheta_D(x)$ denotes the degree f membership of x in D and denotes the degree of non-membership of x in D.

Since Zadeh fuzzy set A of X uses a number in [0, 1] to character the degree f member ship of x in A, we call Zadeh fuzzy sets one-dimensional fuzzy sets of X. Similarly, we call the interval-valued fuzzy sets and intuitionistic fuzzy sets two-dimensional fuzzy sets because they u set two numbers in [0, 1] to character the degree of membership of the element x in A.

In practical applications, we can use triple value form such as (very good, good, more or less good) or (very young, young, more or less young) to describe we the r an object be longs to a notion. From this point of view, an we kind of L-fuzzy sets is introduced, which is called the three-dimensional fuzzy sets in this paper. Next, we give the definition of the three-dimensional fuzzy sets.

Let $I_3D = \{(d_1, d_2, d_3)/0 \leq d_1 \leq d_2 \leq d_3 \leq 0\}$. The Operator over I_3D defined as follows: for $(d_1, d_2, d_3), (e_1, e_2, e_3) \in I_3D, (d_1t, d_2t, d_3t) \in I_3D$, for all $t \in T$.

$$\bigvee_{t\in T}(d_1^t, d_2^t, d_3^t) = \left(\bigvee_{t\in T} d_1^t, \bigvee_{t\in T} d_2^t, \bigvee_{t\in T} d_3^t\right)$$

$$\bigwedge_{t\in T}(d_1^t, d_2^t, d_3^t) = \bigwedge_{t\in T} d_1^t, \bigwedge_{t\in T} d_2^t, \bigwedge_{t\in T} d_3^t$$

$$(d_1, d_2, d_3)^c = (1 - d_1, 1 - d_2, 1 - d_3)$$

$$\underline{1} = (1, 1, 1), \underline{O} = (0, 0, 0, 0).$$

Then $(I_3D, \Delta, \Delta, c, \underline{1}, \underline{O})$ is an F lattice.

Definition 3.1.

The mapping $D : X \to I_3D, D(x) = (D_1(x), D_2(x), D_3(x))$ is called a three dimensional diamond fuzzy set. We set the operations in I_3D, X according to the operations in I_3D and let $\overline{X} = (1, 1, 1), \overline{\varphi} = (0, 0, 0)$ then $I_3^{D, X}, \cup, \cap, \overline{X}, \overline{\varphi})$ is an F lattice.

The cut sets of the two-dimensional fuzzy sets (the interval-valued fuzzy sets and the intuitionistic fuzzy sets) are defined by 3-valued fuzzy sets in [16] and their properties are same as the cut sets of Zadeh fuzzy sets. Similarly, cut sets on the three-dimensional fuzzy sets are defined by the 4-valued fuzzy sets as follows. Let $4^{D, X} = \{D \mid D : X \to \{0, 1/3, 2/3, 1\}\}$. According to Zadeh operators, $4^{D, X}$ is an F lattice.

Definition 3.2. Let $D \in I_3^{D,X}$ and $\alpha \in [0, 1]$. We call

$$A_{\alpha}(x) = \begin{cases} 1 & D_1(x) \geq \alpha \\ \frac{2}{3} & D_1(x) < \alpha \leq D_2(x) \\ \frac{1}{3} & D_2(x) < \alpha \leq D_3(x) \\ 0 & \alpha > D_3(x) \end{cases} \text{ and } A_{\underline{\alpha}}(x) = \begin{cases} 1 & D_1(x) > \alpha \\ \frac{2}{3} & D_1(x) \leq \alpha < D_2(x) \\ \frac{1}{3} & D_2(x) \leq \alpha < D_3(x) \\ 0 & \alpha \geq D_3(x) \end{cases}$$

α-upper cut set and α-strong upper cut set of A, respectively.

Definition 3.3. Let $D \in I_3^{D,X}$ and $\alpha \in [0, 1]$. We call

$$A^{\alpha}(x) = \begin{cases} 1 & D_3(x) \leq \alpha \\ \frac{2}{3} & D_2(x) \leq \alpha \leq D_3(x) \\ \frac{1}{3} & D_2(x) < \alpha \leq D_2(x) \\ 0 & D_1(x) > \alpha \end{cases} \text{ and } A^{\underline{\alpha}}(x) = \begin{cases} 1 & D_3(x) < \alpha \\ \frac{2}{3} & D_2(x) < \alpha \leq D_3(x) \\ \frac{1}{3} & D_1(x) < \alpha \leq D_2(x) \\ 0 & D_1(x) \geq \alpha \end{cases}$$

α-lower cut set and α-strong lower cut set of A, respectively.

Definition 3.4. Let $D \in I_3^{D,X}$ and $\alpha \in [0, 1]$. We call

$$A_{[\alpha]}(x) = \begin{cases} 1 & \alpha + D_1(x) \geq 1 \\ \frac{2}{3} & D_1(x) < 1 - \alpha \leq D_2(x) \\ \frac{1}{3} & D_2(x) < 1 - \alpha \leq D_3(x) \\ 0 & \alpha + D_3(x) < 1 \end{cases} \text{ and } A_{[\underline{\alpha}]}(x) = \begin{cases} 1 & \alpha + D_1(x) < 1 \\ \frac{2}{3} & D_1(x) \leq 1 - \alpha < D_2(x) \\ \frac{1}{3} & D_2(x) \leq 1 - \alpha < D_3(x) \\ 0 & \alpha + D_3(x) \geq 1 \end{cases}$$

α-upper Q-cut set and α-strong upper Q-cut set of A, respectively.

Definition 3.5. Let $D \in I_3^{D,X}$ and $\alpha \in [0, 1]$. We call

$$A^{[\alpha]}(x) = \begin{cases} 1 & \alpha + D_3(x) \leq 1 \\ \frac{2}{3} & D_2(x) \leq 1 - \alpha < D_3(x) \\ \frac{1}{3} & D_1(x) \leq 1 - \alpha < D_2(x) \\ 0 & D_1(x) + \alpha > 1 \end{cases} \text{ and } A^{[\underline{\alpha}]}(x) = \begin{cases} 1 & \alpha + D_1(x) < 1 \\ \frac{2}{3} & D_2(x) < 1 - \alpha < D_3(x) \\ \frac{1}{3} & D_1(x) < 1 - \alpha \leq D_2(x) \\ 0 & \alpha + D_1(x) \geq 1 \end{cases}$$

α-lower Q-cut set and α-strong lower Q-cut set of A, respectively. Next, we give the properties of these kinds of cut sets on three-dimensional diamond fuzzy sets.

Property 3.1.

$$D^{\alpha} = (D_{\underline{\alpha}})^c,\ D^{\underline{\alpha}} = (D_{\alpha})^c,\ D_{[\alpha]} = D_{1-\alpha},\ D_{[\underline{\alpha}]} = D_{\underline{1-\alpha}},$$

$$D^{[\alpha]} = (D_{\underline{1-\alpha}})^c,\ D^{[\underline{\alpha}]} = (D_{1-\alpha})^c.$$

Property 3.2.

(1) $D_{[\underline{a}]} \subset D_a$

(2) *If* $a_1 < a_2$ then $D_{a1} \supset D_{a2},\ D_{\underline{a1}} \supset D_{\underline{a2}},\ D_{\underline{a1}} \supset D_{a2}$

(3) *If* $D \subset E$, then $D_a \subset E_a,\ D_a \subset E_a$

(4) $D_I = \emptyset,\ D_0 = X$.

Property 3.3.

(1) $(D^c)_{a} = (D_{\underline{1-a}})^c,\ (D^c)_{\underline{\alpha}} = (D_{1-a})^c$

(2) $(D \cup E)_{\alpha} = D_{\alpha} \cup E_{\alpha},\ (D \cup E)_{\underline{\alpha}} = D_{\underline{\alpha}} \cup E_{\underline{\alpha}},\ (D \cap E)_{\alpha} = D_{\alpha} \cap E_{\alpha},$
$(D \cap E)_{\underline{\alpha}} D_{\underline{\alpha}} \cap E_{\underline{\alpha}},$

(3) $(\bigcup_{t \in T} D_t)_{\alpha} \supset \bigcup_{t \in T} (D_t)_{\alpha},\ (\bigcup_{t \in T} D_t)_{\underline{\alpha}} = \bigcup_{t \in T} (D_t)_{\underline{\alpha}},$
$(\bigcap_{t \in T} D_t)_{\alpha} = \bigcap_{t \in T} (D_t)_{\alpha},\ (\bigcap_{t \in T} D_t)_{\underline{\alpha}} = \bigcap_{t \in T} (D_t)_{\underline{\alpha}}$

(4) $D_{\alpha \wedge \beta} = D_{\alpha} \cup D_{\beta},\ D_{\underline{\alpha \wedge \beta}} = D_{\underline{\alpha}} \cup D_{\underline{\beta}},\ D_{\alpha \vee \beta} = D_{\alpha} \cap D_{\beta},\ D_{\underline{\alpha \vee \beta}} = D_{\underline{\alpha}} \cap D_{\underline{\beta}}$

(5) *Let* $\alpha_t \in [0, 1] (t \in T),\ a = \bigwedge_{t \in T} \alpha_t,\ b = \bigvee_{t \in T} \alpha_t$. *Then* $\bigcup_{t \in T} D_{\alpha_t} \subset A_a,$
$\bigcap_{t \in T} D_{\alpha_t} = D_b,\ \bigcap_{t \in T} D_{\underline{\alpha_t}} = A_{\underline{\alpha}},\ \bigcap_{t \in T} D_{\underline{\alpha_t}} \supset D_{\underline{b}}.$

Proof.

(1) $(D^c)_{\alpha}(x) = 1 \Leftrightarrow 1 - D_3(x) \geq a \Leftrightarrow 1 - a \geq D_3(x) \Leftrightarrow D_{\underline{1-a}}(x) = 0 \Leftrightarrow$
$(D_{\underline{1-a}})^c(x) = 1.$

$(D^c)_a(x) = \frac{2}{3} \Leftrightarrow 1 - D_3(x) < a \leq 1 - D_2(x) \Leftrightarrow D_2(x) \leq 1 - a < D_3(x)$

$\Leftrightarrow D_{\underline{1-a}}(x)\frac{2}{3} \Leftrightarrow (D_{\underline{\underline{1-a}}})^c(x) = \frac{2}{3}$

$(D^c)_a(x) = 0 \Leftrightarrow a > 1 - D_1(x) \Leftrightarrow D_1(x) > 1 - a \Leftrightarrow D_{\underline{1-a}}(x) = 1$

$\Leftrightarrow (D_{\underline{1-a}})^c(x) = 0.$

Since $(D^c)_a, (D_{\underline{1-a}})^c \in 4^X$, we have $(D^c)_a = (D_{\underline{1-a}})^c$

(2) $(D \cup E)_\alpha(x) = 1 \Leftrightarrow D_1(x) \vee E_1(x) \geq \alpha \Leftrightarrow D_1(x) \geq \alpha$ or $E_1(x) \geq \alpha \Leftrightarrow$ $D_\alpha(x) = 1$ or $E_\alpha(x) = 1 \Leftrightarrow (D_\alpha \cup E_\alpha)(x) = D_\alpha(x) \vee E_\alpha(x) = 1$

$(D \cup E)_\alpha(x) = \frac{2}{3} \Leftrightarrow D_1(x) \vee E_1(x) < \alpha \leq D_2(x) \vee E_2(x)$

$\Leftrightarrow (D_1(x) < \alpha \leq D_2(x), E_1(x) < \alpha), (E_1(x) < \alpha \leq E_2(x), D_1(x) < \alpha)$

$\Leftrightarrow D_\alpha(x) = \frac{2}{3}, E_\alpha(x) \leq \frac{2}{3}$ or $E_\alpha(x) = \frac{2}{3}, D_\alpha(x) \leq \frac{2}{3} \Leftrightarrow (D_\alpha \cup E_\alpha)(x)$ $D_\alpha(x) \vee E_\alpha(x) = \frac{2}{3}, (D \cup E)_\alpha = 0 \Leftrightarrow \alpha > D_3(x) \vee E_3(x) \Leftrightarrow \alpha > D_2(x)$ and $\alpha > E_2(x) \Leftrightarrow D_\alpha(x) = 0$ and $E_\alpha(x) = 0 \Leftrightarrow (D_\alpha \cup E_\alpha)(x) D_\alpha(x) \vee E_\alpha(x) = 0.$ Since $(D \cup E)_\alpha, D_\alpha \cup E_\alpha \in 4^{D, X}$, we have $(D \cup E)_\alpha, D_\alpha \cup E_\alpha$.

(3) Let $D_t(x) = (d_1^t(x), d_2^t(x), d_3^t(t))$ we have $(\bigcup_{t \in T} D_t)_\alpha(x) = 1 \Rightarrow \vee_{t \in T}(D_t)_\alpha(x) = 1 \Rightarrow \exists t \in T, (D_t)_\alpha(x) \Rightarrow \exists t \in T,$ $D_1^t(x) = 1 \Rightarrow (\bigcup_{t \in T} D_t)_\alpha(x) = \frac{2}{3} \Rightarrow \vee_{t \in T}(D_t)_\alpha(x) = \frac{2}{3} \Rightarrow$ $\left(\forall t \in T, (D_t)_\alpha(x) \leq \frac{2}{3}\right)$ and $\exists t \in T, (D_t)_\alpha(x) = \frac{2}{3} \Rightarrow \exists t \in T, D_1^t(x) < \alpha$ and $\exists t \in T, D_1^t(x) < \alpha \leq D_2^t(x) \Rightarrow \vee_{t \in T} D_1^t(x) \leq \alpha \leq \vee_{t \in T} D_2^t(x) \Rightarrow$ $(\bigcup_{t \in T} D_t)_\alpha(x) \geq \frac{2}{3}.$

$$\left(\bigcup_{t \in T} D_t\right)_\alpha (x) = 0 \Rightarrow \alpha > \bigvee_{t \in T} D_3^t(x) \Rightarrow \forall t \in T, \alpha, > D_3^t(x) \Rightarrow \forall t \in T,$$

$$(D_t)_\alpha(x) = 0 \Rightarrow \left(\bigcup_{y \in T} D_t\right)_\alpha (x)$$

$$= \bigvee_{t \in T} (D_t)_\alpha(x) = 0.$$

Since $(\bigcup_{t \in T} D_t)_\alpha(x), \bigcup_{t \in T}(D_t)_\alpha \in 4^{D, X}$ we have $\bigcup_{t \in T} D_t)_\alpha(x) \subset (\bigcup_{t \in T} D_t)_\alpha$.

Next, we prove $\bigcup_{t \in T}(D_t)_{\underline{\alpha}} = (\bigcup_{t \in T} D_t)_{\underline{\alpha}}(x)$ $(\bigcup_{t \in T} D_t)_{\underline{\alpha}}(x) = 1 \Rightarrow$ $\bigvee_{t \in T}(D_t)_{\underline{\alpha}}(x) = 1 \Rightarrow \exists t \in T, (D_t)_{\underline{\alpha}}(x) \Rightarrow \exists t \in T, D_1^t(x) = 1 \Rightarrow$ $(\bigcup_{t \in T} D_t)_{\underline{\alpha}}(x) = \frac{2}{3} \Rightarrow \bigvee_{t \in T}(D_t)_{\underline{\alpha}}(x) = \frac{2}{3} \Rightarrow \left(\forall t \in T, (D_t)_{\underline{\alpha}}(x) \leq \frac{2}{3}\right)$ and $\exists t \in T, (D_t)_{\underline{\alpha}}(x) = \frac{2}{3} \Rightarrow \exists t \in T, D_2^t(x) < \alpha$ and $\exists t \in T, D_1^t(x) < \alpha \leq D_2^t(x) \Rightarrow \bigvee_{t \in T} D_1^t(x) \leq \alpha \leq \bigvee_{t \in T} D_2^t(x) \Rightarrow (\bigcup_{t \in T} D_t)_{\underline{\alpha}}(x) \geq \frac{2}{3}$.

$$\left(\bigcup_{t \in T} D_t\right)_{\underline{\alpha}} (x) = 0 \Rightarrow \alpha > \bigvee_{t \in T} D_3^t(x) \Rightarrow \forall t \in T, \alpha > D_3^t(x) \Rightarrow \forall t \in T,$$

$$(D_t)_{\underline{\alpha}}(x) = 0 \Rightarrow \left(\bigcup_{t \in T} D_t\right)_{\underline{\alpha}} (x)$$

$$= \bigvee_{t \in T} (D_t)_{\underline{\alpha}}(x) = 0.$$

Since $(\bigcup_{t \in T} D_t)_{\underline{\alpha}}, \bigcup_{t \in T}(D_t)_{\underline{\alpha}} \in 4^{D, X}$ we have $(\bigcup_{t \in T} D_t)_{\underline{\alpha}}, \bigcup_{t \in T}(D_t)_{\underline{\alpha}}$

(4) The proof is similar to (3)

The proof is obvious.

4. Conclusion

In this paper, the concept of the three dimensional diamond fuzzy sets is introduced. The cut sets of the three dimensional diamond fuzzy sets are defined by 4-valued fuzzy sets and some of its properties also obtained. We can extend this concept for decomposition theorems and representation theorems.

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