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FUZZY EQUATIONS AND OPERATIONS ON REVERSE ORDER FUZZY NUMBERS

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Abstract

In this paper, a new type of reverse order fuzzy number has been introduced and its complement of fuzzy function is verified. Using the complement and dual α-cuts of fuzzy numbers, properties of reverse order fuzzy numbers are further investigated. Some prepositions for instance maximum and minimum operations on fuzzy equations and Dual α-cuts and reverse order fuzzy numbers have been introduced and proved.

1. Introduction

Lotfi A. Zadeh [15] introduced the notion of fuzziness in 1965. Fuzzy Mathematicians extended the real number system to a fuzzy real number system by studying real numbers in fuzzy settings. An approximation of analytic functions by Dubois, Prade, and Yager [5], encompasses dividing the membership functions of the algebraic operations into a left side and a right sides representation by simple analytic form.

In particular Michael Hans [5], Chen [4], investigated the operations and properties of fuzzy numbers. The properties of α-cuts and strong α-cuts are well developed by George Klir [1]. The concepts of triangular and trapezoidal fuzzy numbers are the special cases of fuzzy numbers. Recently T. Pathinathan and K. Ponnivalavan [10] introduced the notion of a pentagonal fuzzy number which is also a special type of fuzzy number. The same authors

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[11] also introduced the concepts of reverse order triangular, trapezoidal and pentagonal fuzzy numbers.

In a bounded subset of [0, 1], the supremum and infimum always exist by Rockafellar [13]. Because of the supremum and infimum property of a set of real numbers concave fuzzy numbers are defined as reverse order fuzzy numbers. Pathinathan and Anita Dolorosa introduced dual α-cuts and strong dual α-cuts [6]. In this paper, functions of Reverse Order Fuzzy Numbers and their operations are introduced. The complement and Duality principle of α-cuts have been presented and proved for fuzzy functions. Reverse order fuzzy numbers are used to prove for maximum and minimum operations. Some prepositions for a fuzzy equation using fuzzy lattice and hamming distance formula are used to express Dual α-cuts and reverse order fuzzy numbers.

2. Preliminaries

Definition 2.1 [15]**.** A fuzzy set $\underset{\sim}{A}$ of X is defined as $\underset{\sim}{A} = \{x, \mu_{\underset{\sim}{A}}(x)/x \in X\}$, where x is an element in the universe of discourse X and $\mu_{\underset{\sim}{A}} : X \to [0, 1]$ is a mapping called the degree of membership function and $\mu_{\underset{\sim}{A}}(x)$ is the membership value of $x \in X$.

Definition 2.2 [1]**.** Let $\underset{\sim}{A}$ be a fuzzy subset of X and $\alpha \in [0, 1]$. The fuzzy set $\underset{\sim}{A}^{\geq\alpha} = \{x : \underset{\sim}{A}(x) \geq \alpha\}$ is known as the α-cut off $\underset{\sim}{A}$ and the fuzzy set $\underset{\sim}{A}^{\geq\alpha} = \{x : \underset{\sim}{A}(x) \geq \alpha\}$ is the strong α-cut of $\underset{\sim}{A}$.

Supp $(\underset{\sim}{A}) = \underset{\sim}{A}^{>0}$ = the support of $\underset{\sim}{A}$.

Definition 2.3 [1]**.** A fuzzy subset $\underset{\sim}{A}$ of a real vector space X is convex iff $\underset{\sim}{A}(\alpha x_1 + (1 - \alpha)x_2) \geq \min(\underset{\sim}{A}(x_1), \underset{\sim}{A}(x_2))$ for all $x_1, x_2 \in X$ and all $\alpha \in [0, 1]$.

Definition 2.4 [6]**.** The fuzzy set $\underset{\sim}{A}^{\leq\alpha} = \{x : \underset{\sim}{A}(x) \leq \alpha\}$ is called the dual

α-cut of $\underset{\sim}{A}$, $\underset{\sim}{A}^{<\alpha} = \{x : \underset{\sim}{A}(x) < \alpha\}$ is known as the dual strong α-cut of $\underset{\sim}{A}$ and $\underset{\sim}{A}^{=\alpha} = \{x : \underset{\sim}{A}(x) = \alpha\}$ is the α-level cut of $\underset{\sim}{A}$.

Proposition 2.5 [6]**.** $\underset{\sim}{A}$ is a reverse order fuzzy number iff the following conditions hold.

(i) $\underset{\sim}{A}$ is down-normal

(ii) $\underset{\sim}{A}^{>\alpha}$ is a union of two disjoint unbounded open intervals for each $\alpha \in [0, 1)$.

(iii) The level set $\underset{\sim}{A}^{=1}$ is un bounded and $\underset{\sim}{A}^{<1}$ is bounded.

Definition 2.6 [1]**.** Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two continuous fuzzy numbers and $^{*} \in \{+, -, \cdot, /\}$. Then $\underset{\sim}{A} * \underset{\sim}{B}$ is a fuzzy number defined by

$$(\underset{\sim}{A} * \underset{\sim}{B})(z) = \text{Sup}\{\text{Min}\{\underset{\sim}{A}(x), \underset{\sim}{B}(y)\} : z = x * y\}.$$

Definition 2.7 [1]**.** Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two fuzzy numbers. Then $\text{Min}(\underset{\sim}{A}, \underset{\sim}{B})$ and $\text{Max}(\underset{\sim}{A}, \underset{\sim}{B})$ are fuzzy numbers, defined by

$$\text{Min}(\underset{\sim}{A}, \underset{\sim}{B})(z) = \text{Sup}\{\text{Min}\{\underset{\sim}{A}(x), \underset{\sim}{B}(y)\} : z = \text{Min}(x, y)\}.$$

$$\text{Max}(\underset{\sim}{A}, \underset{\sim}{B})(z) = \text{Sup}\{\text{Max}\{\underset{\sim}{A}(x), \underset{\sim}{B}(y)\} : z = \text{Max}(x, y)\}.$$

Definition 2.8 [1]**.** Lattice $\langle R, \text{Min}, \text{Max}\rangle$

It also is expressed as the pair $\langle R, \preccurlyeq\rangle$ where $\preccurlyeq$ is a partial ordering defined as $\underset{\sim}{A} \preccurlyeq \underset{\sim}{B}$ iff $\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{A}$ or $\underset{\sim}{A} \preccurlyeq \underset{\sim}{B}$ iff $\text{Max}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{B}$.

Definition 2.9 [1]**.** Let $\underset{\sim}{A} + X = B$ and $\underset{\sim}{A} * X = B$, where $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are fuzzy numbers, and X is an unknown fuzzy number for which either of the equations is to be satisfied.

Remarks. If $\underset{\sim}{A} + X = \underset{\sim}{B}$ is a fuzzy equation, then $X = \underset{\sim}{B} - \underset{\sim}{A}$ is not the solution.

3. Method based on α-cuts

Fix $\alpha \in [0, 1]$. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are fuzzy numbers, then $\underset{\sim}{A}^{\geq\alpha} = [a_1, a_2]$ and $\underset{\sim}{B}^{\geq\alpha} = [b_1, b_2]$ and $\underset{\sim}{A} \leq_\alpha \underset{\sim}{B}$ if $a_2 \leq b_2$.

Lemma 3.1. *Let* $\underset{\sim}{A}$ *be a fuzzy number*, $\mathbb{S}(f(\underset{\sim}{A}))(y) \leq f(\mathbb{S}\underset{\sim}{A})(y)$.

Proof.

$$f(\underset{\sim}{A})(y) = \text{Sup}\{\underset{\sim}{A}(x) : f(x) = y\}$$

$$\begin{aligned}\mathbb{S}(f(\underset{\sim}{A})(y)) &= \mathbb{S}(\text{Sup}\{\underset{\sim}{A}(x) : f(x) = y\}) \\ &= \text{Inf}\{\mathbb{S}\underset{\sim}{A}(x) : f(x) = y\} \\ &\leq \text{Sup}\{\mathbb{S}\underset{\sim}{A}(x) : f(x) = y\} = f(\mathbb{S}\underset{\sim}{A})(y).\end{aligned}$$

Theorem 3.2. *Let* $\underset{\sim}{A}$ *be a fuzzy number*,

(i) $[f(\underset{\sim}{A})]^{<\alpha} \subseteq f(\underset{\sim}{A}^{<\alpha})$

(ii) $[f(\underset{\sim}{A})]^{\leq\alpha} \supseteq f(\underset{\sim}{A}^{\leq\alpha})$ *if* $\mathbb{S}(f(\underset{\sim}{A})) = f(\mathbb{S}\underset{\sim}{A})$.

Proof.

(i) $$\begin{aligned}[f(\underset{\sim}{A})]^{<\alpha} &= [\mathbb{S}(f(\underset{\sim}{A}))]^{>(1-\alpha)} \\ &\subseteq [f(\mathbb{S}\underset{\sim}{A})]^{>(1-\alpha)} \\ &= f(\underset{\sim}{A})^{>(1-\alpha)} = f(\underset{\sim}{A}^{<\alpha})\end{aligned}$$

(ii) $[f(\underset{\sim}{A})]^{\leq\alpha} = [\mathbb{S}(f(\underset{\sim}{A}))]^{\geq(1-\alpha)}$

$$= [f(\mathbb{S}\underset{\sim}{A})]^{\geq(1-\alpha)}$$

$$\supseteq f(\underset{\sim}{A})^{\geq(1-\alpha)} = f(\underset{\sim}{A}^{\leq\alpha})$$

4. Operations on Revere Order Fuzzy Numbers

Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two continuous reverse order fuzzy numbers and $^{*}\in \{+, -, \cdot, /\}$. Then $\mathbb{S}\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are continuous fuzzy numbers.

Therefore $\mathbb{S}\underset{\sim}{A} * \mathbb{S}\underset{\sim}{B}$ is a continuous fuzzy number, which implies $\mathbb{S}(\mathbb{S}\underset{\sim}{A} * \mathbb{S}\underset{\sim}{B})$ is a continuous reverse order fuzzy number. This motivates us to have the following definition.

Definition 4.1. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two continuous reverse order fuzzy numbers and $^{*}\in \{+, -, \cdot, /\}$. Then $\underset{\sim}{A} * \underset{\sim}{B}$ is a reverse order fuzzy number defined by $\underset{\sim}{A} * \underset{\sim}{B} = \mathbb{S}(\mathbb{S}\underset{\sim}{A} * \mathbb{S}\underset{\sim}{B})$. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two reverse order fuzzy numbers, then $\mathbb{S}\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers.

Hence $\text{Min}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B})$ and $\text{Max}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B})$ are fuzzy numbers, which implies $\mathbb{S}(\text{Min}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}))$ and $\mathbb{S}(\text{Max}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}))$ are reverse order fuzzy numbers. This motivates us to have the following definition.

Definition 4.2. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two reverse order fuzzy numbers.

$\text{Min}(\underset{\sim}{A}, \underset{\sim}{B})$ and $\text{Max}(\underset{\sim}{A}, \underset{\sim}{B})$ are defined as

$\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}) = \mathbb{S}(\text{Min}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}))$ and

$\text{Max}(\underset{\sim}{A}, \underset{\sim}{B}) = \mathbb{S}(\text{Max}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}))$.

Remark 4.3. Let $R\mathbb{R}$ denote the collection of all reverse order fuzzy numbers. Then $(R\mathbb{R}, \text{Min. Max.})$ is a distributive lattice.

Theorem 4.4. *Let* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *be any two reverse order fuzzy numbers. Then the fuzzy equation*

$\underset{\sim}{A} + \mathbb{X} = \underset{\sim}{B}$ *has a solution iff for* $\alpha, \beta \in [0, 1)$

(i) *the interval equations* $\underset{\sim}{A}^{\leq\alpha} + \mathbb{X}^{\leq\alpha} = \underset{\sim}{B}^{\leq\alpha}$ *have solutions*

(ii) $\alpha \leq \beta \Rightarrow \mathbb{X}^{\leq\alpha} \subseteq \mathbb{X}^{\leq\beta}$

(iii) $\mathbb{X} = \bigcap\{({}_{\leq\alpha}\mathbb{X}) : \alpha \in [0, 1)\}$

Proof. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two reverse order fuzzy numbers. Then $\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers. Therefore $\underset{\sim}{A} + \mathbb{S}\mathbb{X} = \mathbb{S}\underset{\sim}{B}$ has a solution iff for $\alpha, \beta \in [0, 1)$

(i) the interval equations $(\mathbb{S}\underset{\sim}{A})^{\geq\alpha} + (\mathbb{S}\mathbb{X})^{\geq\alpha} = (\mathbb{S}\underset{\sim}{B})^{\geq\alpha}$ have solutions

(ii) $\alpha \leq \beta \Rightarrow (\mathbb{S}\mathbb{X})^{\geq\beta} \subseteq (\mathbb{S}\mathbb{X})^{\geq\alpha}$

(iii) $\mathbb{S}\mathbb{X} = \bigcup\{{}_{\geq\alpha}(\mathbb{S}\mathbb{X}) : \alpha \in (0, 1]\}$

That is, iff for $\alpha, \beta \in (0, 1]$

(i) the interval equations $\underset{\sim}{A}^{\leq(1-\alpha)} + \mathbb{X}^{\leq(1-\alpha)} = \underset{\sim}{B}^{\leq(1-\alpha)}$ have solutions

(ii) $\alpha \leq \beta \Rightarrow \mathbb{X}^{\leq(1-\beta)} \subseteq \mathbb{X}^{\leq(1-\alpha)}$

(iii) $\mathbb{S}\mathbb{X} = \bigcup\{{}_{\leq(1-\alpha)}\mathbb{X} : \alpha \in (0, 1]\}$

That is, iff for $\alpha, \beta \in [0, 1)$

(i) the interval equations $\underset{\sim}{A}^{\leq\alpha} + \mathbb{X}^{\leq\alpha} = \underset{\sim}{B}^{\leq\alpha}$ have solutions

(ii) $\alpha \leq \beta \Rightarrow \mathbb{X}^{\leq\alpha} \subseteq \mathbb{X}^{\leq\beta}$

(iii) $\mathbb{S}\mathbb{X} = \bigcup\{_{\leq\alpha}\mathbb{X} : \alpha \in (0, 1]\}$ that is $\mathbb{X} = (\bigcup\{_{\leq\alpha}\mathbb{X} : \alpha \in (0, 1]\})$ that is

$\mathbb{X} = \bigcap\{(_{\leq\alpha}\mathbb{X}) : \alpha \in [0, 1)\}$.

Theorem 4.5. *Let* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *be any two reverse order fuzzy numbers on* $(0, \infty)$. *Then the fuzzy equation* $\underset{\sim}{A} \cdot \mathbb{X} = \underset{\sim}{B}$ *has a solution iff for* $\alpha, \beta \in [0, 1)$

(i) *the interval equations* $\underset{\sim}{A}^{\leq\alpha} + \mathbb{X}^{\leq\alpha} = \underset{\sim}{B}^{\leq\alpha}$ *have solutions*

(ii) $\alpha \leq \beta \Rightarrow \mathbb{X}^{\leq\alpha} \subseteq \mathbb{X}^{\leq\beta}$

(iii) $\mathbb{X} = \bigcap\{(_{\leq\alpha}\mathbb{X}) : \alpha \in [0, 1)\}$.

Proof. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be any two reverse order fuzzy numbers on $(0, \infty)$.

Then $\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers on $(0, \infty)$. Therefore $\mathbb{S}\underset{\sim}{A} \cdot \mathbb{S}\mathbb{X} = \mathbb{S}\underset{\sim}{B}$ has a solution iff for $\alpha, \beta \in (0, 1]$

(i) the interval equations $(\mathbb{S}\underset{\sim}{A})^{\geq\alpha} + (\mathbb{S}\mathbb{X})^{\geq\alpha} = (\mathbb{S}\underset{\sim}{B})^{\geq\alpha}$ have solutions

(ii) $\alpha \leq \beta \Rightarrow (\mathbb{S}\mathbb{X})^{\geq\beta} \subseteq (\mathbb{S}\mathbb{X})^{\geq\alpha}$

(iii) $\mathbb{S}\mathbb{X} = \bigcup\{_{\geq\alpha}(\mathbb{S}\mathbb{X}) : \alpha \in (0, 1]\}$

That is iff for $\alpha, \beta \in [0, 1)$

(i) the interval equations $\underset{\sim}{A}^{\leq(1-\alpha)} + \mathbb{X}^{\leq(1-\alpha)} = \underset{\sim}{B}^{\leq(1-\alpha)}$ have solutions

(ii) $\alpha \leq \beta \Rightarrow \mathbb{X}^{\leq(1-\beta)} \subseteq \mathbb{X}^{\leq(1-\alpha)}$

(iii) $\mathbb{S}\mathbb{X} = \bigcup\{_{\leq(1-\alpha)}\mathbb{X} : \alpha \in (0, 1]\}$

That is iff for $\alpha, \beta \in [0, 1)$

(i) the interval equations $\underset{\sim}{A}^{\leq\alpha} + \mathbb{X}^{\leq\alpha} = \underset{\sim}{B}^{\leq\alpha}$ have solutions

(ii) $\alpha \leq \beta \Rightarrow \mathbb{X}^{\leq\alpha} \subseteq \mathbb{X}^{\leq\beta}$

(iii) $\mathbb{SX} = \bigcup\{_{\leq\alpha}\mathbb{X} : \alpha \in (0, 1]\}$ that is $\mathbb{X} = (\bigcup\{_{\leq\alpha}\mathbb{X} : \alpha \in (0, 1]\})$ that is $\mathbb{X} = \bigcap\{(_{\leq\alpha}\mathbb{X}) : \alpha \in [0, 1)\}$.

Definition 4.6. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers. $\underset{\sim}{A} \leq \underset{\sim}{B}$ if $\mathbb{S}\underset{\sim}{A} \leq \mathbb{S}\underset{\sim}{B}$.

Proposition 4.7. *Let* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *be the reverse order fuzzy numbers. Then* $\underset{\sim}{A} \leq \underset{\sim}{B}$ *iff* $\underset{\sim}{A}(x) \geq \underset{\sim}{B}(x)$.

Proof. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers. Then $\mathbb{S}\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers. Therefore $\mathbb{S}\underset{\sim}{A} \leq \mathbb{S}\underset{\sim}{B}$ iff $(\mathbb{S}\underset{\sim}{A})(x) \leq (\mathbb{S}\underset{\sim}{B})(x)$ for all x

$$\text{iff } \mathbb{S}(\underset{\sim}{A}(x)) \leq \mathbb{S}(\underset{\sim}{B}(x))$$

$$\text{iff } 1 - \underset{\sim}{A}(x) \leq 1 - \underset{\sim}{B}(x)$$

$$\text{iff } \underset{\sim}{A}(x) \geq \underset{\sim}{B}(x)$$

Therefore $\underset{\sim}{A} \leq \underset{\sim}{B}$ iff $\underset{\sim}{A}(x) \geq \underset{\sim}{B}(x)$.

Definition 4.8. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers, then $\underset{\sim}{A} \lesssim \underset{\sim}{B}$ if $\underset{\sim}{A} \lesssim \mathbb{S}\underset{\sim}{B}$.

Proposition 4.9. *Let* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *be the reverse order fuzzy numbers. Then* $\underset{\sim}{A} \lesssim \underset{\sim}{B}$ *iff* $\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{B}$.

Proof. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers. Then $\mathbb{S}\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers.

Therefore $\mathbb{S}\underset{\sim}{A} \lesssim \mathbb{S}\underset{\sim}{B}$ iff $\text{Max}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}) = \mathbb{S}\underset{\sim}{B}$

$$\text{iff } \text{Max}(1 - \underset{\sim}{A}, 1 - \underset{\sim}{B}) = 1 - \underset{\sim}{B}$$

$$\text{iff } (1 - \text{Min}(\underset{\sim}{A}, \underset{\sim}{B})) = 1 - \underset{\sim}{B}$$

$$\text{iff } \text{Min}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{B}$$

Therefore $\underset{\sim}{A} \lesssim \underset{\sim}{B}$ iff $\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{B}$.

Proposition 4.10. *Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers. Then $\underset{\sim}{A} \lesssim \underset{\sim}{B}$ iff* $\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{A}$.

Proof. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers. Then $\mathbb{S}\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers.

Therefore $\mathbb{S}\underset{\sim}{A} \lesssim \mathbb{S}\underset{\sim}{B}$ iff $\text{Max}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}) = \mathbb{S}\underset{\sim}{A}$

$$\text{iff } \text{Min}(1 - \underset{\sim}{A}, 1 - \underset{\sim}{B}) = 1 - \underset{\sim}{A}$$

$$\text{iff } 1 - \text{Max}(\underset{\sim}{A}, \underset{\sim}{B}) = 1 - \underset{\sim}{A}$$

$$\text{iff } \text{Min}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{A}$$

Therefore $\underset{\sim}{A} \lesssim \underset{\sim}{B}$ iff $\text{Max}(\underset{\sim}{A}, \underset{\sim}{B}) = \underset{\sim}{A}$.

Proposition 4.11. *Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers. Then $\underset{\sim}{A} \lesssim \underset{\sim}{B}$ iff* $d(\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}), \underset{\sim}{A}) \geq d(\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}), \underset{\sim}{B})$.

Proof. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers. Then $\mathbb{S}\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers. Therefore $\mathbb{S}\underset{\sim}{A} \preccurlyeq \mathbb{S}\underset{\sim}{B}$

$$\text{iff } d(\text{Max}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}), \mathbb{S}\underset{\sim}{A}) \geq d(\text{Max}(\mathbb{S}\underset{\sim}{A}, \mathbb{S}\underset{\sim}{B}), \mathbb{S}\underset{\sim}{B}).$$

$$\text{iff } d(\mathbb{S}(\text{Min}(\underset{\sim}{A}, \underset{\sim}{B})), \mathbb{S}\underset{\sim}{A}) \geq d(\mathbb{S}(\text{Min}(\underset{\sim}{A}, \underset{\sim}{B})), \mathbb{S}\underset{\sim}{B}).$$

$$\text{iff } d(\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}), \underset{\sim}{A}) \geq d(\text{Min}(\underset{\sim}{A}, \underset{\sim}{B}), \underset{\sim}{B}).$$

Definition 4.12. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers, then $\underset{\sim}{A} \leq_{\alpha} \underset{\sim}{B}$ if $\underset{\sim}{A} \leq_{(1-\alpha)} \mathbb{S}\underset{\sim}{B}$.

Proposition 4.13. *Let* $\underset{\sim}{A}$ *and* $\underset{\sim}{B}$ *be the reverse order fuzzy numbers.* $\underset{\sim}{A}^{\lesssim \alpha} = [a_1, a_2]$ *and* $\underset{\sim}{B}^{\lesssim \alpha} = [b_1, b_2]$ *Then* $\underset{\sim}{A} \leq_{\alpha} \underset{\sim}{B}$ *iff* $a_2 \leq b_2$.

Proof. Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be the reverse order fuzzy numbers, $\underset{\sim}{A}^{\lesssim \alpha} = [a_1, a_2]$ and $\underset{\sim}{B}^{\lesssim \alpha} = [b_1, b_2]$. Then $\mathbb{S}\underset{\sim}{A}$ and $\mathbb{S}\underset{\sim}{B}$ are fuzzy numbers and $(\mathbb{S}\underset{\sim}{A})^{\geq(1-\alpha)} = [a_1, a_2]$ and $(\mathbb{S}\underset{\sim}{B})^{\geq(1-\alpha)} = [b_1, b_2]$ Therefore $\underset{\sim}{A} \leq_{\alpha} \underset{\sim}{B}$ iff $\mathbb{S}\underset{\sim}{A} \leq_{(1-\alpha)} \mathbb{S}\underset{\sim}{B}$ iff $a_2 \leq b_2$.

Conclusion

A reverse order fuzzy function has been introduced and its complement of functions was verified in this paper. The complement and Duality principle of α-cuts have been discussed and proved for fuzzy functions. Using the complement of a fuzzy number, fuzzy operations such as $^{*} \in \{+, -, \cdot, /\}$, was proved for reverse order fuzzy number. Some prepositions for instance maximum and minimum operations on the fuzzy equation and hamming distance for Dual α-cuts and reverse order fuzzy numbers have been introduced and proved.

References

[1] G. J. Klir and Bo Yuan, Fuzzy Sets and Fuzzy logic, Theory and Applications, Pearson Publications, (1995).

[2] A. J. Kamble, Some notes on pentagonal fuzzy numbers, International Journal Fuzzy Mathematical Archives 13(2) (2017), 113-121.

[3] S. Banerjee and T. K. Roy, Arithmetic operations on generalized trapezoidal fuzzy number and its applications, Turkish Journal of Fuzzy Systems 3(1) (2012), 16-44.

[4] S. H. Chen and C. H. Hsieh, Graded mean integration representation of generalized fuzzy number, Journal of the Chinese Fuzzy System Association 5(2) (1999) 1-7.

[5] D. Dubois and H. Prade, Operations on fuzzy numbers, International Journal of Systems Science 9(6) (1978), 613-626.

[6] T. Pathinathan and E. Anita Dolorosa, Dual α-cuts and applications of reverse order fuzzy numbers, Journal of Huazhong University of Science and Technology 50(3) (2021).

[7] Michael Hans, Applied Fuzzy Arithmetic, An Introduction with Engineering Applications, Springer publications (2003), 53-73.

[8] T. Pathinathan and E. Mike Dison, Defuzzification of Pentagonal fuzzy numbers, International Journal of Current Advanced Research 7(1) (2018), 86-90.

[9] T. Pathinathan and E. Mike Dison, Similarity measures of pentagonal fuzzy numbers, International Journal of Pure and Applied Mathematics 119(9) (2018), 165-175.

[10] T. Pathinathan and K. Ponnivalavan, Pentagonal fuzzy numbers, International Journal of Computing Algorithm 3 (2014), 1003-1005.

[11] Zhaohao Sun and Jun Han, Inverse α-Cuts and Interval α-Cuts. In Proc. Intl Conf on Innovative Computing, Information and Control (ICICIC2006), Beijing, IEEE Press, (2006), 441-444.

[12] T. Pathinathan and K. Ponnivalavan, Reverse order triangular, trapezoidal and pentagonal fuzzy numbers, Annals of Pure and Applied Mathematics 9 (2015), 107-117.

[13] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, New Jersey, 1970.

[14] Walter Rudin, Principles of Mathematical Analysis, Third Edition, McGraw-Hill International Editions, (1976).

[15] L. A. Zadeh, Fuzzy sets, Information and Control 8(3) (1965), 338-353.