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CERTAIN INTEGRAL INVOLVING ERROR AND IMAGINARY ERROR FUNCTION

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Abstract

In this paper we have evaluated certain integral associated to Error function and Imaginary error function involving Hypergeometric function.

1. Introduction

Yurry A. Brychkov [Brychkov p.188 (4.4.5.5, 4.4.5.6)] has derived the following formulae

$$\int_0^1 \cos^{-1} x \ \mathrm{erfi}(ax)\, dx = \frac{\sqrt{\pi}}{2a}\left[1 - e^{\frac{a^2}{2}}\left\{(a^2-1)I_0\left(\frac{a^2}{2}\right) - a^2 I_1\left(\frac{a^2}{2}\right)\right\}\right]. \quad (1.1)$$

$$\int_0^1 x^2 \cos^{-1} x \ \mathrm{erfi}(ax)\, dx$$

$$= \frac{\sqrt{\pi}}{36a^3}\left[(4a^4 - 3a^2 + 6)e^{\frac{a^2}{2}} I_0\left(\frac{a^2}{2}\right) - a^2(4a^2+1)e^{\frac{a^2}{2}} I_1\left(\frac{a^2}{2}\right) - 6\right]. \quad (1.2)$$

The error function is defined as.

$$\mathrm{erf}(\zeta) = \frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n \zeta^{2n+1}}{n!\,(2n+1)}. \quad (1.3)$$

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The imaginary error function is defined as.

$$\operatorname{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{\zeta^{2n+1}}{n!\,(2n+1)}. \tag{1.4}$$

A generalized hyper geometric function $pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\ldots(k+a_p)}{(k+b_1)(k+b_2)\ldots(k+b_q)(k+1)} z. \tag{1.5}$$

Where $k+1$ in the denominator is present for historical reasons of notation [Koepf p.12 (2.9)], and the resulting generalized hypergeometric function is written

$$pFq\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix}\; z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k z^k}{(b_1)_k (b_2)_k \ldots (b_q)_k k!} \tag{1.6}$$

where the parameters $b_1, b_2, \ldots, b_q$ are positive integers.

The pFq series converges for all finite z if $p \le q$, converges for $|z| < 1$ if $p = q+1$, diverges for all z, $z \ne 0$ if $p > q+1$ [Luke p.156 (3)].

The function ${}_2F_1(a, b, c, z)$ corresponding to $p = 2, q = 1$, is the first hyper geometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial, falling sequential product, or lower factorial) is defined as the polynomial [Steffensen p.8]

$$(x)_n = x(x-1)(x-2)\ldots(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) \tag{1.7}$$

Dawson's integral [Abramowitz and Stegun, pp. 295 and 319] is defined as

$$F(z) = \frac{1}{2}\sqrt{\pi}e^{-z^2}\operatorname{erfi}(z) \tag{1.8}$$

The first kind modified Bessel function [Abramowitz and Stegun, pp. 376] is defined as

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\, d\theta$$

2. Main Formulae of the Integration

$$\int_0^1 x\ \sin^{-1} x\ \operatorname{erfi}(ax)\ dx$$

$$= \frac{9\pi\ e^{a^2}(2a^2F(a) + F(a) - a) - 16a^3{}_2F_2\left(\frac{1}{2},\ 2,\ \frac{5}{2},\ \frac{5}{2},\ a^2\right)}{36\sqrt{\pi}\ a^2} \tag{2.1}$$

$$\int_0^1 \frac{\sin^{-1} x\ \operatorname{erfi}(ax)}{x}\ dx$$

$$= \frac{a\left(\pi\ {}_2F_2\left(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},\ a^2\right) - 2{}_3F_3\left(\frac{1}{2},\frac{1}{2},1,\frac{3}{2},\frac{3}{2},\frac{3}{2},\ a^2\right)\right)}{\sqrt{\pi}} \tag{2.2}$$

$$\int_0^1 \frac{\sin^{-1} x\ \operatorname{erfi}(ax)}{x}\ dx$$

$$= \frac{a\left(\pi\ {}_2F_2\left(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{3}{2},-a^2\right) - 2{}_3F_3\left(\frac{1}{2},\frac{1}{2},1,\frac{3}{2},\frac{3}{2},\frac{3}{2},-a^2\right)\right)}{\sqrt{\pi}}$$

$$\text{for } \operatorname{Re}(a) > 0 \tag{2.3}$$

$$\int_0^1 x\sin^{-1} x\ \operatorname{erfi}(ax)\ dx$$

$$= \frac{\pi(2\ a^2 - 1)\ \operatorname{erfi}(a) + 2\sqrt{\pi}e^{-a^2}a}{8a^2} - \frac{4\ a{}_2F_2\left(\frac{1}{2},\ 2,\ \frac{5}{2},\frac{5}{2},-a^2\right)}{9\sqrt{\pi}}$$

for $\operatorname{Re}(a) > 0$ (2.4)

$$\int_0^1 \sin^{-1} x \ \operatorname{erfi}\left(\frac{x}{a}\right) dx$$

$$= \frac{\sqrt{\pi}^{2a^2}\sqrt{e}\left(-a^{2a^2}\sqrt{e}\left(a - 2\ F\left(\frac{1}{a}\right)\right) + (a^2 - 1) I_0\left(\frac{1}{2a^2}\right) + I_1\left(\frac{1}{2a^2}\right)\right)}{2a} \tag{2.5}$$

$$\int_0^1 x \sin^{-1} x \ \operatorname{erfi}\left(\frac{x}{a}\right) dx$$

$$= \frac{9\pi a^{a^2}\sqrt{e}((a^2 + 2)\ F\left(\frac{1}{a}\right) - a) - 16\,{}_2F_2\left(\frac{1}{2}, 2, \frac{5}{2}, \frac{5}{2}, \frac{1}{a^2}\right)}{36\sqrt{\pi}\ a} \tag{2.6}$$

$$\int_0^1 x \sin^{-1} x \ \operatorname{erfi}\left(\frac{x}{a}\right) dx = -\frac{4\,{}_2F_2\left(\frac{1}{2}, 2, \frac{5}{2}, \frac{5}{2}, -\frac{1}{a^2}\right)}{9\sqrt{\pi}\ a}$$

$$-\frac{1}{8}\pi(a^2 - 2)\ erf\left(\frac{1}{a}\right) + \frac{1}{4}\sqrt{\pi}e^{-1/a^2}a \text{ for } \operatorname{Re}(a) > 0 \tag{2.7}$$

$$\int_0^1 \sin^{-1} x \ \operatorname{erf}\left(\frac{x}{a}\right) dx$$

$$= \frac{1}{2}\sqrt{\pi}\left(e^{-1/a^2}a - \frac{e^{-1/(2a^2)}\left((a^2 + 1) I_0\left(\frac{1}{2a^2}\right) + I_1\left(\frac{1}{2a^2}\right)\right)}{a} + \sqrt{\pi}\ erf\left(\frac{1}{2}\right)\right)$$

for $\operatorname{Re}(a) > 0$ (2.8)

$$\int_0^1 x \sin^{-1} x \ \operatorname{erfi}\left(\frac{x^2}{a}\right) dx$$

$$= \frac{1}{4}\sqrt{\pi}\left[a\ {}_3F_3\left(-\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{a^2}\right) - {}^{a^2}\sqrt{e}\left(a - 2\ F\left(\frac{1}{a}\right)\right)\right] \tag{2.9}$$

$$\int_0^1 x^3 \sin^{-1} x \ \operatorname{erfi}\left(\frac{x^2}{a}\right) dx$$

$$= \frac{\sqrt{\pi}\left[12\ a\ \sqrt[a^2]{e}\left((a^2+2)F\left(\frac{1}{a}\right)-a\right)-5\,{}_3F_3\left(\frac{1}{2},\frac{7}{4},\frac{9}{4},\frac{5}{2},\frac{5}{2},2,\frac{1}{a^2}\right)\right]}{96\ a} \tag{2.10}$$

$$\int_0^1 x^3 \sin^{-1} x\ \text{erfi}\left(\frac{x^2}{a}\right)\ dx = \frac{1}{96}\sqrt{\pi}\left[-\frac{5\,{}_3F_3\left(\frac{1}{2},\frac{7}{4},\frac{9}{4},\frac{5}{2},\frac{5}{2},2,-\frac{1}{a^2}\right)}{a}\right.$$

$$\left.-6\sqrt{\pi}(a^2-2)\ erf\left(\frac{1}{a}\right)+12\ e^{-1/a^2}a\right] \text{ for } \operatorname{Re}(a)>0 \tag{2.11}$$

$$\int_0^1 \cos^{-1} x\ \text{erfi}\left(\frac{x}{a}\right)\ dx = \frac{\sqrt{\pi}\left[a^2-\sqrt[2a^2]{e}\left((a^2-1)I_0\left(\frac{1}{2a^2}\right)+I_1\left(\frac{1}{2a^2}\right)\right)\right]}{2a} \tag{2.12}$$

$$\int_0^1 x\cos^{-1} x\ \text{erfi}\left(\frac{x}{a}\right)\ dx = \frac{4\,{}_2F_2\left(\frac{1}{2},2,\frac{5}{2},\frac{5}{2},\frac{1}{a^2}\right)}{9\sqrt{\pi}\ a} \tag{2.13}$$

$$\int_0^1 x^2\cos^{-1} x\ \text{erfi}\left(\frac{x}{a}\right)\ dx$$

$$= \frac{\sqrt{\pi}\left[\sqrt[2a^2]{e}\left((6a^4-3a^2+4)I_0\left(\frac{1}{2a^2}\right)-(a^2+4)I_1\left(\frac{1}{2a^2}\right)\right)-6a^4\right]}{36a} \tag{2.14}$$

$$\int_0^1 x^3\cos^{-1} x\ \text{erfi}\left(\frac{x}{a}\right)\ dx$$

$$= -\frac{4\left[{}_2F_2\left(\frac{5}{2},3,\frac{7}{2},\frac{7}{2},\frac{1}{a^2}\right)-5\,{}_2F_2\left(\frac{1}{2},3,\frac{3}{2},\frac{7}{2},\frac{1}{a^2}\right)\right]}{75\sqrt{\pi}\ a} \tag{2.15}$$

$$\int_0^1 x^5\cos^{-1} x\ \text{erfi}\left(\frac{x}{a}\right)\ dx$$

$$= -\frac{16\left[{}_2F_2\left(\frac{7}{2},4,\frac{9}{2},\frac{9}{2},\frac{1}{a^2}\right)-7\,{}_2F_2\left(\frac{1}{2},4,\frac{3}{2},\frac{9}{2},\frac{1}{a^2}\right)\right]}{735\sqrt{\pi}\ a} \tag{2.16}$$

$$\int_0^1 x^7 \cos^{-1} x \ \text{erfi}\left(\frac{x}{a}\right) dx$$

$$= -\frac{32\left[{}_2F_2\left(\frac{9}{2}, 5, \frac{11}{2}, \frac{11}{2}, \frac{1}{a^2}\right) - 7\,{}_2F_2\left(\frac{1}{2}, 5, \frac{3}{2}, \frac{11}{2}, \frac{1}{a^2}\right)\right]}{2835\sqrt{\pi}\ a} \qquad (2.17)$$

References

[1] A. Abramowitz, Milton, Stegun and Irene, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, (1972).

[2] Aomoto, Kazuhiko, Kita, Michitake, Theory of Hypergeometric functions, Springer Tokyo Dordrecht Heidelberg London New York, (2011).

[3] Y. A. Brychkov, Handbook of Special Functions: Derivatives, Integrals, Series and Other Formulas, CRC Press, Taylor and Francis Group, London, UK, (2008).

[4] F. Bowman, Introduction to Bessel functions, Dover Publications Inc., New York, (1958).

[5] G. Gasper, Rahman, Mizan, Basic Hypergeometric Series, Cambridge University Press, Cambridge CB2 2RU, UK, (2004).

[6] C. F. Gauss, Disquisitiones generales circa seriem infinitam..., Comm. soc. reg. sci. Gott. rec. 2 (1813), 123-162.

[7] W. Koepf, Hypergeometric Summation, An Algorithmic Approach to Summation and Special Function Identities, Braunschweig, Germany: Vieweg, (1998).

[8] Y. L. Luke, Mathematical functions and their approximations, Academic Press Inc., London, (1975).

[9] A. M. Mathai and H. J. Haubold, Special Functions for Applied Scientists, Springer, New York, (2008).

[10] P. Appell, Sur une formule de M. Tisserand et sur les fonctions hypergomtriques de deux variables, J. Math. Pures Appl. 10(3) (1884), 407-428.

[11] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integral and Series More Special Functions, Nauka, Moscow 3 (2003).

[12] Salahuddin, Husain and Intazar, Certain New Formulae Involving Modified Bessel Function of First Kind, Global Journal of Science Frontier Research (F) 13(10) (2013), 13-19.

[13] Salahuddin and R. K. Khola, New hypergeometric summation formulae arising from the summation formulae of Prudnikov, South Asian Journal of Mathematics 4 (2014), 192-196.

[14] J. F. Steffensen, Interpolation (2nd ed.), Dover Publications, USA, (2006).