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A NEW TECHNIQUE TO FIND LAPLACE TRANSFORM

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Abstract

Recently a new integral transform named β-Laplace integral transform has been introduced which is a new form of generalization of classical Laplace transform. This new generalized transform has many characteristic properties over Laplace transform. In this work, we purpose a new technique to find Laplace transform by applying the β-Laplace integral transform associated with Leibnitz rule which is simple and efficient than the direct Laplace transform technique.

1. Introduction

Integral transforms always been a very useful mathematical tool to solve many kind of differential equation, partial differential equation, integral equtions. We have a wide range of integral transform such as Laplace, Fourier, Hankel, Mellin, Radon, Gabor, Hilbert, Weiestrauss, Abel, Sumudu, etc. They all have a wide range of applications in the fields of Physical science, Mathematics, Statistics, and engineering [2, 9, 11].

Laplace integral transform is one of the oldest and famous integral transforms purposed by P. S. Laplace in its most celebrated work 'Théorie analytique des probabilités' [8]. Laplace transform has a wide range of applications in many fields of science and engineering [1, 3, 10, 12].

Recently a new form of generalization of Laplace Transform has been introduced and named β-Laplace integral transform which has many characteristic properties over Laplace transform [4, 5, 6].

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One more beautiful aspect of this new β-Laplace integral transform is that it is generalization of most of the recently introduced transforms, for instance, Sumudu, Kamal, Natural, Polynomial, Tarig, Elzaki, Aboodh, Laplace-Carson (Mahgoub), Mohand, Sawi, Sadik integral transform [6].

In this paper, we establish a connection between β-Laplace integral transform and Laplace integral transform via Leibnitz rule to find Laplace integral transform of a function. Here we purpose a new efficient and simple technique to find Laplace transform by using this new generalized β-Laplace transform via Leibnitz rule. We illustrate this technique by some examples.

2. Definitions and Prepositions

(i) Laplace Integral Transform

Laplace integral transform of a suitable function $\psi(t),\ t \geq 0$ is defined by

$$L\{\psi\}_{(p)} = \int_0^\infty e^{-pu}\psi(u)du,\ p \in \mathbb{C},\ \mathrm{Re}(p) > 0. \tag{1}$$

Where suitable function means a function for which improper integral of right side converges.

(ii) β-Laplace Integral Transform [4]

β-Laplace integral transform of a suitable function $\psi(t),\ t \geq 0$ is defined by

$$L_\beta\{\psi\}_{(p)} = \int_0^\infty \beta^{-pu}\psi(u)du,\ \beta > 1,\ p \in \mathbb{C},\ \mathrm{Re}(p) > 0. \tag{2}$$

(iii) Leibnitz Integral Rule [7]

Leibnitz rule (Differentiation under integral sign) for an integral in the form $\int_{\phi_1(x)}^{\phi_2(x)}\psi(x, y)dy$, states that:

Let $\psi(x,\ y)$ be a continuous function with continuous partial derivative $\frac{\partial\psi(x,\ y)}{\partial x}$ in some region of the $(x,\ y)$-plane which include region $\phi_1(x) \leq y \leq \phi_2(x),\ x_0 \leq x \leq x_1$, where both the functions $\phi_1(x)$, and $\phi_2(x)$

are continuous and have continuous derivatives for $x_0 \leq x \leq x_1$. Then, for $x_0 \leq x \leq x_1$,

$$\frac{d}{dx}\left\{\int_{\phi_1(x)}^{\phi_2(x)} \psi(x, y)dy\right\} = \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial}{\partial x}\{\psi(x, y)\}dy + \psi(x, \phi_1(x)) \cdot \frac{d}{dx}\{\phi_1(x)\}$$

$$-\psi(x, \phi_2(x)) \cdot \frac{d}{dx}\{\phi_2(x)\}. \quad (3)$$

Preposition 1 ([4]). *Let* $\psi(t)$, $t \geq 0$ *be a suitable function then*

$$\lim_{\beta \to e} L_\beta\{\psi\}_{(p)} = L\{\psi\}_{(p)}. \quad (4)$$

Preposition 2 ([4]). *Let* $\psi(t)$, $t \geq 0$ *be a suitable function then*

$$\lim_{\beta \to e} L_\beta\{\psi\}_{(p)} = 0. \quad (5)$$

3. Main Results

In this section we introduce a new way to find Laplace transform with the help of applying Leibnitz rule on β-Laplace integral transform.

Let $\psi(t)$ be the function under consideration of finding Laplace integral transform, and we start from the definition of β-Laplace integral transform

$$L_\beta\{\psi\}_{(p)} = \int_0^\infty \beta^{-pu}\psi(u)du.$$

Taking partial derivative both sides

$$\frac{\partial}{\partial \beta}\{L_\beta\{\psi\}_{(p)}\} = \frac{\partial}{\partial \beta}\int_0^\infty \beta^{-pu}\psi(u)du.$$

After applying Leibnitz rule, we get

$$\frac{\partial}{\partial \beta}\{L_\beta\{\psi\}_{(p)}\} = \int_0^\infty \left(-\frac{pu}{\beta}\right)\beta^{-pu}\psi(u)du.$$

Let find the β-Laplace transform of the right side of the equation, plug into equation

$$\frac{\partial}{\partial \beta}\{L_\beta\{\psi\}_{(p)}\} = -\frac{p}{\beta}G(\beta, p).$$

Where $\int_0^\infty u\beta^{-pt}\psi(u)du = G(\beta, p)$.

Taking integration both sides with respect to β

$$L_\beta\{\psi\}_{(p)} = -\int \frac{p}{\beta} G(\beta,\ p)d\beta.$$

$L_\beta\{\psi\}_{(p)} = \overline{G}(\beta,\ p) + c,\ c$ is an integral constant

Where $-\int \frac{p}{\beta} G(\beta, p)d\beta = \overline{G}(\beta, p)$.

By preposition (2)

$$\lim_{\beta\to\infty} L_\beta\{\psi\}_{(p)} = 0.$$

After taking Limit $\beta \to \infty$

$$\lim_{\beta\to\infty} L_\beta\{\psi\}_{(p)} = \lim_{\beta\to\infty} (\overline{G}(\beta,\ p) + c)$$

$$\Rightarrow c = -\lim_{\beta\to\infty} \overline{G}(\beta,\ p).$$

After plugging the value of c in equation

$$L_\beta\{\psi\}_{(s)} = \overline{G}(\beta,\ p) - \lim_{\beta\to\infty} \overline{G}(\beta,\ p).$$

Finally, after taking limit $\beta \to e,$ we obtain our desired Laplace transform

$$\lim_{\beta\to e} L_\beta\{\psi\}_{(p)} = \lim_{\beta\to e}\left(\overline{G}(\beta,\ p) - \lim_{\beta\to e} \overline{G}(\beta,\ p)\right)$$

$$L\{\psi\}_{(p)} = \lim_{\beta\to e} \overline{G}(\beta,\ p) - \lim_{\beta\to e}\left(\lim_{\beta\to\infty} \overline{G}(\beta,\ p)\right)$$

$$L\{\psi\}_{(p)} = \lim_{\beta\to e} \int \left(-\frac{p}{\beta}\right) G(\beta,\ p)d\beta - \lim_{\beta\to e}\left\{\lim_{\beta\to\infty} \int \left(-\frac{p}{\beta}\right) G(\beta,\ p)d\beta\right\}$$

$$L\{\psi\}_{(p)} = \lim_{\beta\to e} \int \left(\int_0^\infty \left(-\frac{p}{\beta}\right)\beta^{-pu}\psi(u)du\right)d\beta$$

$$-\lim_{\beta\to e}\left\{\lim_{\beta\to\infty} \int \left(\int_0^\infty \left(-\frac{pu}{\beta}\right)\beta^{-pu}\psi(u)du\right)d\beta\right\}.$$

Now We have obtained our desired result.

4. Illustrative Examples

Example 1. The Laplace transform of $\frac{\sin(t)}{t}$, $t > 0$. By the definition of β-Laplace transform

$$L_\beta\{\psi\}_{(p)} = \int_0^\infty \beta^{-pu}\psi(u)du.$$

Taking $\psi(t) = \frac{\sin t}{t}$, $t > 0$, we have

$$L_\beta\{\psi\}_{(p)} = \int_0^\infty \frac{\beta^{-su}\sin(u)}{u}du.$$

Taking partial derivative with respect to β (Partial derivative under integral sign) We obtain

$$\frac{\partial}{\partial\beta}(L_\beta\{\psi\}_{(p)}) = \frac{\partial}{\partial\beta}\int_0^\infty \frac{\beta^{-pu}\sin(u)}{u}du = \int_0^\infty \left(-\frac{pu}{\beta}\right)\frac{\beta^{-pu}\sin(u)}{u}du$$

$$= -\frac{p}{\beta}\int_0^\infty \beta^{-pu}\sin(u)du$$

$$= -\frac{p}{\beta}\left(\frac{1}{(p\ln\beta)^2+1}\right).$$

Integrating with respect to β

$$L_\beta\{\psi\}_{(p)} = \int\left(-\frac{p}{\beta}\left(\frac{1}{(p\ln\beta)^2+1}\right)\right)d\beta.$$

After solving, we have $L_\beta\{\psi\}_{(p)} = -\tan^{-1}(p\ln\beta) + c$ where c is an integral constant.

Taking limit $\beta \to \infty$, we get

$$\lim_{\beta\to\infty} L_\beta\{\psi\}_{(p)} = \lim_{\beta\to\infty}(-\tan^{-1}(p\ln\beta) + c)$$

$$0 = -\frac{\pi}{2} + c$$

$$c = \frac{\pi}{2}.$$

And we get

$$L_\beta\{\psi\}_{(p)} = -\tan^{-1}(p\ln\beta) + \frac{\pi}{2} = \cot^{-1}(p\ln\beta)$$

$$L_\beta\{\psi\}_{(p)} = \tan^{-1}\left(\frac{1}{p\ln\beta}\right).$$

Again, taking limit as $\beta \to e$

$$\lim_{\beta\to e} L_\beta\{\psi\}_{(p)} = \lim_{\beta\to e}\tan^{-1}\left(\frac{1}{p\ln\beta}\right)$$

$$L\left\{\psi(t) = \frac{\sin t}{t}\right\}_{(p)} = \tan^{-1}\left(\frac{1}{p}\right).$$

Example 2. The Laplace integral transform $\frac{e^{-\frac{1}{t}}}{t^{\frac{3}{2}}}$, $t > 0$. By the definition of β-Laplace transform

$$L_\beta\{\psi\}_{(p)} = \int_0^\infty \beta^{-pu}\psi(u)du.$$

Taking $\psi(t) = \frac{e^{-\frac{1}{t}}}{e^{\frac{3}{2}}}$, $t > 0$, we have

$$L_\beta\{\psi\}_{(p)} = \int_0^\infty \beta^{-pu}\frac{e^{-\frac{1}{u}}}{u^{\frac{3}{2}}}du.$$

Taking partial derivative with respect to β (Partial derivative under integral sign) We obtain

$$\frac{\partial}{\partial \beta}\left(L_{\beta}\{\psi\}_{(p)}\right) = \frac{\partial}{\partial \beta}\int_{0}^{\infty} \beta^{-pu} \frac{e^{-\frac{1}{u}}}{u^{\frac{3}{2}}} du = \int_{0}^{\infty}\left(-\frac{pu}{\beta}\right)\beta^{-pu}\frac{e^{-\frac{1}{u}}}{u^{\frac{3}{2}}}du$$

$$= -\left(\frac{p}{\beta}\right)\int_{0}^{\infty} \beta^{-pu} \frac{e^{-\frac{1}{u}}}{u^{\frac{1}{2}}}du$$

$$= -\left(\frac{p}{\beta}\right)\sqrt{\frac{\pi}{p \ln \beta}}e^{-2\sqrt{p \ln \beta}}.$$

Substitute $p \ln \beta = v$, $\frac{p}{\beta} d\beta = dv$ and integrating with respect to β

$$L_{\left(e^{\frac{v}{p}}\right)}\{\psi\}_{(p)} = -\int\left(\sqrt{\frac{\pi}{v}}e^{-2\sqrt{v}}\right)dv$$

$$L_{\left(e^{\frac{v}{p}}\right)}\{\psi\}_{(p)} = \sqrt{\pi} \cdot e^{-2\sqrt{v}} + c$$

$$L_{\beta}\{\psi\}_{(p)} = \sqrt{\pi} \cdot e^{-2\sqrt{p \ln \beta}+c}.$$

Taking $\beta \to \infty$, we have

$$\lim_{\beta \to \infty} L_{\beta}\{\psi\}_{(p)} = \lim_{\beta \to \infty}(\sqrt{\pi} \cdot e^{-2\sqrt{p \ln \beta}} + c)$$

$$0 = 0 + c \Rightarrow c = 0$$

$$L_{\beta}\{\psi\}_{(p)} = \sqrt{\pi} \cdot e^{-2\sqrt{p \ln \beta}}.$$

Taking limit $\beta \to e$, we get Laplace transform of the given function

$$L_{\beta}\{\psi\}_{(p)} = \sqrt{\pi} \cdot e^{-2\sqrt{p}}.$$

5. Conclusion

It is obvious that if we apply β-Laplace integral transform via Leibnitz rule, then it is applicable to find Laplace integral transform of many of the functions without including much more calculation.

References

[1] R. E. Bellman and R. S. Roth, The Laplace Transform, World Scientific, Singapore, 1984.

[2] L. Debnath and D. Bhatta, Integral Transforms and Their Applications, 3rd edition, Chapman and Hall/CRC, 2015.

[3] G. Doetsch, Introduction to the Theory and Applications of the Laplace Transformation, Springer-Verlag, New York, 1970.

[4] A. Gaur and G. Agarwal, Application of β-Laplace Integral Transform, Talent Development and Excellence 12(3s) (2020), 1058-1068.

[5] A. Gaur and G. Agarwal, On β-Laplace Integral Transforms and Their Properties, International Journal of Advanced Science and Technology 29 (2020), 1481-1491.

[6] A. Gaur and G. Agarwal, Relation of β-Laplace Integral Transforms with Other Integral Transforms, Test Engineering and Management 83(2020), 8653-8659.

[7] H. M. Protter and B. Jr .M. Charles, Differentiation under the Integral Sign, Intermediate Calculus (Second ed.), New York: Springer, 1985.

[8] P. S. Laplace, Theorie Analytique des Probabilities, Vol. I, Part 2, Lerch, Paris. 1820.

[9] I. Podlubny, Fractional Differential Equations, Academic Press, United States, 1999.

[10] J. L. Schiff, The Laplace Transforms, Springer, New York, 1999.

[11] I. N. Sneddon, The Use of Integral Transforms, McGraw-Hill Book Company, New York, United States, 1972.

[12] E. J. Watson, Laplace Transforms and Applications, Van Nostrand Reinhold, New York, 1981.