RICCI SOLITONS IN $f$-KENMOTSU MANIFOLDS WITH THE QUARTER-SYMMETRIC NON-METRIC CONNECTION

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Abstract

In this paper, we have some curvature conditions in 3-dimensional $f$-Kenmotsu manifolds with the quarter-symmetric non-metric connection. We also have that this manifold is not always $\xi$-projective flat. And we have shown that 3-dimensional $f$-Kenmotsu manifold with the quarter-symmetric non-metric connection is also an $\eta$-Einstein manifold and the Ricci soliton defined on this manifold is said to be expanding or shrinking with respect to values of $f$ and $\lambda$ constant.

1. Introduction

In 1972, Kenmotsu [6] studied a class of contact Riemannian manifold satisfying some special conditions and named this manifold as Kenmotsu manifold.

The manifold $M$, with the structure $(\phi, \xi, \eta, g)$ is called normal if $[\phi, \phi] + 2d\eta \otimes \xi = 0$ and it is almost cosymplectic if $d\eta = 0$ and $d\phi = 0$. A normal and almost cosymplectic manifold is called cosymplectic. Olszak and Rosca [10] studied geometrical aspect of $f$-Kenmotsu manifolds and gave some curvature conditions. Also the other mathematicians proved that a Ricci-symmetric $f$-Kenmotsu Manifold is an Einstein Manifold. Later on, in 2010, authors also proved that Ricci semi-symmetric $\alpha$-Kenmotsu manifolds are Einstein manifolds.
In 1983, Sharma and Sinha [13] started to study of the Ricci Solitons. Later on Ricci Solitons in contact manifolds were extensively studied by Cornelia Livia Bejan and Mircea Crasmareanu [2].

In 2012, the theory of Ricci solitons on Kenmotsu manifolds were studied by Nagaraja and Premalatha [2] and a deep study was done by S. C. Rastogi [11], [12] on quarter-symmetric non-metric connection.

Starting with the introduction, we have some fundamental notions used in this study, in section 2. In section 3, we have the introduction of $f$-Kenmotsu Manifold. In the next section 4 we study $f$-Kenmotsu manifold with quarter-symmetric non-metric connection and proved that this manifold is not always $\xi$-projective flat. In the last section we prove that $f$-Kenmotsu manifold with the quarter-symmetric non-metric connection is $\eta$-Einstein manifold and the Ricci soliton defined on this manifold is classified with respect to the values of $f$ and $\lambda$ constant.

2. Preliminaries

Consider a 3-dimensional differentiable manifold $M$ with an almost contact structure $(\phi, \xi, \eta, g)$ satisfying

$$\phi^2 X = -X + \eta(X)\xi,$$

$$\eta \circ \phi = 0,$$

$$\phi\xi = 0,$$

$$\eta(\xi) = 1,$$

$$g(X, \xi) = \eta(X),$$

$$g(X, \phi Y) = -g(\phi X, Y),$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \quad (2.1)$$

for any vector fields $X, Y \in \chi(M)$, where $\phi$ is a (1,1) tensor field, $\xi$ is a vector field, $\eta$ is a 1-form and $g$ is Riemannian metric. Then $M$ is called an almost contact manifold. For an almost contact manifold $M$, we have [16]

$$(\nabla_X \phi)Y = \nabla_X \phi Y - \phi(\nabla_X Y) \quad (2.2)$$
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\[
(\nabla_X \eta)Y = \nabla_X \eta Y - \eta(\nabla_X Y).
\] (2.3)

Let \( \{e_1, e_2, e_3, \ldots, e_n\} \) be orthonormal basis of \( T_p(M) \). \( R \) be Riemannian curvature tensor, \( S \) be Ricci curvature tensor, \( Q \) be Ricci operator, then \( \forall X, Y \in \chi(M) \) it follows that [5]

\[
S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i),
\] (2.4)

\[
QX = -\sum_{i=1}^{n} R(e_i, X)e_i
\] (2.5)

\[
S(X, Y) = g(QX, Y)
\] (2.6)

In \( f \)-Kenmotsu manifold, if the Ricci tensor \( S \) satisfy the condition

\[
S(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)
\] (2.7)

\( \alpha, \beta \) be certain scalars, then the manifold \( M \) is said to be \( \eta \)-Einstein manifold. If \( \beta = 0 \), the manifold is Einstein manifold.

In a three dimensional Riemannian manifold, the curvature tensor \( R \) is defined as

\[
R(X, Y)Z = S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY
\]

\[
-\frac{\tau}{2}[g(Y, Z)X - g(X, Z)Y]
\] (2.8)

where \( S \) is the Ricci tensor, \( Q \) is Ricci operator and \( \tau \) is the scalar curvature.

Now, let \( M \) be an \( n \)-dimensional Riemannian manifold with the Riemannian connection \( \nabla \). A linear connection \( \tilde{\nabla} \) is said to be a quarter-symmetric connection on \( M \) if its tortion tensor \( \tilde{T} \) satisfies

\[
\tilde{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y
\] (2.9)

where \( \tilde{T} \neq 0 \) and \( \eta \) is a 1-form. If moreover \( \tilde{\nabla}g = 0 \) then the connection is called quarter-symmetric metric connection.
If \( \tilde{\nabla} g \neq 0 \) then the connection is called quarter-symmetric non-metric connection [17].

For \( n \geq 1 \), the manifold \( M \) is locally projectively flat iff the projective curvature tensor \( P \) vanishes. We define the projective curvature tensor \( P \) as

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y]
\]

(2.10)
for any \( X, Y, Z \in \chi(M) \) where \( S \) is the Ricci tensor and \( R \) is the curvature tensor of \( M \). If \( P(X, Y)\xi = 0 \) for any \( X, Y \in \chi(M) \), the manifold \( M \) is called \( \xi \)-projective flat [16].

A Ricci Soliton is defined on a Riemannian manifold \((M, g)\) as a natural generalization of an Einstein metric. We define Ricci Soliton as a triple \((g, V, \lambda)\) with \( g \) a Riemannian metric, \( V \) a vector field and \( \lambda \) be a real scalar such that

\[
L_V g + 2S + 2\lambda g = 0
\]

(2.11)
where \( L_V \) denotes the Lie derivative operator along the vector field \( V \) and \( S \) is a Ricci tensor of \( M \). The Ricci soliton is said to be shrinking, steady and expanding accordingly \( \lambda \) is -ve, 0, +ve respectively.

3. \( f \)-Kenmotsu manifolds

A three dimensional almost contact manifold \( M \) with the structure \((\phi, \xi, \eta, g)\) is an \( f \)-Kenmotsu manifold if the covariant derivative of \( \phi \) satisfies [16],

\[
(\nabla_X \phi)Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X]
\]

(3.1)
where \( f \in C^\infty(M, R) \) such that \( df \wedge \eta = 0 \). If \( f^2 + f' \neq 0 \) where \( f' = \xi f \), then \( M \) is called Regular [3]. If \( f = \alpha = \text{constant} \neq 0 \), \( M \) is called \( \alpha \)-Kenmotsu manifold. If \( f = 1 \) then manifold is called 1-Kenmotsu manifold also called Kenmotsu Manifold.

By (2.1) and (2.3), we have

\[
(\nabla_X \eta)Y = fg(\phi X, \phi Y),
\]

(3.2)
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from (3.1), we have [14]

$$\nabla_X \xi = f[X - \eta(X)]$$

(3.3)

Also from (2.7), in a 3-dimensional $f$-Kenmotsu manifold

$$R(X, Y)Z = \left( \frac{1}{2} + 2f^2 + 2f' \right)(X \wedge Y)Z$$

$$- \left( \frac{1}{2} + 3f^2 + 3f' \right)[\eta(X)(\xi \wedge Y) + \eta(Y)(X \wedge \xi)Z]$$

(3.4)

and

$$S(X, Y) = \left( \frac{1}{2} + f^2 + f' \right)g(X, Y) - \left( \frac{1}{2} + 3f^2 + 3f' \right)\eta(X)\eta(Y).$$

(3.5)

Thus from (3.5), we get

$$S(X, \xi) = -2(f^2 - f')\eta(X).$$

(3.6)

By (3.4) and (3.5), we get

$$R(X, Y)\xi = -(f^2 + f')[\eta(Y)X - \eta(X)Y]$$

(3.7)

$$R(X, Y)\xi = -(f^2 + f')(\eta(X)\xi - X),$$

(3.8)

$$QX = \left( \frac{1}{2} + f^2 + f' \right)X - \left( \frac{1}{2} + 3f^2 + 3f' \right)\eta(X)\xi$$

(3.9)

From (2.10) and using (3.7) and (3.6), we have that

**Theorem 1.** A 3-dimensional $f$-Kenmotsu manifold is always $\xi$-projectively flat.

**4. $f$-Kenmotsu Manifolds with the Quarter-symmetric Non-metric Connection**

Let $\nabla$ be a Riemannian connection of $f$-Kenmotsu manifold and $\nabla^{\perp}$ be a linear connection then this linear connection $\nabla^{\perp}$ defined as

$$\nabla^{\perp}_X Y = \nabla_X Y - \eta(X)g(Y) - g(X, Y)\xi$$

(4.1)

where $X, Y \in \chi(M)$ be any vector field and $\eta$ be 1-form, is called the quarter-symmetric non-metric connection [15]. Now, using (2.2), (3.1) and (4.1) we
have

\[(\tilde{\nabla}_X \phi) Y = f[g(\phi X, Y)\xi - \eta(Y)\phi X] + g(\phi X, Y)\xi\]  \hspace{1cm} (4.2)

for any vector field \( X, Y \in \chi(M) \) where \( \phi \) be \((1,1)\) tensor field, \( \eta \) is a vector field, \( \xi \) is 1-form and \( f \in C^\infty(M, R) \) so that \( df \wedge \eta = 0 \). As a result of \( df \wedge \eta = 0 \), we have

\[df = f', \hspace{1cm} X(f) = f' \eta(X)\]  \hspace{1cm} (4.3)

where \( f' = \xi f \) \cite{10}. If \( f = 0 \), the manifold is cosymplectic. If \( f = \alpha \neq 0 \), then the manifold is \( \alpha \)-Kenmotsu. An \( \alpha \)-Kenmotsu manifold with quarter-symmetric non-metric connection is called regular, if \( f'^2 + f' + f - 2f\phi \neq 0 \).

From (2.2), (4.2) we have

\[\tilde{\nabla}_X \xi = f[X - \eta(X)\xi] - \eta(X)\xi.\]  \hspace{1cm} (4.4)

Using (2.2), (4.1) and (3.2), we get

\[(\tilde{\nabla}_X \eta) = fg(\phi X, \phi Y).\]  \hspace{1cm} (4.5)

We define the curvature tensor \( \tilde{R} \) of any \( \alpha \)-Kenmotsu manifold \( M \) with respect to quarter-symmetric non-metric connection \( \tilde{\nabla} \) as

\[\tilde{R}(X, Y)\xi = \tilde{\nabla}_X \tilde{\nabla}_Y \xi - \tilde{\nabla}_Y \tilde{\nabla}_X \xi - \tilde{\nabla}_{[X,Y]} \xi\]  \hspace{1cm} (4.6)

using (4.1), (4.4) and (3.3) we obtain

\[\tilde{\nabla}_X \tilde{\nabla}_Y \xi = X(f)Y - X(f)\eta(Y)\xi + f\tilde{\nabla}_X Y - fX\eta(Y)\xi - \eta(X)f\phi Y\]

\[- fg(X,Y)\xi - \eta(Y)f^2 X - \eta(Y)fX + 2\eta(X)\eta(Y)f\xi\]

\[+ \eta(X)\eta(Y)\xi + f^2 \eta(X)\eta(Y)\xi - X\eta(Y)\xi,\]  \hspace{1cm} (4.7)

and

\[- \tilde{\nabla}_{[X,Y]} \xi = -f\tilde{\nabla}_X Y + f\tilde{\nabla}_Y X + f \eta(Y)\phi X - f\eta(X)\phi Y\]

\[+ fX\eta(Y)\xi - fY\eta(X)\xi + X\eta(Y)\xi - Y\eta(X)\xi.\]  \hspace{1cm} (4.8)
Using (4.7) and (4.8) in (4.6), we have
\[ \tilde{R}(X, Y)\xi = X(f)Y - Y(f'X - X(f)\eta(Y)\xi + Y(f)\eta(X)\xi \]
\[ - f^2\eta(Y)X + f^2\eta(X)Y - f\eta(Y)X + f\eta(X)Y \]
\[ + 2\eta(Y)f\phi X - 2\eta(X)f\phi Y. \] (4.9)

By using (4.3) in (4.9), we have
\[ \tilde{R}(X, Y)\xi = -(f^2 + f' + f - 2f\phi)\eta(Y)X - \eta(X)Y. \] (4.10)

From (4.10), we get
\[ \tilde{R}(\xi, Y)\xi = -(f^2 + f' + f - 2f\phi)\eta(Y)\xi - Y. \] (4.11)

and
\[ \tilde{R}(X, \xi)\xi = -(f^2 + f' + f - 2f\phi)(X - \eta(X)\xi). \] (4.12)

In (4.10), taking inner product with Z, we get
\[ g(\tilde{R}(X, Y)\xi, Z) = -(f^2 + f' + f - 2f\phi)\eta(Y)g(X, Z) - \eta(X)g(Y, Z). \] (4.13)

With the help of these result we have the following lemma.

**Lemma 1.** Let M be 3-dimensional f-Kenmotsu manifold with the quarter-symmetric non-metric connection. \( \tilde{S} \) be Ricci curvature and \( \tilde{Q} \) be Ricci operator, then
\[ \tilde{S}(X, \xi) = -(f^2 + f' + f - 2f\phi)\eta(X), \] (4.14)

and
\[ \tilde{Q}\xi = -(f^2 + f' + f - 2f\phi)\xi. \] (4.15)

**Proof.** Contracting (4.13) with Y and Z and taking summation over \( i = 1, 2, 3, ..., n \), using (2.4) we have (4.14). And also by using (2.6) and (2.1) in (4.14), we get (4.15).

**Lemma 2.** Let M be 3-dimensional f-Kenmotsu manifold with quarter symmetric non-metric connection. \( \tilde{S} \) be Ricci tensor, \( \tau \) be scaler curvature tensor and \( \tilde{Q} \) be Ricci operator. Then it follows that
\[ \tilde{S}(X, Y) = \left( \frac{\tau}{2} + f'^2 + f' + f - 2f\phi \right) g(X, Y) - \left( \frac{\tau}{2} + 3f'^2 + 3f' + 3f - 6f\phi \right) \eta(X)\eta(Y), \] 
(4.16)

and

\[ \tilde{Q}X = \left( \frac{\tau}{2} + f'^2 + f' + f - 2f\phi \right) X - \left( \frac{\tau}{2} + 3f'^2 + 3f' + 3f - 6f\phi \right) \eta(X)\xi. \] 
(4.17)

**Proof.** Taking inner product of (4.12) with \( Y \), we get

\[ g(\tilde{R}(X, \xi), Y) = -(f^2 + f' + f - 2f\phi)(g(X, Y) - \eta(X)\eta(Y)). \] 
(4.18)

By putting \( X = \xi, Y = X, Z = Y \) in (2.8), using (4.14) and taking contraction with \( \xi \), we obtain

\[ g(\tilde{R}(\xi, X)Y, \xi) = \tilde{S}(X, Y) + 4(f^2 + f' + f - 2f\phi)\eta(X)\eta(Y) \]

\[ - 2(f^2 + f' + f - 2f\phi)g(X, Y) - \frac{\tau}{2} [g(X, Y) - \eta(X)\eta(Y)]. \] 
(4.19)

With the help of (4.18) and (4.19), we have (4.16). Now using (4.16) and (2.6), we get

\[ g\left( \tilde{Q}X - \left[ \left( \frac{\tau}{2} + f'^2 + f' + f - 2f\phi \right) X - \left( \frac{\tau}{2} + 3f'^2 + 3f' + 3f - 6f\phi \right) \eta(X)\xi \right], Y \right) = 0. \] 
(4.20)

Since \( Y \neq 0 \) in (4.20), this leads the proof of (4.17).

**Example (a 3-dimensional \( f \)-Kenmotsu manifold with quarter-symmetric non-metric connection).** Let us consider the 3-dimensional manifold \( M = (x, y, z) \in R^3, z \neq 0 \) where \((x, y, z)\) are the standard coordinates in \( R^3 \). The vector fields

\[ e_1 = z^2 \frac{\partial}{\partial x}, \quad e_2 = z^2 \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z} \]

are linearly independent at each point of \( M \). Let \( g \) be the Riemannian metric defined as

\[ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1, \quad g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0. \]
Now consider a \((1, 1)\) tensor field \(\phi\) defined by \(\phi(e_1) = -e_2, \phi(e_2) = e_1, \phi(e_3) = 0\) then using linearity of \(g\) and \(\phi\), for any \(Z, W \in \chi(M)\) we have

\[
\eta(e_3) = 1,
\]

\[
\phi^2(Z) = -Z + \eta(Z)e_3,
\]

\[
g(\phi Z, \theta W) = g(Z, W) - \eta(Z)\eta(W).
\]

Now by computation directly, we get

\[
[e_1, e_2] = 0, [e_2, e_3] = -\frac{2}{z} e_2, [e_1, e_3] = -\frac{2}{z} e_1.
\]

By the use of these above equations, we have

\[
\nabla_{e_1} e_1 = \frac{2}{z} e_3, \nabla_{e_2} e_2 = \frac{2}{z} e_3, \nabla_{e_3} e_3 = 0,
\]

\[
\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = \nabla_{e_3} e_1 = \nabla_{e_3} e_3 = 0. \tag{4.21}
\]

Now in this example we consider for quarter-symmetric non-metric connection, using (4.1) and (4.21) we have

\[
\nabla_{i} e_i = \left(\frac{2}{z} - 1\right)e_3, \nabla_{e_3} e_3 = -e_3, \nabla_{e_i} e_3 = -\frac{2}{z} e_i, \nabla_{e_i} e_j = 0 = \nabla_{e_3} e_i \tag{4.22}
\]

where \(i \neq j = 1, 2\).

We know that

\[
\bar{R}(X, Y)Z = \nabla_{X} \nabla_{Y} Z - \nabla_{Y} \nabla_{X} Z - \nabla_{[X, Y]} Z. \tag{4.23}
\]

Using (4.22) and (4.23) we get

\[
\bar{R}(e_i, e_3)e_3 = \left(\frac{2}{z} - \frac{6}{z^2}\right)e_i, \bar{R}(e_i, e_j)e_3 = 0
\]

\[
\bar{R}(e_i, e_j)e_j = \left(\frac{2}{z} - \frac{6}{z^2}\right)e_i, \bar{R}(e_i, e_3)e_j = 0, \tag{4.24}
\]

\[
\bar{R}(e_3, e_i)e_i = \left(\frac{2}{z} - \frac{6}{z^2}\right)e_3
\]

where \(i \neq j = 1, 2\).
Using (2.4) and (4.24), we verify that
\[ S(e_i, e_i) = \frac{-10}{z^2} + \frac{2}{z^2} + 1, \quad i = 1, 2, 3. \] (4.25)

Now using (2.10), (4.24) and (4.25), we have that
\[ \tilde{P}(e_1, e_2)e_3 = 0, \quad \tilde{P}(e_i, e_3)e_3 = \left( \frac{4}{3z} - \frac{8}{z^2} \right)e_i. \]

This leads to the following Lemma:

**Lemma 3.** A 3-dimensional $f$-Kenmotsu manifold with the quarter-symmetric non-metric connection is not necessarily $\xi$-projectively flat.

### 5. Ricci Solitons in $f$-Kenmotsu Manifold with the quarter-symmetric non-metric connection

Consider a 3-dimensional $f$-Kenmotsu manifold with the quarter-symmetric non-metric connection. Let $V$ be pointwise collinear with $\xi$ (i.e. $V = b\xi$, where $b$ is a function). Then $(L_V g + 2S + 2\lambda g)(X, Y) = 0$ implies
\[ (Xb)\eta(Y) + bg(\tilde{V}_X \xi, Y) + (Yb)\eta(X) + bg(X, \tilde{V}_Y \xi) + 2\tilde{S}(X, Y) + 2\lambda g(X, Y) = 0. \] (5.1)

Using (4.4) in (5.1), we get
\[ 0 = (Xb)\eta(Y) + (Yb)\eta(X) + 2bfg(X, Y) - 2bef(X)\eta(Y) - b\eta(X)\eta(Y) + 2\tilde{S}(X, Y) + 2\lambda g(X, Y) \] (5.2)

substitute $Y$ with $\xi$ in (5.2), we obtain
\[ Xb - 2b\eta(X) + \xi b\eta(X) - 4(f^2 + f' + f - 2f\phi)\eta(X) + 2\lambda \eta(X) = 0 \] (5.3)

again substituting $X$ with $\xi$ in (5.3),
\[ \xi b = 2(f^2 + f' + f - 2f\phi) + b - \lambda. \] (5.4)

putting (5.3) in (5.4), we have
\[ b = [2(f^2 + f' + f - 2f\phi) + b - \lambda] \eta. \] (5.5)
applying $d$ on (5.5)

$$0 = db = [2(f^2 + f' + f - 2f\phi) + b - \lambda]d\eta$$  (5.6)

since $d\eta \neq 0$, we have

$$[2(f^2 + f' + f - 2f\phi) + b - \lambda] = 0.  \quad \text{(5.7)}$$

Now using (5.5) and (5.7), it is obtain that $b$ is constant. Hence from (5.2), we can verify

$$\bar{S}(X, Y) = -[(bf + \lambda)g(X, Y) + b(f - 1)\eta(X)\eta(Y)$$  (5.8)

which results that $M$ is $\eta$-Einstein manifold. This gives a following theorem:

**Theorem 2.** If in a 3- dimensional $f$-Kenmotsu manifold with quarter-symmetric non-metric connection, the metric $g$ is a Ricci solitons and $V$ is a pointwise collinear with $\xi$, then $V$ is a constant multiple of $\xi$ and $M$ is $\eta$-Einstein manifold of the form (5.8) and Ricci Solitons is expanding or shrinking according as $\lambda = 2f^2 + f' + f - 2f\phi + b$ is positive or negative.

**References**


