Advances and Applications in Mathematical Sciences
Volume 20, Issue 10, August 2021, Pages 2447-2457

DIRICHLET BOUNDARY VALUE PROBLEM ON THE QUARTER PLANE

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Abstract

We have written explicit representation for Cauchy Pompeiu formula on the Quarter plane and making use of this formula, solution of higher order Dirichlet boundary value problem on the Quarter plane is explicitly written.

1. Introduction

Solution of inhomogeneous polyanalytic equation with Dirichlet boundary conditions on the Quarter plane cannot be obtained using iteration technique due to involvement of unbounded and divergent integrals [14, 15, 16] as it is done in case of bounded domains [3, 4, 7, 11, 12, 13]. Therefore, we will first write expression for Cauchy theorem and Cauchy-Pompeui formula for higher orders on the Quarter plane and then we will write the solution of Dirichlet boundary value problem for polyanalytic functions on the quarter plane. In case of Upper half plane these problems are also studied [8, 9, 10]. In case of first order partial differential equations, Gauss theorem and the Cauchy-Pompeiu formulas are used to find fundamental solutions of these equations [1, 2, 5, 6].

2010 Mathematics Subject Classification: 32A30, 30G20, 31A30, 35J55, 30E25, 35J55.
Keywords: Dirichlet boundary condition, Boundary value problems, Cauchy-Pompeiu representation, Gauss theorem.
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Received June 1, 2021; Accepted July 22, 2021

2. Dirichlet Boundary Value Problem

The Dirichlet problem is one of the fundamental problems in potential theory. It is related to the Riemann jump problem.

Theorem 2.1. *The Dirichlet problem for analytic functions:* $w_{\bar{z}} = 0$ *in* $\mathbb{Q}_1$, *with boundary conditions* $w = \gamma_1$ *for* $0 \le x,\ y = 0,\ w = \gamma_2$ *for* $0 \le y,\ x = 0$ *for* $\gamma_1, \beta_1 \in C(\mathbb{R}; \mathbb{C})$ *such that* $(1+s)^{\delta}\gamma_1(s), (1+s)^{\delta}\beta_1(s)$ *are bounded for some* $0 < \delta$ *and satisfying the compatibility condition* $\gamma_1(0) = \beta_1(0)$ *is uniquely weakly solvable in the class* $C^1(\mathbb{Q}_1; \mathbb{C}) \cap C(\overline{\mathbb{Q}}_1; \mathbb{C})$ *if and only if*

$$\frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-\bar{z}} - \frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+i\bar{z}} = 0, \tag{2.1}$$

$$\frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-\bar{z}} - \frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+i\bar{z}} = 0, \tag{2.2}$$

$$\frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-z} - \frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+iz} = 0. \tag{2.3}$$

The solution then is given as

$$w(z) = \frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-z} - \frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+iz}. \tag{2.4}$$

In case of the inhomogeneous Dirichlet boundary value problem the solution is given as follows:

Theorem 2.2. *The Dirichlet problem for inhomogeneous Cauchy Riemann equation:* $w_{\bar{z}} = f$ *in* $\mathbb{Q}_1$, *with boundary conditions* $w = \gamma_1$ *for* $0 \le x,\ \ y = 0$, $w = \beta_1$ *for* $0 \le y,\ \ x = 0$ *for* $f \in L_{p,2}(\mathbb{Q}_1; \mathbb{C}),\ 2 < p,\ \gamma_1,\ \beta_1 \in C(\mathbb{R}; \mathbb{C})$ *such that* $(1+s)^{\delta}\gamma_1(s), (1+s)^{\delta}\beta_1(s)$ *are bounded for some* $0 < \delta$ *and satisfying the compatibility condition* $\gamma_1(0) = \beta_1(0)$ *is uniquely weakly solvable in the class* $C^1(\mathbb{Q}_1; \mathbb{C}) \cap C(\overline{\mathbb{Q}}_1; \mathbb{C})$ *if and only if*

$$\frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-\bar{z}}-\frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+i\bar{z}}-\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)\frac{d\xi d\eta}{\varsigma-\bar{z}}=0, \tag{2.5}$$

$$\frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-\bar{z}}-\frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+i\bar{z}}-\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)\frac{d\xi d\eta}{\varsigma-\bar{z}}=0, \tag{2.6}$$

$$\frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-z}-\frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+iz}-\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)\frac{d\xi d\eta}{\varsigma-z}=0. \tag{2.7}$$

The solution then is given as

$$w(z)=\frac{1}{2\pi i}\int_0^{+\infty}\gamma_1(s)\frac{ds}{s-z}-\frac{1}{2\pi i}\int_0^{+\infty}\beta_1(s)\frac{ds}{s+iz}-\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)\frac{d\xi d\eta}{\varsigma+z}. \tag{2.8}$$

Proof. If this Dirichlet problem has a solution in the class $w\in C^1(\mathbb{Q}_1;\mathbb{C})\cap C(\overline{\mathbb{Q}}_1;\mathbb{C})$ satisfying $W_{\bar{z}}\in L_{p,2}(\mathbb{Q}_1;\mathbb{C}), 2\le p,$ then it is of the form (2.8) and can be verified to be a solution. From the assumption on γ_1, β_1 and f it follows by direct computation and $(1+R)^{\delta}M(R, w)$ on as is bounded on $\mathbb{R}^+$ with. Moreover, w obviously satisfies the differential equation. For checking the boundary conditions the solvability conditions (2.5) to (2.7) are needed. Adding (2.6), (2.8) and subtracting (2.5), (2.7) leads to

$$w(z)=\frac{2}{\pi}\int_0^{+\infty}\gamma_1(s)\frac{y}{|s-z|^2}\frac{s^2+|z|^2}{|s+z|^2}ds+\frac{2i}{\pi}\int_0^{+\infty}\beta_1(s)\frac{y}{|s-iz|^2}\frac{s^2-|z|^2}{|s-iz|^2}ds$$
$$-\frac{2}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)\left(\frac{z}{\varsigma^2-z^2}-\frac{\bar{z}}{\varsigma^2-\bar{z}^2}\right)d\xi d\eta \tag{2.9}$$

adding (2.7), (2.8) and subtracting (2.5), (2.6) leads to

$$w(z)=\frac{2}{\pi i}\int_0^{+\infty}\gamma_1(s)\frac{x}{|s-z|^2}\frac{s^2-|z|^2}{|s+z|^2}ds+\frac{2}{\pi i}\int_0^{+\infty}\beta_1(s)\frac{y}{|s-iz|^2}\frac{s^2+|z|^2}{|s-iz|^2}ds$$
$$-\frac{2}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)\left(\frac{z}{\varsigma^2-z^2}+\frac{\bar{z}}{\varsigma^2-\bar{z}^2}\right)d\xi d\eta \tag{2.10}$$

Observing that

$$\frac{1}{2\pi i}\frac{z-\bar{z}}{|s-z|^2}=\frac{1}{\pi}\frac{y}{|s-z|^2} \quad z=x+iy,\ x,\ s\in\mathbb{R},\ 0<y$$

is the Poisson kernel for the upper half plane [8] for $s_0 \in \mathbb{R}^+$

$$\lim_{z\to s_0} w(z)=\lim_{z\to s_0}\frac{2}{\pi}\int_0^{+\infty}\gamma_1(s)\frac{y}{|s-z|^2}\frac{s^2+|z|^2}{|s+z|^2}\,ds=\gamma_1(s_0)$$

and from (2.10) likewise for $s_0 \in \mathbb{R}^+$

$$\lim_{z\to s_0} w(z)=\lim_{z\to s_0}\frac{2}{\pi i}\int_0^{+\infty}\beta_1(s)\frac{y}{|s-iz|^2}\frac{s^2+|z|^2}{|s+iz|^2}\,ds=\beta_1(s_0). \quad \square$$

Before deducing the solution of Dirichlet problem for the higher order we will write a representation for general Cauchy Pompeiu formula on $\mathbb{Q}_1$.

Theorem 2.3. *Let* F_k *be the space of functions in* $W^{k,1}(\mathbb{Q}_1, \mathbb{C})$ *for which* $\lim\limits_{R\leftarrow\infty} R^{v} M(\partial_{\bar{z}}^{v} w, R) = 0,\ 0 \le v \le k-1$ *where* $M(\partial_{\bar{z}}^{v} w, R) = \max\limits_{\substack{|z|=R \\ 0\le I_m Z}} |\partial_{\bar{z}}^{v} w(z)|$ *and* $\bar{z}^{k-2}\partial_{\bar{z}}^{k} w \in L^1(\mathbb{Q}_1, \mathbb{C})$. *Then every* $w \in F_k$ *is representable as*

$$w(z)=\sum_{v=0}^{k-1}\frac{1}{2\pi i}\int_0^{+\infty}\frac{1}{v!}\frac{(\overline{z-s})^{v}}{(s-z)}\partial_{\bar{\varsigma}}^{v}w(s)\,ds-\sum_{v=0}^{k-1}\frac{1}{2\pi i}\int_0^{+\infty}\frac{1}{v!}\frac{(\bar{z}+is)^{v}}{(s+iz)}\partial_{\bar{\varsigma}}^{v}w(s)\,ds$$

$$-\frac{1}{\pi}\int_{\mathbb{Q}_1}\frac{1}{(k-1)!}\frac{(\overline{z-\varsigma})^{k-1}}{(\varsigma-z)}\partial_{\bar{\varsigma}}^{k}w(\varsigma)\,d\xi d\eta, \quad (2.11)$$

for $z \in \mathbb{Q}_1$.

Proof. If $k=1$, then

$$w(z)=\frac{1}{2\pi i}\int_0^{+\infty}\frac{w(s)}{s-z}\,ds-\frac{1}{2\pi i}\int_0^{+\infty}\frac{w(is)}{s+iz}\,ds-\frac{1}{\pi}\int_{\mathbb{Q}_1}\frac{\partial_{\bar{\varsigma}}w(\xi)}{\varsigma-z}\,d\xi d\eta$$

which is Cauchy-Pompeiu representation on $\mathbb{Q}_1$.

Let again $\mathbb{Q}_{1R}=\{z=x+iy : |z|<R,\ 0<x,\ 0<y\}=\mathbb{Q}_{1R}\cap\{|z|<R\}$.

Then assuming for $z \in \mathbb{Q}_{1R}$

$$w(z) = \sum_{v=0}^{k-1} \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \frac{1}{v!} \frac{(\overline{z-\varsigma})^{v}}{(\varsigma - z)} \partial_{\bar{\varsigma}}^{v} w(\varsigma)\, d\varsigma - \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{1}{(k-1)!} \frac{(\overline{z-\varsigma})^{k}}{(\varsigma - z)} \partial_{\bar{\varsigma}}^{k} w(\varsigma)\, d\xi d\eta$$

To hold, this formula will be proved for $(k+1)$ instead of k. An application of Cauchy Pompeiu formula for $\varsigma \in \mathbb{Q}_{1R}$

$$\partial_{\bar{\varsigma}}^{k} w(\varsigma) = \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \frac{1}{\tilde{\varsigma} - \varsigma} \partial_{\bar{\tilde{\varsigma}}}^{k} w(\tilde{\varsigma})\, d\tilde{\varsigma} - \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{1}{\tilde{\varsigma} - \varsigma} \partial_{\bar{\tilde{\varsigma}}}^{k+1} w(\tilde{\varsigma}) d\xi d\eta \tag{2.12}$$

Inserting this in the preceding formula (2.12) shows that

$$w(z) = \sum_{v=0}^{k-1} \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \frac{1}{v!} \frac{(\overline{z-\varsigma})^{v}}{(\varsigma - z)} \partial_{\bar{\varsigma}}^{v} w(\varsigma)\, d\varsigma$$

$$- \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \partial_{\bar{\tilde{\varsigma}}}^{k} w(\tilde{\varsigma}) \Big(\frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{1}{(k-1)!} \frac{(\overline{z-\varsigma})^{k-1}}{(\varsigma - z)} \frac{1}{(\tilde{\varsigma} - \varsigma)} d\xi d\eta \Big)\, d\tilde{\varsigma}$$

$$- \frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \partial_{\bar{\varsigma}}^{k} w(\varsigma) \Big(\frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{1}{(k-1)!} \frac{(\overline{z-\varsigma})^{k}}{(\varsigma - z)} \frac{1}{(\tilde{\varsigma} - \varsigma)} d\xi d\eta \Big)\, d\tilde{\xi} d\tilde{\eta}. \tag{2.13}$$

Now, let

$$\frac{1}{\pi} \int_{\mathbb{Q}_{1R}} \frac{1}{(k-1)!} \frac{(\overline{z-\varsigma})^{k}}{(\varsigma - z)} \frac{1}{(\tilde{\varsigma} - \varsigma)} d\xi d\eta = \varphi(z, \tilde{\varsigma}). \tag{2.14}$$

On substituting (2.15) in (2.14), we have

$$w(z) = \sum_{v=0}^{k-1} \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \frac{1}{v!} \frac{(\overline{z-\varsigma})^{v}}{(\varsigma - z)} \partial_{\bar{\varsigma}}^{v} w(\varsigma)\, d\varsigma - \frac{1}{2\pi i} \int_{\partial \mathbb{Q}_{1R}} \partial_{\bar{\varsigma}}^{k} w(\varsigma)\, \varphi(z, \tilde{\varsigma})\, d\tilde{\varsigma}$$

$$+ \frac{1}{\pi} \int_{\partial \mathbb{Q}_{1R}} \partial_{\bar{\tilde{\varsigma}}}^{k+1} w(\tilde{\varsigma})\, \varphi(z, \tilde{\varsigma})\, d\tilde{\xi} d\tilde{\eta} \tag{2.15}$$

Now, using Cauchy Pompeiu formula for $\frac{1}{k!} \frac{(\overline{z-\tilde{\varsigma}})^{k}}{(\tilde{\varsigma} - z)}$, we have

$$\frac{1}{k!}\frac{(\overline{z-\tilde{\varsigma}})^k}{(\tilde{\varsigma}-z)}$$

$$= \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{1}{k!}\frac{(\overline{z-\tilde{\varsigma}})^{k-1}}{(\tilde{\varsigma}-z)}\frac{d\varsigma}{(\varsigma-\tilde{\varsigma})}+\frac{1}{\pi}\int_{\partial\mathbb{Q}_{1R}}\frac{1}{(k-1)!}\frac{(\overline{z-\varsigma})^{k-1}}{(\varsigma-z)}\frac{d\xi d\eta}{(\varsigma-\tilde{\varsigma})}$$

$$= \tilde{\psi}(z,\ \tilde{\varsigma})-\varphi(z,\ \tilde{\varsigma}), \tag{2.16}$$

where

$$\tilde{\psi}(z,\ \tilde{\varsigma}) = \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{1}{k!}\frac{(\overline{z-\varsigma})^{k-1}}{(\varsigma-z)}\frac{d\varsigma}{(\varsigma-\tilde{\varsigma})}$$

As obviously $\partial_{\bar{\tilde{\varsigma}}}\tilde{\psi}(z,\ \tilde{\varsigma}) = 0,$ then

$$\frac{1}{\pi}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}^{k}w(\tilde{\varsigma})\tilde{\psi}(z,\ \tilde{\varsigma})\,d\tilde{\xi}d\tilde{\eta} = \frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}\{\partial_{\bar{\tilde{\varsigma}}}^{k-1}w(\tilde{\varsigma})\tilde{\psi}(z,\ \tilde{\varsigma})\}\,d\tilde{\xi}d\tilde{\eta}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}^{k-1}w(\tilde{\varsigma})\tilde{\psi}(z,\ \tilde{\varsigma})\,d\tilde{\varsigma}$$

i.e.

$$\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}^{k-1}w(\tilde{\varsigma})\tilde{\psi}(z,\ \tilde{\varsigma})\}\,d\tilde{\varsigma}-\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}^{k}w(\tilde{\varsigma})\tilde{\psi}(z,\ \tilde{\varsigma})\,d\tilde{\xi}d\tilde{\eta} = 0. \tag{2.17}$$

Now substituting the values from (2.16), (2.17) and using (2.18), we have

$$w(z) = \sum_{v=0}^{k-1}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{1}{v!}\frac{(z-\varsigma)^v}{(\varsigma-z)}\partial_{\bar{\varsigma}}^{v}w(\varsigma)\,d\varsigma$$

$$-\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}^{k}w(\tilde{\varsigma})\{\tilde{\psi}(z,\ \tilde{\varsigma})-\frac{1}{k!}\frac{(\overline{z-\tilde{\varsigma}})^v}{(\tilde{\varsigma}-z)}\}\,d\tilde{\varsigma}$$

$$+\frac{1}{\pi}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}^{k+1}w(\tilde{\varsigma})\{\tilde{\psi}(z,\ \tilde{\varsigma})-\frac{1}{k!}\frac{(\overline{z-\tilde{\varsigma}})^v}{(\tilde{\varsigma}-z)}\}\,d\tilde{\xi}d\tilde{\eta}$$

$$= \sum_{v=0}^{k-1}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{1}{v!}\frac{(z-\varsigma)^v}{(\varsigma-z)}\partial_{\bar{\varsigma}}^{v}w(\varsigma)\,d\varsigma-\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\tilde{\varsigma}}}^{k}w(\tilde{\varsigma})\tilde{\psi}(z,\ \tilde{\varsigma})\,d\tilde{\varsigma}$$

$$+\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\partial_{\tilde{\bar{\varsigma}}}^{k+1}w(\tilde{\varsigma})\,\tilde{\psi}(z,\ \tilde{\varsigma})\,d\tilde{\xi}d\tilde{\eta}+\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{\partial_{\tilde{\bar{\varsigma}}}^{k}w(\tilde{\varsigma})}{k!}\frac{(\overline{z-\tilde{\varsigma}})^{k}}{(\tilde{\varsigma}-z)}d\tilde{\varsigma}$$

$$-\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\frac{\partial_{\tilde{\bar{\varsigma}}}^{k+1}w(\tilde{\varsigma})}{k!}\frac{(\overline{z-\tilde{\varsigma}})^{k}}{(\tilde{\varsigma}-z)}d\tilde{\xi}d\tilde{\eta}$$

$$w(z)=\sum_{v=0}^{k}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{1}{\nu i}\frac{(\overline{z-\varsigma})^{v}}{(\varsigma-z)}\,\partial_{\bar{\varsigma}}^{v}w(\varsigma)\,d\varsigma-\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\frac{\partial_{\tilde{\bar{\varsigma}}}^{k}w(\tilde{\varsigma})}{k!}\frac{(\overline{z-\varsigma})^{k}}{(\varsigma-z)}\,d\tilde{\xi}d\tilde{\eta}$$

which is (2.12) for $k+1$.

From the estimate

$$\left|\frac{1}{2\pi i}\int_{\substack{|\varsigma|=R\\0<x\\0<y}}\frac{1}{v!}\frac{(\overline{z-\varsigma})^{v}}{(\varsigma-z)}\,\partial_{\bar{\varsigma}}^{v}w(\varsigma)d\varsigma\right|\le\frac{1}{2\pi}\frac{1}{v!}\int_{|\varsigma|=R}\frac{|(\overline{z-\varsigma})^{v}|}{|\varsigma-z|}|\,\partial_{\bar{\varsigma}}^{v}w(\varsigma)\,||\,d\varsigma\,|$$

$$=\frac{1}{2\pi}\frac{1}{v!}\int_{|\varsigma|=R}|\,(\overline{z-\varsigma})^{v-1}\,||\,\partial_{\bar{\varsigma}}^{v}w(\varsigma)\,||\,d\varsigma\,|$$

$$\le\frac{1}{4\pi}\frac{1}{v!}(|\,z\,|+R)^{v-1}M(\partial_{\bar{\varsigma}}^{v}w,\ R)R=\frac{1}{4\pi}\frac{1}{v!}(2R)^{v-1}M(\partial_{\bar{\varsigma}}^{v}w,\ R)R$$

$$=\frac{2^{v-1}}{4}\frac{1}{v!}R^{v}M(\partial_{\bar{\varsigma}}^{v}w,\ R)\to 0\ \text{ as }\ R\to\infty.$$

Applying $R\to\infty,$ we have

$$\sum_{v=0}^{k}\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\frac{1}{v!}\frac{(\overline{z-\varsigma})^{v}}{(\varsigma-z)}\partial_{\bar{\varsigma}}^{v}w(\varsigma)\,d\varsigma$$

$$=\sum_{v=0}^{k}\left[\frac{1}{2\pi i}\int_{0}^{+\infty}\frac{1}{v!}\frac{(\overline{z-t})^{v}}{(t-z)}\partial_{\bar{\varsigma}}^{v}w(t)\,dt-\frac{1}{2\pi i}\int_{0}^{+\infty}\frac{1}{v!}\frac{(\bar{z}+it)^{v}}{(t+iz)}\,\partial_{\bar{\varsigma}}^{v}w(it)\,dt\right]$$

which exists by the respective assumption (2.11) follows from (2.12). □

We now investigate Dirichlet boundary conditions for the inhomogeneous polyanalytic equation:

Theorem 2.4. *Let w be as in theorem 2.3, then the Dirichlet problem for polyanalytic equation*

$$\partial_{\bar{z}}^{n} w = f \ \ in \ \mathbb{Q}_1. \ \partial_{\bar{z}}^{\lambda} w = \gamma_\lambda \ \ for \ 0 \le x, \ y = 0 \ \partial_{\bar{z}}^{\lambda} w = \beta_\lambda \ \ for \ 0 \le y, \ x = 0 \tag{2.18}$$

is uniquely weakly solvable and the solution is given by

$$w(z) = \frac{1}{2\pi i}\sum_{\lambda=0}^{k-1}\int_0^{+\infty}\frac{1}{\lambda!}\frac{(\overline{z-s})^{\lambda}}{(s-z)}\gamma_\lambda(s)\,ds - \frac{1}{2\pi i}\sum_{\lambda=0}^{k-1}\int_0^{+\infty}\frac{1}{\lambda!}\frac{(\bar{z}+is)^{\lambda}}{(s+iz)}\beta_\lambda(s)\,ds$$

$$-\frac{1}{\pi}\int_{\mathbb{Q}_1}\frac{1}{(n-1)!}\frac{(\overline{z-\varsigma})^{n-1}}{(\varsigma-z)}f(\varsigma)\,d\xi d\eta \tag{2.19}$$

If and only if for $0 \le v \le n-1$

$$\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_0^{+\infty}\gamma_\lambda(s)(s-z)^{\lambda-v}\frac{ds}{(s-\bar{z})}$$

$$-\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{1}{(\lambda-v)!}\int_0^{+\infty}\beta_\lambda(s)(s-z)^{\lambda-v}\frac{ds}{(s+i\bar{z})}$$

$$+\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)(\bar{\varsigma}-z)^{n-v-1}\frac{d\xi d\eta}{\varsigma-\bar{z}} = 0, \tag{2.20}$$

$$\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_0^{+\infty}\gamma_\lambda(s)(s-z)^{\lambda-v}\frac{ds}{(s+i\bar{z})}$$

$$-\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_0^{+\infty}\beta_\lambda(s)(z-is)^{\lambda-v}\frac{ds}{(s+i\bar{z})}$$

$$+\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)(\bar{\varsigma}-z)^{n-v-1}\frac{d\xi d\eta}{\varsigma-\bar{z}} = 0, \tag{2.21}$$

$$\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_0^{+\infty}\gamma_\lambda(s)(s-\bar{z})^{\lambda-v}\frac{ds}{(s+iz)}$$

$$-\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{1}{(\lambda-v)!}\int_0^{+\infty}\beta_\lambda(s)(is-\bar{z})^{\lambda-v}\frac{ds}{(s+iz)}$$

$$+\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}f(\varsigma)\left(\overline{\varsigma+z}\right)^{n-v-1}\frac{d\xi d\eta}{\varsigma-z}=0. \tag{2.22}$$

Proof. In order to show (2.20), we first note that $\gamma_\lambda(s)\in L^p(\mathbb{R})$, $0\le\lambda\le n-1$. The representation (2.20) is indeed a solution. This follows from (2.11). In order to show (2.21), we show the following for $0\le v\le n-1$

$$\frac{(-1)^{n-v}}{(n-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_1}\partial_{\bar{\varsigma}}^n w(\varsigma)(\bar{\varsigma}-z)^{n-v-1}\frac{d\xi d\eta}{\varsigma-\bar{z}}$$

$$=-\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_0^{+\infty}\partial_{\bar{\varsigma}}^v w(s)(s-z)^{\lambda-v}\frac{ds}{(s+\bar{z})}$$

$$+\sum_{\lambda=v}^{n-1}\frac{1}{2\pi i}\frac{1}{(\lambda-v)!}\int_0^{+\infty}\partial_{\bar{\varsigma}}^v w(is)(is-z)^{\lambda-v}\frac{ds}{(s+i\bar{z})}. \tag{2.23}$$

This will be proved using induction on n. For $n=1$, (2.23) is essentially the Gauss theorem [3] which holds since $\lim\limits_{R\to\infty}M(w,R)=0$.

Let $\mathbb{Q}_{1R}=\mathbb{Q}_1\cap\{z:|z|<R\}$. Then assuming for $z\in\mathbb{Q}_{1R}$, $m\le n-1$,

$$\frac{(-1)^{m-v}}{(m-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\partial_{\bar{\varsigma}}^m w(\varsigma)(\bar{\varsigma}-z)^{m-v-1}\frac{d\xi d\eta}{\varsigma-\bar{z}}$$

$$=-\sum_{\lambda=v}^{m-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(\lambda-v)!}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\varsigma}}^v w(\varsigma)(\bar{\varsigma}-z)^{\lambda-v}\frac{ds}{(\varsigma+\bar{z})} \tag{2.24}$$

to hold, this formula will be proved for $(m+1)$ instead of m. Applying Gauss theorem for regular domains [2] and substituting the value from (2.24), we obtain

$$\frac{(-1)^{m+1-v}}{(m-v)!}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\partial_{\bar{\varsigma}}^{m+1}w(\varsigma)(\bar{\varsigma}-z)^{m-v}\frac{d\xi d\eta}{\varsigma-\bar{z}}$$

$$= (-1)^{m+1-v}\left[\frac{1}{(m-v)!}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\frac{\partial}{\partial\bar{\varsigma}}\left(\frac{\partial_{\bar{\varsigma}}^{m} w(\varsigma)(\bar{\varsigma}-z)^{m-v}}{\varsigma-\bar{z}}\right)d\xi d\eta\right.$$

$$\left.-\frac{1}{(m-v-1)!}\frac{1}{\pi}\int_{\mathbb{Q}_{1R}}\partial_{\bar{\varsigma}}^{w} w(\varsigma)\frac{(\bar{\varsigma}-z)^{m-v-1}}{(\varsigma-\bar{z})}d\xi d\eta\right]$$

$$= \frac{(-1)^{m+1-v}}{(m-v)!}\left[\frac{1}{2\pi i}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\varsigma}}^{m} w(\varsigma)\frac{(\bar{\varsigma}-z)^{m-v}}{(\varsigma-\bar{z})}d\varsigma\right.$$

$$\left.+\sum_{\lambda=v}^{m-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-v}}{(m-v)!}\int_{\partial\mathbb{Q}_{1R}}\partial_{\bar{\varsigma}}^{\lambda} w(\varsigma)(\bar{\varsigma}-z)^{\lambda-v}\frac{d\varsigma}{\varsigma-\bar{z}}\right]$$

which is (2.24) for $(m+1)$.

Moreover

$$\left|\frac{1}{2\pi i}\int_{|\varsigma|=R,\ 0<\operatorname{Im}\varsigma}\partial_{\bar{\varsigma}}^{\lambda} w(\varsigma)(\bar{\varsigma}-z)^{\lambda-v}\frac{d\varsigma}{(\varsigma-\bar{z})}\right| \le (R+|z|)^{\lambda-v-1} M(R,\partial_{\bar{z}}^{\lambda} w)R$$

which tends to zero as R tends to ∞. Thus (2.23) follows from (2.24). On the similar way (2.21), (2.22) can be proved using Gauss theorem as above. For checking the boundary conditions the solvability conditions (2.21) to (2.23) are needed and may be verified on the similar pattern as in theorem 2.2. So we leave the calculations here.

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