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PROPERTIES OF SIMPLICIAL VERTICES IN SOME FUZZY GRAPHS

K. RADHA[1] and P. INDUMATHI[2]

[1]Associate Professor, [2]Research Scholar
PG and Research Department of Mathematics
Periyar E.V.R College (Autonomous and
Affiliated to Bharathidasan University)
Trichy-23, India
E-mail: radhagac@yahoo.com
indujune23@gmail.com

Abstract

In this paper, simplicial vertices in the direct sum of two fuzzy graphs are discussed. Some of their properties are studied. The conditions for a vertex to be simplicial in the complement of a fuzzy graph are obtained. Also the conditions for a vertex to be simplicial in the fuzzy graphs whose underlying graphs are complete bipartite graph, path, star, cycle and tree are discussed.

1. Introduction

Azriel Rosenfeld [9] introduced Fuzzy graph theory in 1975. He has developed fuzzy analogue of many graph theoretic concepts. Since then it has been growing fast and has numerous applications in various fields. M. S. Sunitha and A. Vijayakumar [10] discussed about complement of these operations. The degree of a vertex in fuzzy graphs which are obtained from two fuzzy graphs using these operations were discussed by A. Nagoor Gani and K. Radha [6]. K. Radha and S. Arumugam [4] discussed about the On Direct sum of two fuzzy graphs. K. Radha and P. Indumathi [3] discussed about properties of simplicial vertices in some graphs. In this paper, simplicial vertices in the direct sum of two fuzzy graphs are discussed. Some of their properties are studied. The conditions for a vertex to be simplicial in

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the complement of a fuzzy graph are obtained. Also the conditions for a vertex to be simplicial in the fuzzy graphs whose underlying graphs are complete bipartite graph, path, star, cycle and tree are discussed.

2. Basic Concepts

First let us recall some preliminary definitions that can be found in [1] to [13].

A complete graph is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on n vertices is denoted by K_n. A clique of a graph is a complete subgraph. A vertex is simplicial if its neighborhood is a clique. A simplicial vertex of a graph G is a vertex v such that the neighbours of v form a clique in G. A set of pairwise non-adjacent vertices is called an independent set. A graph G is said to be bipartite if the vertex set can be partitioned into two disjoint subsets V_1 and V_2 such that for every edge $e_k = v_i v_j \in E(G)$, $v_i \in V_1$ and $v_j \in V_2$. A complete bipartite graph is a bipartite graph in which all the vertices in V_1 are adjacent with all the vertices of V_2. If $|V_1| = m$ and $|V_2| = n$, then the corresponding complete bipartite graph is denoted as $K_{m,n}$. A complete bipartite graph $K_{1,n}$ is known as star graph. Here the vertex of degree n is called the apex vertex. Bistar is the graph obtained by joining the apex vertices of two copies of star $K_{1,n}$ by an edge.

Definition 2.1 [5]**.** Let V be a non-empty finite set and $E \subseteq V \times V$. A fuzzy graph $G : (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : E \to [0, 1]$ such that $\mu(x, y) \leq \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$. Underlying crisp graph is denoted by $G^* : (V, E)$.

Definition 2.2 [5]**.** A fuzzy graph $H : (P, \sigma', \mu')$ is called fuzzy subgraph of $G : (V, \sigma, \mu)$ if $\sigma'(u) \leq \sigma(u), \forall u \in P$ and $\mu'(uv) \leq \mu''(uv), \forall u, v \in P$. (σ', μ') is a spanning fuzzy subgraph of (σ, μ) if $\sigma = \sigma'$ and $\mu' \subseteq \mu$, that is, if $\sigma(u) = \sigma'(u)$ for every $u \in V$ and $\mu'(e) \leq \mu(e)$ for every $e \in E$.

(σ', μ') is an induced fuzzy subgraph of (σ, μ) if $\sigma = \sigma'$ and $\mu' = \mu$, P, that is, if $\sigma(u) = \sigma'(u)$ for every $u \in P$ and $\mu'(uv) = \mu(uv)$ for every $u, v \in P$.

Definition 2.3 [5]**.** An edge uv is effective if $\mu(xy) = \sigma(x) \wedge \sigma(y)$. G is an effective fuzzy graph if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $xy \in E$. G is a complete fuzzy graph if $\mu(xy) = \sigma(x) \wedge \sigma(y)$ for all $x, y \in V$.

Definition 2.4 [4]**.** If $\mu(xy) > 0$, then x and y are called neighbours, x and y are said to lie on the edge $e = xy$. $N_G(u)$ denotes the set of all vertices adjacent to u. $N_G[u]$ denotes the set of all vertices adjacent to u including u. $G[N_G[u]]$ denotes the subgraph induced by $N_G[u]$. We use $N_G[u]$ to denote the subgraph induced by $N_G[u]$.

Definition 2.5 [10]**.** The complement of a fuzzy graph $G : (\sigma, \mu)$ is graph $G^C = (\sigma^C, \mu^C)$ where $\sigma^C = \sigma$ and $\mu^C(uv) = 0$ and $\mu^C(uv) = \sigma(u) \wedge \sigma(V)$ otherwise.

Definition 2.6 [7]**.** Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ respectively. Let $V = V_1 \cup V_2$ and Let $E = \{uv/u, v \in V; uv \in E_1$ or $uv \in E_2$ but not both$\}$. The direct sum of G_1 and G_2 is a fuzzy graph $G_1 \oplus G_2 = G : (\sigma, \mu)$ given by

$$\sigma(u) = \begin{cases} \sigma_1(u), & \text{if } u \in V_1 \\ \sigma_2(u), & \text{if } u \in V_2 \\ \sigma_1(u) \vee \sigma_2(u), & \text{if } u \in V_1 \cup V_2 \end{cases}$$

and

$$\mu(uv) = \begin{cases} \mu_1(uv), \text{ if } uv \in E_1 \\ \\ \mu_2(uv), \text{ if } uv \in E_2. \end{cases}$$

3. Simplicial Vertex in Fuzzy Graphs

Definition 3.1. A vertex u in a fuzzy graph G is simplicial in G if $N_G[u]$ is a complete fuzzy graph. In other words, u is simplicial in G if u is simplicial in G^* and each edge of $N[u]$ is an effective edge.

Example 3.2. The vertices x, v and w in G_1 of Figure 4.1 are simplicial in G_1. The vertex u is not simplicial in G_1.

Theorem 3.3. *Let* $G : (\sigma, \mu)$ *be an effective fuzzy graph with underlying crisp graph* $G^* : (V, E)$*. Then a vertex u is simplicial in G if and only if it is simplicial in* G^*.

Proof. If u is a simplicial vertex of G, then by the definition 3.1, u is simplicial in G^*.

Conversely, let u be a simplicial vertex in G^*. Then $N[u]$ is complete in G^*. Since G is an effective fuzzy graph, $N[u]$ is a complete fuzzy graph. Hence u is simplicial in G.

Theorem 3.4. *A connected graph* $G^* : (V, E)$ *is complete if and only if each vertex of* G^* *is simplicial.*

Proof. If G^* is complete, then for any vertex u of G^*, $N[u]$ is V and $G^*[N[u]]$ is G^* itself which is complete. Hence each vertex of G^* is simplicial.

Conversely assume that each vertex of G^* is simplicial. We have to prove that there is an edge between any two vertices of G^*. Let u and v be any two vertices of G^*. If there is an edge between u and v, there is nothing to prove. Otherwise there exists a path $u = v_0 v_1 v_2 \dots v_n = v$ between u and v of length greater than one. In G^* the neighbours of v_1 contains v_0 and v_2. Since v_1 is simplicial, there must be an edge between v_0 and v_2. Now the neighbours of v_2 contains v_0, v_1 and v_3. Since v_2 is simplicial, there must be an edge between v_0 and v_3. Therefore the neighbours of v_3 contain v_0, v_2 and v_4. Since v_3 is simplicial, there must be an edge between v_0 and v_4. Proceeding in this way, we obtain an edge between v_0 and v_n, that is, between u and v. Hence G^* is complete.

Theorem 3.5. *A connected fuzzy graph G is complete if and only if each vertex is simplicial in G.*

Proof. Assume that G is a complete fuzzy graph. By Theorem 3.4, each vertex of G is a simplicial vertex of G^*. Since G is a complete fuzzy graph, all its edges are effective. Therefore each vertex is simplicial in G.

Conversely assume that each vertex of G is simplicial in G. Then each vertex is simplicial in G^*. Therefore by theorem 3.4, G^* is a complete. Let u be any vertex of G. Then u is simplicial in G and hence all the edges of $N[u]$ are effective. Since G^* is complete, $N[u]^*$ is G^* itself. Therefore all the edges of G are effective. Hence G is a complete fuzzy graph.

4. Simplicial Vertex in the Direct Sum of Two Fuzzy Graphs

In this section, properties of simplicial vertices in the direct sum of two fuzzy graphs are discussed. If a vertex is simplicial in G_1 or in G_2, then it need not be simplicial in $G_1 \oplus G_2$. This can be seen from the example given in figure 4.1. If a vertex is simplicial in $G_1 \oplus G_2$, then it need not be simplicial in G_1 or in G_2. This can be seen from the example given in figure 4.1. Here we obtain some necessary as well as sufficient conditions for a vertex to be simplicial in $G_1 \oplus G_2$.

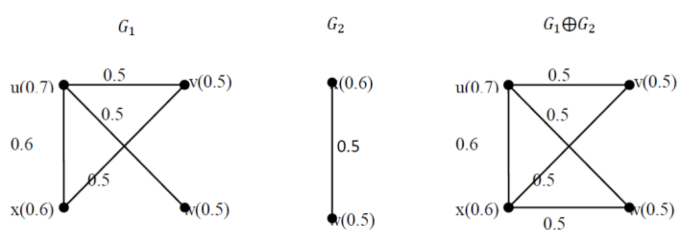

Figure 4.1.

Example 4.1. Even though a vertex is simplicial in both G_1 and G_2, it need not be simplicial in $G_1 \oplus G_2$. For example, consider the following figure 4.1. The vertex x is simplicial in both G_1 and G_2. But it is not simplicial in $G_1 \oplus G_2$.

Theorem 4.2. *Let* $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ *be two fuzzy graphs with underlying crisp graphs* $G_1^* : (V_1, E_1)$ *and* $G_2^* : (V_2, E_2)$ *such that* $V_1 \cap V_2 = \phi$. *Then* $u \in V_i$ *is simplicial in* G_i, $i = 1, 2$ *if and only if u is simplicial in* $G_1 \oplus G_2$. *In this case, the number of simplicial vertices in* $G_1 \oplus G_2$ *is the sum of the number of simplicial vertices in* G_1 *and the number of simplicial vertices in* G_2.

Proof. Let u be a simplicial vertex in G_1.

Since $V_1 \cap V_2 = \phi$, $E_1 \cap E_2 = \phi$.

Therefore all the edges of G_1 and G_2 appear in $G_1 \oplus G_2$. Also these are the only of $G_1 \oplus G_2$. Hence $G_1 \oplus G_2$ is a disconnected fuzzy graph with G_1 and G_2 as its components.

Therefore, for $i = 1, 2,$ a vertex u is simplicial in G_i if and only if it is simplicial in $G_1 \oplus G_2$.

Hence the number of simplicial vertices in $G_1 \oplus G_2$ is the sum of the number of simplicial vertices in G_1 and the number of simplicial vertices in G_2.

Theorem 4.3. *Let* $G_1 : (V_1, \sigma_1, \mu_1)$ *and* $G_2 : (V_2, \sigma_2, \mu_2)$ *be two fuzzy graphs with underlying crisp graphs* $G_1^* : (V_1, E_1)$ *and* $G_2^* : (V_2, E_2)$. *Let* $u \in V_1 - V_2$ *be a vertex in* G_1 *such that no edge of* $N_{G_1}[u]$ *is in* E_2. *If u is a simplicial vertex of* G_1, *then u is a simplicial vertex of* $G_1 \oplus G_2$.

Proof. Let u be a simplicial vertex of G_1. Then $N_{G_1}[u]$ is complete fuzzy graph. Since no edge of $N_{G_1}[u]$ is in E_2, by the definition of direct sum, all the edges of $N_{G_1}[u]$ appear in $G_1 \oplus G_2$.

Since $u \notin V_2$, no other edge will be incident at u in $G_1 \oplus G_2$.

Therefore $N_{G_1 \oplus G_2}[u]$ is $N_{G_1}[u]$.

Since $N_{G_1}[u]$ is complete fuzzy graph, then $N_{G_1 \oplus G_2}[u]$ is also a complete fuzzy graph.

Therefore u is simplicial in $G_1 \oplus G_2$.

Remark 4.4. The converse of the above theorem 4.3 need not be true.

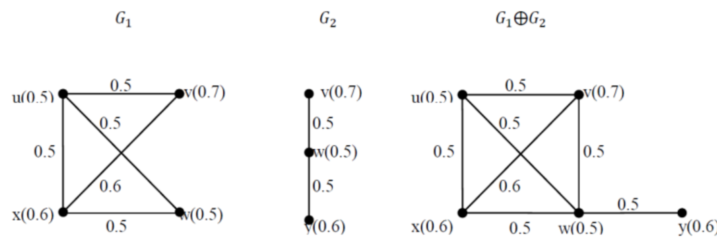

Figure 4.2.

Theorem 4.5. *Let* $G_1 : (V_1, \sigma_1, \mu_1)$ *and* $G_2 : (V_2, \sigma_2, \mu_2)$ *be two fuzzy graphs with underlying crisp graphs* $G_1^* : (V_1, E_1)$ *and* $G_2^* : (V_2, E_2)$. *Let* $u \in V_1 - V_2$ *be a vertex in* G_1 *such that no edge of* $N_{G_1}[u]$ *is in* E_2. *Then u is a simplicial vertex of* $G_1 \oplus G_2$ *if and only if the complement of* $N_{G_1}[u]$ *is a fuzzy subgraph of* G_2.

Proof. Let u be a simplicial vertex of $G_1 \oplus G_2$. Then $N_{G_1 \oplus G_2}[u]$ is complete. Since no edge of $N_{G_1}[u]$ is in E_2, all the edges of $N_{G_1}[u]$ are in $N_{G_1 \oplus G_2}[u]$.

By the definition of direct sum, the remaining edges of $N_{G_1 \oplus G_2}[u]$ must all be in E_2.

Also since $N_{G_1 \oplus G_2}[u]$ is complete, these remaining edges are the edges which are not in $N_{G_1}[u]$. Hence the complement of $N_{G_1}[u]^*$ is a subgraph of G_2^*.

Conversely, assume that the complement of $N_{G_1}[u]$ is a subgraph of G_2. Since these edges are in E_2 only, they will be in $G_1 \oplus G_2$. Also all the edges of $N_{G_1}[u]$ are in $G_1 \oplus G_2$. Since $u \notin V_2$, no other edge is incident at u in $G_1 \oplus G_2$. Therefore $N_{G_1 \oplus G_2}[u]$ is complete. Hence u is simplicial in $G_1 \oplus G_2$.

Corollary 4.6. *Let* $G_1 : (V_1, \sigma_1, \mu_1)$ *and* $G_2 : (V_2, \sigma_2, \mu_2)$ *be two fuzzy graphs with underlying crisp graphs* $G_1^* : (V_1, E_1)$ *and* $G_2^* : (V_2, E_2)$. *Such that* $E_1 \cap E_2 = \phi$. *Let* $u \in V_1 - V_2$. *If u is a simplicial vertex of* G_1, *then u is a simplicial vertex of* $G_1 \oplus G_2$.

Proof. Since $E_1 \cap E_2 = \phi$, no edge of $N_{G_1}[u]$ will be in E_2. Therefore by the above theorem 4.5, if u is a simplicial vertex of G_1 then u is a simplicial vertex of $G_1 \oplus G_2$.

5. Simplicial Vertex in the Complement of the Fuzzy Graphs

In this section, we obtain a necessary and sufficient condition for a vertex to be simplicial in the complement of a fuzzy graph.

Theorem 5.1. *Let* $G : (\sigma, \mu)$ *be a fuzzy graph on* $G^* : (V, E)$. *A vertex u is simplicial in* G^C *if and only if the vertices of* $N_G[u]$ *is an independent set.*

Proof. Suppose u is a simplicial vertex in G^C.

Then $N_{G^C}[u]$ is complete fuzzy graph.

Therefore there is an edge between any two vertices of $N_{G^C}[u]$ in G^C. By the definition of the complement of a fuzzy graph, there is no edge between any two vertices of $N_G[u]$ in G.

Therefore the vertices of $N_G[u]$ in G is an independent set.

Conversely assume that the vertices of $N_{G^C}[u]$ is an independent set in G.

Then there is no edge between any two vertices of $N_G[u]$ in G. Therefore any two vertices of $N_{G^C}[u]$ are adjacent in G^C. Hence u is simplicial in $(G^C)^*$.

Theorem 5.2. *If* $G : (\sigma, \mu)$ *be a fuzzy graph on a trivial graph* $G^* : (V, E)$, *then each vertex of* G^C *is simplicial in* G^C.

Proof. If G is a trivial fuzzy graph, then G^C is complete. Hence for each vertex u, $N_{G^C}[u]$ is G^C itself which is complete fuzzy graph. Therefore each vertex of G^C is simplicial.

Remark 5.3. The converse of the above theorem 5.2 need not be true. For example consider the fuzzy graph and its complement in the following figure 5.1. Here each vertex of G^C is simplicial but G is not a trivial fuzzy graph.

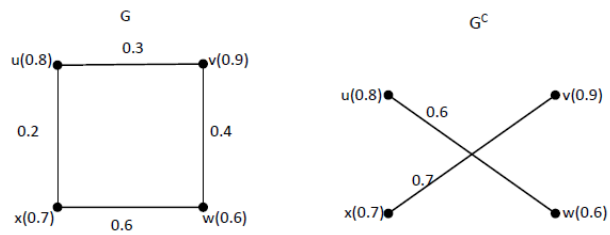

Figure 5.1.

Theorem 5.4. *Let* $G:(\sigma, \mu)$ *be a Fuzzy graph on* $G^*:(V, E)$*. Where* G^* *is* $K_{m,n}$*. Each vertex of* $K_{m,n}^C$ *is simplicial.*

Proof. Since $K_{m,n}$ is a complete bipartite graph, $K_{m,n}^C$ is the union of two disjoint complete graphs K_m and K_n. Also G^* is $K_{m,n}$. Since each vertex of a complete graph is simplicial, each vertex of $K_{m,n}^C$ is simplicial.

6. Simplicial Vertex in the Tree, Path, Star and Cycle Graphs

Theorem 6.1. *The end vertex* u *of any fuzzy graph* $G:(\sigma, \mu)$ *is a simplicial vertex if the edge incident at* u *is an effective edge.*

Proof. Let u be an end vertex of a graph G and let $e = uv$ be the edge incident at u. Then $N_G[u]$ is K_2 with vertices u and v. Hence u is a simplicial vertex in G^*. Therefore if uv is effective edge, then u is simplicial in G.

Theorem 6.2. *The simplicial vertices of an effective fuzzy graph, whose underlying graph is a tree, are its end vertices. Hence an effective fuzzy graph* *on a tree has at least two simplicial vertices.*

Proof. Let T be an effective fuzzy graph on a tree. Then the edge incident at an end vertex is an effective edge.

Therefore the above Theorem 6.1, all the end vertices of T are simplicial vertices. Let u be any vertex with $d(u) > 1$. Then u is adjacent to at least two vertices.

Let the vertices v and w be adjacent to u. If v and w are adjacent, then $uvwv$ is a cycle in T which is a contradiction, since T has no cycle.

Therefore v and w are not adjacent. Hence $N_T[u]$ is not complete. Therefore u is not simplicial in T^* and hence u is not simplicial in T. This shows that only the end vertices are the simplicial vertices of T.

Since a tree has at least two end vertices, T has at least two simplicial vertices.

Corollary 6.3. *An effective fuzzy graph on a path has exactly two simplicial vertices.*

Proof. Since a path is a tree with exactly two end vertices, an effective fuzzy graph on it has exactly two simplicial vertices.

Corollary 6.4. *An effective fuzzy graph on a star $K_{1,n}$ has exactly n simplicial vertices.*

Proof. Since a star $K_{1,n}$ is a tree with n end vertices, it has exactly n simplicial vertices.

Corollary 6.5. *An effective fuzzy graph on a bistar with $2n+2$ vertices has exactly 2n simplicial vertices.*

Proof. Since a bistar is a tree with $2n$ end vertices, it has exactly $2n$ simplicial vertices.

Theorem 6.6. *A fuzzy graph on a cycle C_n has simplicial vertices if and only if $n = 3$ and all its edges are effective edges.*

Proof. Let G be a fuzzy graph on a cycle C_n.

Suppose C_n has simplicial vertices.

Let v be a simplicial vertex of G.

Let $vv_2v_3 \dots v_nv$ be the cycle C_n on n vertices. Here v is adjacent to v_2 and v_n. Since v is simplicial, v_2 and v_n must be adjacent. This is possible only if $n = 3$.

Conversely suppose $n = 3$ and all its edges are effective edges.

Also, C_3 is a complete graph and $N_{C_3}[u]^*$ is C_3 for each vertex u of G. Hence each vertex of G is simplicial.

7. Conclusion

In this paper, simplicial vertices in the direct sum of two fuzzy graphs are discussed. Some of their properties are studied. The conditions for a vertex to be simplicial in the complement of a fuzzy graph are obtained. Also the conditions for a vertex to be simplicial in the fuzzy graphs whose underlying graphs are complete bipartite graph, path, star, cycle and tree are discussed.

References

[1] G. A. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg 25 (1961), 71-76.

[2] B. Dougals, West, Introduction to the Graph Theory, 2nd Ed., PHI Learning Private Limited, New Delhi (2011).

[3] A. J. J. Kloks, D. Kratsch and H. Muller, Listing Simplicial Vertices and recognizing diamond-free graphs, Computing Science Reports; Vol.9451, Eindhoven: Technische Universiteit Eindhoven. (1994),

[4] A. Nagoorgani and K. Radha, The degree of a vertex in some fuzzy graphs, International Journal of Algorithms, Computing and Mathematics 2(3) (2009), 107-

[5] K. Radha and S. Arumugam, On Direct sum of two fuzzy graphs, International Journal of Scientific and Research Publications, Volume 3, Issue 5, May 2013.

[6] K. Radha and P. Indumathi, Properties of simplicial vertices in some graphs, Our Heritage 68(4) (2020) 349-356.

[7] K. Radha and N. Kumaravel, The degree of an edge in union and join of two fuzzy graphs, International Journal of Fuzzy Mathematical Archive 4(1) (2014), 8-19.

[8] K. Radha and N. Kumaravel, The degree of an edge in Cartesian product and composition of two fuzzy graphs, Intern. J. Applied Mathematics and Statistical Sciences 2(2) (2013), 65-78.

[9] A. Rosenfeld, Fuzzy graphs, in: L. A. Zadeh, K. S. Fu, K. Tanaka and M. Shimura (editors), Fuzzy Sets and their Applications to Cognitive and Decision Process, Academic Press, New York, (1975), 77-95.

[10] M. S. Sunitha and A. Vijayakumar, Complement of a fuzzy graphs, Indian Journal of Pure and Applied Mathematics 33(9) (2002), 92-110.

[11] M. S. Sunitha and A. Sunil Mathew, Survey of fuzzy graph theory, Annals of Pure and Applied Mathematics 4(1) (2013), 92-110.

[12] R. T. Yeh and S. Y. Bang, Fuzzy relations, Fuzzy graphs and their applications to clustering analysis, in L. A. Zadeh, K. S. Fu, K. Tanaka and M. Shimura (editors), Fuzzy Sets and Their Applications to Cognitive and Decision Process, Academic Press, New York (1975), 125-149.

[13] L.A. Zadeh, Fuzzy sets, Inform. Control 8 (1965), 338-53.