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FIXED POINT THEOREMS FOR *Z*-CONTRACTION IN STRONG FUZZY METRIC SPACE USING FUZZY SIMULATION FUNCTION

A. NAGOOR GANI, P. MURUGANANTHAM and B. SHAFINA BANU

PG and Research Department of Mathematics
Jamal Mohamed College (Autonomous)
(Affiliated to Bharathidasan University)
Tiruchirappalli-620 020, Tamilnadu, India
E-mail: ganijmc@yahoo.co.in
shafinabanu@yahoo.co.in

Abstract

In this paper, we discuss some fixed point results for Z-contraction using fuzzy simulation function in strong fuzzy metric space.

1. Introduction

The theory of fuzzy set has been developed extensively by many authors in different fields such as control theory, engineering sciences neural networks, etc. In 1965, Zadeh [6] presented the theory of fuzzy sets. The concept of fuzzy metric space was introduced by Kramosil and Michalek. After that George and Veeramani gives the modified notion of fuzzy metric spaces. In 1988, Grabiec proved an analog of the Banach contraction theorem and fuzzy version of Cauchy sequence in fuzzy metric spaces. Farshid Khojasteh, Satish Shukla and Stojan Radenovi introduced the simulation function ζ and Z-contraction with respect to ζ which generalized the Banach contraction principle [1].

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2. Preliminaries

In this section some basic definitions are given.

Definition 2.1. An ordered triple $(X, M, *)$ is called a fuzzy metric space if it satisfies the following conditions: $\forall x, y, z \in X$ and $s, t > 0$

(GV1) $M(x, y, t) > 0, \forall t > 0$

(GV2) $M(x, y, t) = 1$ if and only if $x = y$ and $t > 0$

(GV3) $M(x, y, t) = M(y, x, t)$

(GV4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$

(GV5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous,

where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty) \to [0, 1]$.

Definition 2.2. Let $(X, M, *)$ be a fuzzy metric space. M is said to be Strong if it satisfies the following additional condition:

(GV4') $M(x, z, t) \geq M(x, y, t) * M(y, z, t) \forall x, y, z \in X$ and $t > 0$.

Definition 2.3. A fuzzy metric space in which every Cauchy sequence is convergent is said to be Complete.

Definition 2.4. The mapping $\zeta : [0, \infty) \times [0, \infty) \to R$ is called a Simulation Function if it satisfies the following conditions:

(ζ1) $\zeta(0, 0) = 0$;

(ζ2) $\zeta(t, s) < s - t$ for all $s, t > 0$;

(ζ3) If $\{t_n\}, \{s_n\} \subseteq (0, \infty)$ such that $\lim_{n\to\infty}\{t_n\} = \lim_{n\to\infty}\{s_n\} > 0$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of simulation functions by Z.

Definition 2.5. Let (X, d) be a metric space and $T : X \to X$ a mapping and $\zeta \in Z$. Then T is called a Z-contraction with respect to ζ if the following condition is satisfied: $\zeta(d(Tx, Ty), d(x, y)) \geq 0$, for all $x, y \in X$.

3. Main Results

Definition 3.1. A mapping $\zeta : [1, \infty) \times [1, \infty) \to [0, 1]$ is called a Fuzzy Simulation Function if it satisfies the following conditions:

(ζ1) $\zeta(t, s) \geq \frac{1}{t}, \frac{1}{s}$, for all $1 \leq t \leq s$;

((ζ2) If $\{t_n\}, \{s_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty}\{t_n\} = \lim_{n\to\infty}\{s_n\} > 1$ in $[1, \infty)$ then $\limsup\limits_{n\to\infty} \zeta(t_n, s_n) \leq 1$.

We denote the set of fuzzy simulation functions by Z.

Definition 3.2. Let $(X, M, *)$ be a strong fuzzy metric space and T be a self-map on $X = [0, 1]$ and $\zeta \in Z$. Then T is called a Z-contraction with respect to ζ in $(X, M, *)$ if the following condition is satisfied:

$\zeta(M(Tx, Ty, t), M(x, y, t)) \leq 1$, for all $x, y \in X$.

Examples 3.3.

(1) Let $X = [0, 1]$ and the self-map $Tx = 1 - x$, for all $x \in X$. Let $M(x, y, t) = \frac{t}{(|x - y| + t)}$ and $\zeta(s, t) = \frac{1}{t} - \lambda \frac{1}{s}$, where $\lambda \in [0, 1]$ and $t \leq s \in [1, \infty)$.

Solution:

Let $t = 99$, $s = 100$ and $\lambda = 0.7$. Then $\zeta(99, 100) = \frac{1}{99} - 0.7\frac{1}{100}$ $> \frac{1}{99} - \frac{1}{100}$.

Now Z-contraction with respect to ζ in $(X, M, *)$:

$$\zeta(M(Tx, Ty, Ty, t)), M(x, y, t) = \frac{1}{M(Tx, Ty, t)} - \lambda \frac{1}{M(x, y, t)}, \text{ for all } x, y \text{ in } X.$$

$$= \frac{1}{\left(\frac{t}{|(1-x)-(1-y)|+t}\right)} - \lambda \frac{1}{\left(\frac{t}{t+|x-y|}\right)}.$$

Let $x = 0.8,\ y = 0.9$ and $t = \min(x,\ y) = 0.8$.

Then $(M(Tx,\ Ty,\ t),\ M(x,\ y,\ t)) = \dfrac{1}{M(Tx,\ Ty,\ t)} - \lambda \dfrac{1}{M(x,\ y,\ t)} \leq 1$.

(2) Let $X = [0, 1]$ and the self-map $Tx = x/1 + x$, for all $x \in X$. Let $M(x, y, t) = \dfrac{t}{(|x - y| + t)}$ and $\zeta(s,\ t) = \dfrac{1}{t} - \dfrac{1}{s + 1}$, where $t \leq s \in [1, \infty)$.

(3) Let $X = [0, 1]$ and the self-map $Tx = x/1 + x$, for all $x \in X$. Let $M(x, y, t) = e^{-\left\{\frac{|x-y|}{t}\right\}}$ and $\zeta(s,\ t) = \dfrac{1}{t} - \dfrac{1}{s + 1}$, where $t \leq s \in [1, \infty)$.

(4) Let $X = [0, 1]$ and the self-map $Tx = 1 - x$, for all $x \in X$. Let $M(x,\ y,\ t) = \dfrac{t}{t + |\ x - y\ |}$ and $\zeta(s,\ t) = k\left(\dfrac{1}{t} - 1\right) - \left(\dfrac{1}{s} - 1\right)$, where $k \in [0, 1]$ and $t \leq s \in [1, \infty)$.

4. Fixed Point Theorem in Strong Fuzzy Metric Space using Fuzzy Simulation Function

Theorem 4.1. *Let T be a Z-contraction with respect to ζ in a complete strong fuzzy metric space $(X,\ M,\ *)$. Then there exists a unique fixed point of T in X.*

Proof.

Case 1. Let $x_0 \in X$ be any arbitrary point. By Picard's Sequence, the sequence $\{x_n\}$ such that $x_n = Tx_{n-1}$, for all $n \in N$.

Let $C_n = \sup\left\{\dfrac{1}{M(x_i,\ y_j,\ t)};\ i,\ j \geq n\right\}$.

$\Rightarrow$ $\{C_n\}$ is a monotonically decreasing sequence and $\limsup_{n\to\infty} C_n \leq 1$ and so $\{C_n\}$ is convergent then there exist $C \geq 1$ such that $\lim_{n\to\infty} C_n = C$.

Thus $\{x_n\}$ is a Cauchy Sequence in X.

Assume that $C > 1$.

Then there exist m_k, n_k such that $m_k \geq n_k > k$, for every $k \in N$ such that $C_k < \frac{1}{M(x_{m_k}, x_{n_k}, t)} < C_k + \frac{1}{k}$.

$$\Rightarrow \lim_{k \to \infty} \frac{1}{M(x_{m_k}, x_{n_k}, t)} = C > 1.$$

We know that $\frac{1}{M(x_{m_k}, x_{n_k}, t)} \leq \frac{1}{M(x_{m_k - 1}, x_{n_k - 1}, t)}$.

$$\Rightarrow \lim_{k \to \infty} \frac{1}{M(x_{m_k - 1}, x_{n_k - 1}, t)} = C > 1.$$

Therefore, $\lim_{k \to \infty} \left(\frac{1}{M(x_{m_k - 1}, x_{n_k - 1}, t)} - \frac{1}{M(x_{m_k}, x_{n_k}, t)} \right) = C > 1.$

$\Rightarrow \limsup_{k \to \infty} \zeta(M(x_{m_k - 1}, x_{n_k - 1}, t), M(x_{m_k}, x_{n_k}, t)) > 1,$ which is a contradiction to the condition (ii) of fuzzy simulation function definition and so if the only possible is $C = 1$.

Since $(X, M, *)$ is a strong fuzzy metric space, then there exist $x \in X$ such that $\lim_{n \to \infty} x_n = x$.

(i.e.) $\lim_{n \to \infty} \frac{1}{M(x_n, x, t)} = 1$, since $x_n \to x$ and $M(x, x, t) = 1$.

Suppose $Tx \neq x$ then $\frac{1}{M(Tx, x, t)} > 1$.

$\Rightarrow \limsup_{k \to \infty} \zeta(M(Tx_k, Tx, t), M(x_k, x, t)) > 1,$ which is a contradiction to the condition (ii) of fuzzy simulation function definition.

Therefore, $Tx = x$.

Hence x is a fixed point of T in X.

Let $v \in X$ be another fixed point of T and $v \neq x, Tv = v$. Then $\frac{1}{M(Tx, Tv, t)} > 1$.

$\Rightarrow \zeta(M(Tx, Tv, t), M(x, v, t)) > 1$, which is a contradiction to the Z-contraction of fuzzy simulation function.

Therefore, $v = x$.

Hence there exist a unique fixed point x of T in X.

Case 2:

Let $C_n = \sup\{\frac{1}{1 - M(x_i, y_j, t)}; i, j \geq n$ and $0 \leq M(x_i, y_j, t) < 1\}$.

$\Rightarrow \{C_n\}$ is a monotonically increasing sequence and $\limsup_{n\to\infty} C_n \leq 1$ and so $\{C_n\}$ is convergent then there exist $C \geq 1$ such that $\limsup_{n\to\infty} C_n = C$.

Thus $\{x_n\}$ is a Cauchy sequence in X.

Assume that $C > 1$.

Then there exist m_k, n_k such that $m_k \geq n_k > k$, for every $k \in N$ such that $C_k - \frac{1}{k} < \frac{1}{1 - M(x_{m_k}, x_{n_k}, t)} < C_k$.

$$\Rightarrow \lim_{n\to\infty} \frac{1}{1 - M(x_{m_k}, x_{n_k}, t)} = C > 1.$$

We know that $\frac{1}{1 - M(x_{m_k}, x_{n_k}, t)} \geq \frac{1}{M(x_{m_k}, x_{n_k}, t)} \geq \frac{1}{M(x_{m_k - 1}, x_{n_k - 1}, t)}$.

Therefore, $\limsup_{k\to\infty} \left(\frac{1}{M(x_{m_k - 1}, x_{n_k - 1}, t)}, \frac{1}{M(x_{m_k}, x_{n_k}, t)} \right) = C > 1$.

$\Rightarrow \limsup_{k\to\infty} \zeta(M(x_{m_k - 1}, x_{n_k - 1}, t), M(x_{m_k}, x_{n_k}, t)) > 1$, which is a contradiction to the condition (ii) of fuzzy simulation function definition and if the only possible is $C = 1$.

Since $(X, M, *)$ is a strong fuzzy metric space, then there exist $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

$$\Rightarrow \lim_{n\to\infty} \frac{1}{M(x_n, x, t)} = 1, \text{ since } x_n \to x \text{ and } M(x, x, t) = 1.$$

Suppose $Tx \neq x$ then $\frac{1}{M(Tx, x, t)} > 1$.

$\limsup_{k\to\infty} \zeta(M(Tx_k, Tx, t), M(Tx_k, x, t)) > 1$, which is a contradiction to the condition (ii) of fuzzy simulation function definition.

Therefore, $Tx = x$.

Hence x is a fixed point of T in X.

Let $v \in X$ be another fixed point of T and $v \neq x$, $Tv = v$. Then $\frac{1}{M(Tx, Tv, t)} > 1$.

$\Rightarrow \zeta(M(Tx, Tv, t), M(x, v, t)) > 1$, which is a contradiction to the Z-contraction of fuzzy simulation function.

Therefore, $v = x$.

Hence there exist a unique fixed point x of T in X.

Example 4.2.

(I) For Monotonically decreasing sequence,

(1) Let $X = [0, \infty)$, $\{x_n\} \to x$, where $x_n = \frac{1}{n^2}$ and $x = 0$, for all n in N.

Let $M(x_n, x, t) = \frac{t}{t + x_n}$, where $t = \frac{1}{n} \in [0, 1]$. Then $\lim_{n\to\infty}\left(\frac{1}{M(x_n, x, t)}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{\left(\frac{t}{t+x_n}\right)}\right) = 1, \text{ where } \lim_{k\to\infty} M\left(\frac{1}{n^2}, 0, \frac{1}{n}\right) = \lim_{n\to\infty}\left(\frac{\frac{1}{n}}{\frac{1}{n} + \frac{1}{n^2}}\right) = 1.$$

(2) Let $X = [0, \infty)$, $\{x_n\} \to x$, where $x_n = \frac{1}{n}$ and $x = 0$, for all n in N.

Let $M(x_n, x, t) = \frac{t}{t + x_n}$, where $t = \frac{1}{n} \in [0, 1]$.

Then $\lim_{n\to\infty}\left(\frac{1}{M(x_n, x, t)}\right) = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{t}{t + x_n}\right)}\right) = 1.$

(3) Let $X = Z$, $\{x_n\} \to x$, where $x_n = \frac{1}{n + x^2}$ and $x = 0$, for all n in N and x in X.

Let $M(x_n, x, t) = \frac{t}{t + x_n}$, where $t = \frac{1}{n} \in [0, 1]$.

Then $\lim_{n\to\infty}\left(\frac{1}{M(x_n, x, t)}\right) = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{t}{t + x_n}\right)}\right) = 1.$

(II) For Monotonically increasing sequence

(1) Let $X = [0, \infty)$, $\{x_n\} \to x$, where $x_n = 1 - \frac{1}{n}$ and $x = 1$, for all n in N.

Let $M(x_n, x, t) = \frac{t}{t + x_n}$, where $t = \frac{1}{n} \in [0, 1]$.

Then $\lim_{n\to\infty}\left(\frac{1}{1 - M(x_n, x, t)}\right) = \lim_{n\to\infty}\left(\frac{1}{1 - \left(\frac{t}{t + x_n}\right)}\right) = 1.$

(2) Let $X = [0, \infty)$, $\{x_n\} \to x$, where $x_n = 1 - \frac{1}{n^2}$ and $x = 1$, for all n in N.

Let $M(x_n, x, t) = \frac{t}{t + x_n}$, where $t = \frac{1}{n} \in [0, 1]$. Then $\lim_{n\to\infty}\left(\frac{1}{1 - M(x_n, x, t)}\right)$

$$= \lim_{n\to\infty}\left(\frac{1}{1 - \left(\frac{t}{t + x_n}\right)}\right) = 1.$$

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