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TOTAL GEODETIC GLOBAL DOMINATION NUMBER OF A GRAPH

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Abstract

In this paper we learn the new idea total geodetic global domination number of a graph. A set $R \subseteq V(G)$ is termed as a total geodetic global dominating set if R is both a total geodetic set and a global dominating set. The minimum cardinality among all total geodetic global dominating sets of G is called total geodetic global domination number and it is designated by $\bar{\gamma}_{gt}(G)$. For a connected graph G, if $\bar{\gamma}(G) = k$, and $g_t(G) = l$ then $\bar{\gamma}_{g_t}(G) = k + l - 2$ with $k, l \geqslant 2$ where k, l are two positive integers.

1. Introduction

Throughout this article we scrutinize a simple graph $G = (V, E)$. For fundamental graph theory expressions see [2], [3]. Here $\overline{G}$ is the complement of $\overline{G}$ with point set V and two points are adjacent in $\overline{G}$ if and only if they

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are not adjacent in G. The distance between two points u and v is the length of a shortest $u-v$ path in a connected graph G. A point u of G is known as a full point if u is adjacent to all other points of G. $N(x)=\{y \in V(G): xy \in E(G)\}$ is called the neighborhood of the point x in G. A point x is an extreme point of a graph G if $\langle N(x) \rangle$ is complete. The eccentricity $e(v)$ of a point v in G is the maximum distance from v and a point of G. The minimum eccentricity among the points of G is the radius, $rad\, G$ or $r(G)$ and the maximum eccentricity is the diameter, $diam\, G$ of G. A cut point of G is a point whose removal results a disconnected graph. A subset $B \subseteq V$ is a dominating set of G if each point of $V-B$ is adjacent with at least one point of B. The domination number, $\gamma(G)$ is the minimum cardinality out of all dominating sets of G. A dominating set B of G is a global dominating set of G if every point in $\overline{G}$ is adjacent with a point in B. The global domination number, $\bar{\gamma}(G)$ is the minimum cardinality out of all global dominating sets of G [6]. An $u-v$ path of length $d(u, v)$ is known as $u-v$ geodesic. A point x is said to lie on a $u-v$ geodesic Q if x is a point of Q including the points u and v. The corona product of two graphs $G \circ H$ is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th point of G to every point in the i-th copy of H. A geodetic set of G is a set $R \subseteq V(G)$ such that every point of G is contained in a geodesic joining some couple of points in R. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a geodetic basis. The geodetic number of a graph was introduced in [4]. A geodetic set $R \subseteq V(G)$ is a total geodetic set if the subgraph $G[R]$ induced by R has no isolated points. The total geodetic number $g_t(G)$ is the minimum cardinality out of all total geodetic sets of G and it was introduced by Abdollahzadeh Ahangar and Vladimir Samodivkin [1]. A set $R \subseteq V$ is a geodetic global dominating set if it is both a geodetic set and a global dominating set. The geodetic global domination number $\bar{\gamma}_g(G)$ is the minimum cardinality among all the geodetic global dominating sets of G [5]. In this paper we define and study total geodetic global domination number of a graph.

Theorem 1.1 [5]. *Each extreme point of a connected graph G belongs to every geodetic global dominating set of G.*

Theorem 1.2 [5]. *Every full point of a connected graph G belongs to every geodetic global dominating set of G.*

Theorem 1.3 [5]. *For any connected graph G with cut point u, every geodetic global dominating set contains at least one point from each component of* $G - \{u\}$.

Theorem 1.4 [5]. *Let G be a connected graph of order p. Then,* $\bar{\gamma}_g(G) = 2$ *if and only if* $G = K_2$ *or there exist a geodetic set* $R = \{x, y\}$ *such that* $d(x, y) = 3$.

2. Total Geodetic Global domination Number of a Graph

Definition 2.1. A set $R \subseteq V(G)$ is termed as a total geodetic global dominating set if R is both a total geodetic set and a global dominating set. The total geodetic global domination number, $\bar{\gamma}_{gt}(G)$ is the minimum cardinality among all total geodetic global dominating sets of G.

Example 2.2. Scrutinize the graph G given in Figure 2.1 Here $R_1 = \{a_1, a_3, a_6\}$ is a total geodetic set and $R_2 = \{a_1, a_3, a_6\}$ is a global dominating set. It is clear that $R_3 = \{a_1, a_2, a_3, a_6\}$ is a minimum total geodetic global dominating set. Hence $\bar{\gamma}_{gt}(G) = 4$.

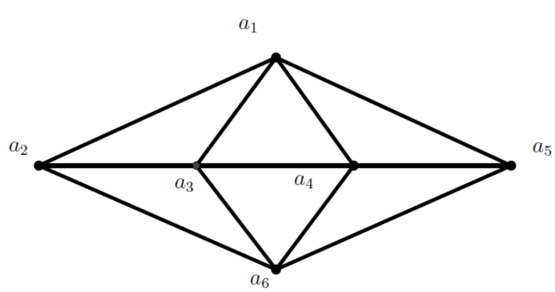

Figure 2.1. Graph G with $\bar{\gamma}_{gt}(G) = 4$.

Observation 2.3. For a connected graph G of order $p \geqslant 2$, $\max\{\bar{\gamma}(G), g_t(G)\} \leq \bar{\gamma}_{gt}(G) \leq g_t(G) + \bar{\gamma}(G)$.

Observation 2.4. For a complete graph $K_p (p \geqslant 2)$, $\bar{\gamma}_{gt}(G) = p$.

Observation 2.5. For a complete bipartite graph $G = k_{p, q}$,

$$\gamma_{gt}(k_{p, q}) = \begin{cases} \min\{p, q\} + 1 & \text{if } 2 \leq p, q \leq 3, \\ 4 & \text{if } p, q \geqslant 4. \end{cases}$$

Observation 2.6. For a star graph $K_{1, p-1}$ with p points $\bar{\gamma}_{gt}(K_{1, p-1}) = p$.

Theorem 2.7. *Every total geodetic global dominating set of a connected graph G contains all its extreme points.*

Proof. Since every total geodetic global dominating set of G is also a geodetic global dominating set of G. Hence the result follows from theorem 1.1. □

Theorem 2.8. *Let G be a connected graph with cut points and let R be a total geodetic global dominating set of G. If u is a cut point of G, then every component of* $G - \{u\}$ *contains at least one element of R.*

Proof. Let u be a cut point of G and R be a total geodetic global dominating set of G. Since R is also a geodetic global dominating set of G. By theorem 1.3. every component of $G - \{u\}$ contains at least one element of R. □

Theorem 2.9. *Each full point and cut point of a connected graph G belongs to every total geodetic global dominating set of G.*

Proof. Since every total geodetic global dominating set is a geodetic global dominating set. By theorem 1.2 each full point belongs to every total geodetic global dominating set of G. Let R be the total geodetic set of G and let u be a cut point of G. Then take $G_1, G_2, G_3, \ldots, G_p (p \geqslant 2)$ be the components of $G - \{u\}$. By theorem 2.8 R contains at least one point from each $G_1, G_2, \ldots, G_p$. Since every points in G is connected, it follows that $u \in R$. □

Theorem 2.10. *For any non-complete connected graph G with m extreme points and n full points,* $\max\{2, m + n\} \leq \bar{\gamma}_{gt}(G)$.

Proof. This follows from theorem 2.7 and theorem 2.9. □

Theorem 2.11. *For a connected graph G of order* $p \geqslant 2$, $2 \leq g_t(G) \leq \gamma_{gt}(G) \leq p$.

Proof. Any total geodetic set has at least two points. Therefore, $g_t(G) \geqslant 2$. By our definition we know that every total geodetic global dominating set is a total geodetic set. So $g_t(G) \leq \bar{\gamma}_{gt}(G)$. Clearly set of all points of G is a total geodetic global dominating set. Thus $\bar{\gamma}_{gt}(G) \leq p$. □

Theorem 2.12. *For a connected graph G of order* $p \geqslant 2$, $2 \leq \gamma_{gt}(G) \leq \bar{\gamma}_{gt}(G) \leq p$.

Proof. Since every total geodetic dominating set contain at least two points. So $\gamma_{gt}(G) \geqslant 2$. Since every total geodetic global dominating set is also a total geodetic global dominating set. From that $\gamma_{gt}(G) \leq \bar{\gamma}_{gt}(G)$. Clearly set of all points of G is a total geodetic global dominating set. Thus $\bar{\gamma}_{gt}(G) \leq \bar{\gamma}_{gt}(G) \leq p$. □

Theorem 2.13. *For a connected graph G of order* $p \geqslant 2$, $2 \leq \bar{\gamma}_g(G) \leq \bar{\gamma}_{gt}(G) \leq p$.

Remark 2.14. The bound given in theorem 2.13 are sharp. For the complete graph K_p, $\bar{\gamma}_{gt}(K_p) = p$ so the above equality hold. For the graph G given in Figure 2.2, $R = \{a_1, a_3, a_5\}$ is the minimum geodetic global dominating set, so that $\bar{\gamma}_g(G) = 3$. Also $R_1 = \{a_1, a_2, a_3, a_5, a_6\}$ is the minimum total geodetic global dominating set. Hence $\bar{\gamma}_{gt}(G) = 5$. Therefore, $2 < \bar{\gamma}_g(G) < \bar{\gamma}_{gt}(G)$.

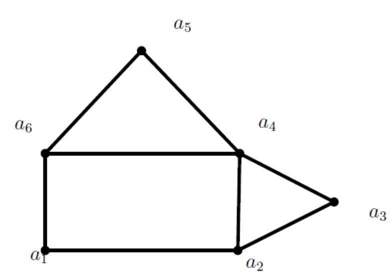

Figure 2.2. Graph G with $p = 6$, $\bar{\gamma}_g(G) = 3$ and $\bar{\gamma}_{gt}(G) = 5$.

Theorem 2.15. *For a connected graph G of order* $p \geqslant 2$, $2 \leq \bar{\gamma}_g(G) = p$ *if and only if every point of G is either a extreme point or a cut point or a full point.*

Proof. Let us assume $\bar{\gamma}_{gt}(G) = p$ for all $p \geqslant 2$. To prove every point of G is either a extreme point or a cut point or a full point. Suppose we assume that G contains a point u which is not a full or cut or extreme point. Since u is not a extreme point, then $G \neq K_p$ and so $V(G) - \{u\}$ is a geodetic set of G. Also, u is not a full point, this implies $G \neq K_p$. Since G is connected, $V(G) - \{u\}$ is a global dominating set of G. Moreover u is not a cut point of G, so $V(G) - \{u\}$ has no isolated points. Hence $V(G) - \{u\}$ is a total geodetic global dominating set of G. Therefore, $\bar{\gamma}_{gt}(G) \leq |V(G) - \{u\}| = p - 1$, which is a contradiction to our assumption. Conversely, we assume that every point of G is either a full point or a cut point or an extreme point. If $G = K_p$, then by observation 2.4., $\bar{\gamma}_{gt}(G) = p$. If $G \neq K_p$ the result follows from theorem 2.10 and theorem 2.12. □

Corollary 2.16. *For a connected graph G, if* $\bar{\gamma}_{gt}(G) = 2$, *then* $\bar{\gamma}_g(G) = 2$.

Corollary 2.17. *For a connected graph G, if* $\bar{\gamma}_g(G) = p$, *then* $\bar{\gamma}_{gt}(G) = p$.

Theorem 2.18. *If* $G = K_p - \{e\}$ *is the graph obtained from* K_p *by removing a line* e, $p \geqslant 4$ *then* $\bar{\gamma}_{gt}(G) = p$.

Proof. Let $G = K_p - \{e\}$, where e is a line of K_p. Let $e = ab$, where $a, b \in V(G)$ then $R_1 = \{a, b\}$ is the geodetic set with minimum cardinality. Also $N[R_1] = V(G)$. Hence R_1 is a dominating set of G. But $\langle R_1 \rangle$ has isolated points. Therefore R_1 is not a total geodetic set. Since $\deg(c_i) = p - 1$ for all $c_i \in V(G) - R_1$, $(1 \leq i \leq p-2)$ here $\deg(a) = \deg(b) = p - 2$. Take $R_2 = R_1 \cup \{c_1\}$. Now, R_2 is a total geodetic set and a dominating set of G. Because each $c_i (2 \leq i \leq p-2)$ has degree $p - 1$. So each $c_i (2 \leq i \leq p-2)$ are isolate in $\overline{G}$. Hence R_2 is not a global dominating set of G. Consider $B = \{c_i / 2 \leq i \leq p-2\}$. Now $R_3 = R_2 \cup B$ is a minimum total geodetic global dominating set of G. Hence $\bar{\gamma}_{gt}(G) = |R_3| = |R_2| + |B| = 3 + p - 3 = p$. □

Theorem 2.19. *Let G be a trivial graph and H be any graph. If R is a minimum total geodetic global dominating set in $G \circ H$, then $R \cap V(G) \neq \phi$.*

Proof. Let $y \in V(G)$ and R be a minimum total geodetic global dominating set of $G \circ H$. We know that by definition of $G \circ H$, y is adjacent to each point of H in $G \circ H$. So that y is an isolate point in G. Hence y must be an point of R. Therefore, $R \cap V(G) \neq \phi$. □

Theorem 2.20. *Let G be any connected graph of order $p \geqslant 2$ and H be any graph. If R is a total geodetic global dominating set in $G \circ H$, then $R \cap V(H_{a_i}) \neq \phi$ for every $a_i \in V(G)$.*

Proof. Let $V(G) = \{a_i, a_2, \ldots, a_p\}$ and $\{b_1^i, b_2^i, \ldots, b_n^i\}$ be the point set of i^{th} copy of H. To prove that $R \cap V(H_{a_i}) \neq \phi$ for some $a_i \in V(G)$. Suppose $R \cap V(H_{a_i}) = \phi$ for some $a_i \in V(G)$. Since every point in $V(H_{a_i})$ is adjacent to exactly one point a_i in $V(G)$ in $G \circ H$. So that $V(H_{a_i})$ does not lies on any geodesic path in R. Hence R is not a total geodetic global dominating set of G. Which is a contradiction to our assumption. Therefore, $R \cap V(H_{a_i}) \neq \phi$ for every $a_i \in V(G)$. □

Theorem 2.21. *Let G be any connected graph of order $n \geqslant 2$ and K_p be a complete graph of order $p \geqslant 2$, then $\bar{\gamma}_{gt}(G \circ K_p) = np$.*

Proof. Let $V(G) = \{a_1, a_2, \ldots, a_n\}$ and $\{b_1^i, b_2^i, \ldots, b_p^i\}$ be the point set of i^{th} copy of K_p. Consider $R = \{b_1^1, b_2^1, \ldots, b_p^1, b_2^2, \ldots, b_p^2, b_1^3, b_2^3, \ldots, b_p^3, \ldots, b_1^n, b_2^n, \ldots, b_p^n\}$. Clearly R is a total geodetic set and dominating set of G. Also R is a dominating set of $\overline{G}$. Which is also minimum. Hence $\bar{\gamma}_{gt}(G \circ K_p) = |R| = np$. □

3. Realization Result

Theorem 3.1. *For a connected graph* G, *if* $\bar{\gamma}(G) = k$, *and* $g_t(G) = l$ *then* $\bar{\gamma}_{gt}(G) = k + l - 2$ *with* $k, l \geqslant 2$ *where* k, l *are two positive integers.*

Proof. Let $C : x_1, x_2, x_3, x_4, x_5, x_6$ be a cycle of order 6. Let H be a graph obtained from C by adding the new points $y_1, y_2, \ldots, y_{l-3}$ to the point x_1. Let G be the graph obtained from H by taking a copy of the path on $3(k-2)+1$ points $v_0, v_1, v_2, \ldots, v_{3(k-2)}$ and joining v_0 to the point x_6 as shown in Figure 3.1. Let $R_1 = \{x_1, x_6, v_2, v_5, \ldots, v_{3(k-2)-1}\}$ is a minimum global dominating set of G. Clearly R_1 contains k points and so $\bar{\gamma}(G) = k$. Take $R_2 = \{y_1, y_2, \ldots, y_{l-3}, x_1, v_{3(k-2)-1}, v_{3(k-2)}\}$. Then R_2 is a minimum total geodetic set of G. Hence $g_t(G) = l$. Now $R_3 = R_1 \cup \{y_1, y_2, \ldots, y_{l-3}, v_{3(k-2)}\}$. Clearly R_3 is a minimum total geodetic global dominating set of G. Hence $\bar{\gamma}_{gt}(G) = | R_3 | = | R_1 | + l - 2 = k + l - 2$. □

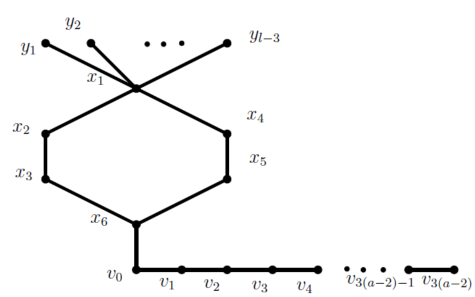

Figure 3.1. Graph G with $\bar{\gamma}(G) = k$, $g_t(G) = l$ and $\bar{\gamma}_{gt}(G) = k + l - 2$

Conclusion

In this paper we discussed the total geodetic global domination number $\bar{\gamma}_{gt}(G)$. We have found some general results of total geodetic global domination number. This work can be extended to find total edge geodetic global domination number of a graph, upper total geodetic global domination number of a graph, upper total edge geodetic global domination number of a graph.

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