Advances and Applications in Mathematical Sciences
Volume 21, Issue 2, December 2021, Pages 651-657

RELATIVELY PRIME EDGE DOMINATION NUMBER OF GRAPHS

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Abstract

In this paper we introduced the concept of relatively prime edge domination of a graph G. Let $G(V, E)$ be a graph. For a set $M \subseteq E(G)$ is said to be relatively prime edge dominating set, if M is an edge dominating set with at least two edges and for each pair of edge e_i and e_j in M such that $(\deg e_i, \deg e_j) = 1$. The minimum cardinality of a relatively prime edge dominating set (rpd-set) is called relatively prime edge domination number and it is denoted by $\beta_{rpd}(G)$. If there is no such pair exist then $\beta_{rpd}(G) = 0$.

1. Introduction

The graph related to the prime elements of G has been discussed since 1981. Williams was the first person who introduced the prime group of a group where the vertices are the primes dividing the order of G and two vertices p and q are joined by an edge if and only if G contains an element of order pq. The significance of the prime graphs of finite groups can be found in Yamiki (1993) and Williams (1981). In this paper we have introduced a relatively prime edge domination number of graphs. The basic definitions have been referred from S. Arumugam and S. Velammal [2], G. Chartrand and Ping Zhang [5], F. Harary [6]. For basic results we have referred C. Jayasekaran and A. Jancy Vini [8], V. R. Kulli and B. Janakiram [9].

2020 Mathematics Subject Classification: 05C20.
Keywords: Edge domination, Relatively prime edge domination.
Received May 17, 2021; Accepted June 7, 2021

2. Definition and Main Results

Definition 2.1. A set $S \subseteq E$ is said to be relatively prime edge dominating set if it is a dominating set with at least two elements and for every pair of edges u and v in S such that $(\deg u, \deg v) = 1$

Definition 2.2. The minimum cardinality of a relatively prime edge dominating set is called relatively prime edge domination number and is denoted by $\beta_{rpd}(G)$.

Theorem 2.1. *Let P_n be a path graph of order $n \geq 3$. then*

$$\beta_{rpd}(P_n) = \begin{cases} 2 & \text{if } 3 \leq n \leq 5 \\ 3 & \text{if } n = 6, 7 \\ 0 & \text{otherwise}. \end{cases}$$

Proof of Theorem 2.1. Let $e_1, e_2, e_3, \ldots, e_n$ be the edge set of path graph P_n.

Case 1. for $3 \leq n \leq 5$

If $n = 3$, then $M = \{e_1, e_2\}$ is the required minimal relatively prime edge dominating set and hence $\beta_{rpd}(P_n) = 2$. Let $n > 3$ in this case $M = \{e_1, e_n\}$ is the dominating set and also $d(e_2), d(e_n) = (2, 1) = 1$ therefore $M = \{e_2, e_n\}$ is the relatively prime edge dominating set and hence $\beta_{rpd}(P_n) = 2$.

Case 2. $n = 6, 7$

In this case $M = \{e_1, e_4, e_n\}$ is the edge dominating set also $d(e_1), d(e_n) = (1, 2) = 1$, and $d(e_4), d(e_n) = (1, 1) = 1$ and $d(e_n), d(e_1) = (1, 2) = 1$ therefore $M = \{e_1, e_4, e_n\}$ is relatively prime edge dominating set and hence $\beta_{rpd}(P_n) = 3$.

Case 3. $n \geq 8$

Any dominating set contains at least two internal edges (e_1, e_j) such that $2 \leq i \neq j \leq n-1$ and $d(e_i), d(e_j) = 2$. Hence $\beta_{rpd}(P_n) = 0$.

Theorem 2.2. *If* $\overline{P_n}$ *is a complement of a path graph, then*

$$\beta_{rpd}(\overline{P_n}) = \begin{cases} 2, & if\ 4 \leq n \leq 7 \\ 0, & otherwise. \end{cases}$$

Proof of Theorem 2.2. Let $e_1, e_2, e_3, \ldots, e_n$ be the edge set of path graph $\overline{P_n}$.

Case 1. If $n = 3$

Here $\overline{P_3}$ is a path of degree 1, but by the definition of relatively prime edge dominating set there should be at least one edge of degree 2. Hence $\beta_{rpd}(\overline{P_3}) = 0$. If $n = 4$, let $M = \{e_1, e_2, e_3\}$ be a path $\overline{P_4}$, v_1 is adjacent to all vertices except v_2 clearly $M = \{e_1, e_2\}$ is the edge dominating set of $\overline{P_4}$. In $\overline{P_4}$ e_1 has degree $(n-2)$ and e_2 has degree $(n-3)$, since $d(e_1), d(e_2) = (1, 2) = 1$, $M = \{e_1, e_2\}$ is relatively prime edge dominating set for $\overline{P_4}$ and hence $\beta_{rpd}(\overline{P_4}) = 2$. For $n \geq 5$, let m be the set of all even numbers, clearly for any two relatively prime edge dominating sets have degree $(n-(2m-1))$ and $(n-2m)$ respectively, since $d(n-(2m-1))$, $d(n-2m) = 1$. Hence $\beta_{rpd}(\overline{G_n}) = 2$.

Case 2. Clearly any dominating sets contain at least two internal edges of same degree and $d(e_i), d(e_j) \neq 1$ which implies $\beta_{rpd}(\overline{P_n}) = 0$. Hence the Theorem.

Theorem 2.3. *If* $G_1 \cong G_2$. *then* $\beta_{rpd}(G_1) = \beta_{rpd}(G_2)$.

Proof of Theorem 2.3. Let f be an isomorphism between graphs G_1 and G_2. Let $E(G_1) = \{e_1, e_2, \ldots, e_n\}$ since $f : E(G_1)E(G_2)$ is a bijection, Let $E(G_2) = \{f(e_1), f(e_2), \ldots, f(e_n)\}$. Let $\{e_1, e_2, \ldots, e_n\}$ be a relatively prime edge dominating set of G_1, since f is an isomorphism, $\{f(e_1), f(e_2), \ldots, f(e_m)\}$. is a also dominating set of G_2. Since isomorphism preserves adjacency of edges $d(e_i), d(e_j) = 1$ for $i \neq j$. Therefore $\{f(e_1), f(e_2), \ldots, f(e_m)\}$ is a relatively prime edge dominating set of G_2. Hence $\beta_{rpd}(G_1) = \beta_{rpd}(G_2)$.

Theorem 2.4. *For a pan graph* P_m

$$\beta_{rpd}(P_m) = \begin{cases} 2, & if\ 4 \le m \le 7 \\ 0, & otherwise. \end{cases}$$

Proof of Theorem 2.4. A cycle C_m will be attached to a path graph P_n by a bridge. C_m is regular graph with edge set $\{e_1, e_2, e_3, \ldots, e_m\}$ and any 2 edges say $\{e_1, e_2\}$ is connected to a path P_m and the degree of any of these two edges will result in 3, and the remaining edges in the cycle C_m will have degree 2. Hence by the definition of relatively prime edge dominating set $\beta_{rpd}(P_m) = 2$.

Theorem 2.5. *Let* $K_{m,n}$ *be a complete bipartite graph,* $\beta_{rpd}(K_{m,n}) = 0$.

Proof of Theorem 2.5. Let v_1 and v_2 be the bipartition vertex of $K_{m,n}$ with $|v_1| = m$ and $|v_2| = n$. Clearly $d(u) = n$ and $d(u) = m$ for $u \in v_1$ and $u \in v_2$ any minimal dominating set of $K_{m,n}$ has one vertex in v_1 and another vertex in v_2, since the degree of each edge in u is same as the degree of each edge in v, the minimal dominating set of $K_{m,n}$ does not become relatively prime dominating set and hence $\beta_{rpd}(K_{m,n}) = 0$.

Theorem 2.6. *Let* $T_{m,n}$ *be a tadpole graph. Then*

$$\beta_{rpd}(L_{m,n}) = \begin{cases} 2 & if\ \frac{q-1}{2} \le 3 \\ 3 & if\ m + n > 7 \\ 0 & otherwise. \end{cases}$$

Proof of Theorem 2.6. Consider a tadpole graph $T_{m,n}$ by definition of $T_{m,n}$ is obtained by joining a cycle graph C_m to a path graph P_n with a bridge.

Case 1. $\frac{q-1}{2} \le 3$

Let $\{u_1, u_2, u_3, \ldots, u_m\}$ be the edges on the cycle graph C_m and $\{v_1, v, v_3, \ldots, v_m\}$ be the edges on the path graph P_n. Let $\{v_1\}$ be the common edge for C_m and P_n and $\{v_m\}$ be the terminal edge of P_n. This implies that for $n = 1$ any 2 edges in the cycle graph will have degree 2 and 3, since the degree of any edge in the cycle will have degree 2 and an edge adjacent to path will have degree 3, $\deg(u_1), \deg(u_m) = (2, 3) = 1$ or $n = 2, 3, 4$.

In this case $M = \{u_1, v_{n-1}\}$ is the required minimal relatively prime dominating set also $\deg(u_1), \deg(v_{n-1}) = (2, 3) = 1$.

Therefore $\{u_1, v_{n-1}\}$ is relatively prime dominating set and hence $\beta_{rpd}(T_{m,n}) = 2$.

Case 2. $m + n > 7$

For $m = 3$, here $\{u_1, v_3, v_n\}$ is the dominating set, also $d(u_1, v_3) = (3, 2) = 1, d(v_3, v_n) = (2, 1) = 1$ and $d(u_1, v_n) = (1, 3) = 1$. Therefore $\{u_1, v_3, v_n\}$ is relatively prime dominating set and hence $\beta_{rpd}(T_{m,n}) = 3$

For $m = 4, 5$.

In this case $\{u_1, v_1, v_m\}$ is the required minimal relatively prime dominating set also $d(u_1), d(v_1) = (2, 3) = 1, d(v_1), d(v_m) = (3, 1) = 1$, $d(v_m), d(u_1) = (1, 3) = 1$. Hence $\beta_{rpd}(T_{m,n}) = 3$ therefore $\{u_1, v_1, v_m\}$ is relatively prime dominating set.

For $m = 6$

In this case $\{u_2, u_5, v_m\}$ is the required relatively prime dominating set also $d(u_1), d(u_5) = (2, 3) = 1, d(u_5), d(v_m) = (3, 1) = 1$ and $d(v_m), d(u_2) = (1, 2) = 1$. Hence $\beta_{rpd}(T_{m,n}) = 3$. Therefore $\{u_1, u_5, v_m\}$ is relatively prime dominating set.

Theorem 2.7. *Let C_n be a centipede graph then*

$$\beta_{rpd}(C_n) = \begin{cases} 2 & if\ n = 3, 5 \\ 3 & if\ n = 7, 8 \\ 1 & otherwise. \end{cases}$$

Proof of Theorem 2.7. Consider a centipede graph C_n, by definition C_n is a tree on $2n$ vertices obtained by joining the bottom of n-copies of the path P_2 laid in a row with edges.

Let the edges on the top of C_n be $u_i; i = 1, 2, 3, \ldots, n$ and the corresponding edges on the bottom be $v_i; i = 1, 2, 3, \ldots, n$.

Case 1. for $n = 3, 5$

In this case $\{u_1, v_n\}$ is the required minimal relatively prime edge dominating set, also $d(u_1), d(v_n) = (2, 1) = 1$. Hence $\beta_{rpd}(C_n) = 2$.

Case 2. for $n = 7, 8$

The edges v_1 and v_n have degree 1 and the edges on the top of C_n have degree 3 and the corresponding edges on the bottom will have degree 2. This is because the edges on the bottom of C_n that is $\{v_1, v_2, \ldots, v_n\}$ will be pendent edges and each edge $\{v_1, v_2, v_{3,} \ldots, v_m\}$ will be connected to the corresponding top edges $\{u_1, u_2, u_3, \ldots, v_n\}$. Hence the edges $\{u_1, v_n, v_{n-1}\}$ forms the required minimal edge dominating set. Also $d(u_1), d(v_{n-1}) = (3, 2) = 1$, $d(v_{n-1}), d(v_n) = (2, 1) = 1$, $d(v_n), d(u_1) = (1, 3) = 1$. Hence $\beta_{rpd}(C_n) = 3$.

3. Conclusion

In this paper, we surveyed selected results on relatively prime edge domination in graphs. These results establish key relationships between the relatively prime numbers and the dominating sets in graphs. We find results for some standard graphs and special graphs.

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