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A STUDY ON NATURAL DIFFERENCE LABELING OF CYCLE GRAPH, LADDER GRAPH AND TADPOLE GRAPH $T(3, n)$

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Abstract

A graph G admits natural difference labeling, if its vertices are labeled by non-negative integers such that the edge labels induced, are obtained by finding the absolute value of the difference of the labels of the end vertices. The edge labels obtained are the first n natural numbers. A graph admitting natural difference labeling is called a natural difference graph. Natural Difference Labeling has its applications in time-bound networks, namely temporal networks. Temporal networks has its applications in communication networks. In this paper Tadpole graph $T(3, n)$ admit natural difference labeling is proved. Also Cycle graph C_n and Ladder graph L_n does not admit natural difference labeling is proved.

1. Introduction

Graph Labeling is assigning labels to vertices or edges or both, subject to certain conditions. Graph Labeling has many applications in coding theory, circuit design, communication networks and so on.

Joseph A. Gallian [4] updates a dynamic survey on Graph Labeling regularly. In [1] G. Amuda and S. Meena have proved Square Difference Labeling of Graphs. In [6] M. Prema and K. Murugan have proved Oblong

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sum labeling of some special graphs. The concept of Natural Difference Labeling arised from the concepts of Triangular Sum Labeling introduced by S. M. Hegde and P. Shankaran [3] and Centered Triangular Sum Labeling [5]. This concept of Natural Difference Labeling has been introduced by A. S. Shanthi and Fathima Azher.

Definition 1 [7]**.** A natural difference labeling of a graph G is a one to one function *f* from $V(G)$ to the set of all non-negative integers that induces a bijection f^* from $E(G)$ to the set of all natural numbers defined by $f^*(uv) = |f(u) - f(v)| \forall e = uv \in E(G)$. The edge labels are the first n natural numbers. The graph admitting such labeling is called natural difference graph.

Definition 2. Tadpole graph $T(3, n)$ is a graph in which the path P_n is attached to any one vertex of cycle C_3.

Definition 3. A Ladder graph is obtained as the Cartesian product of two path graphs, one of which has only one edge i.e $L_{(n,\,1)} = P_n \times P_2$ and is denoted by L_n.

2. Main Results

Theorem 1. *Tadpole graph* $T(3, n)$ *admits natural difference labeling.*

Proof. Let u_0, u_1, u_2 be the vertices of C_3 and $u_3, u_4, \ldots, u_{n+1}$ be the vertices of the path P_n. The mapping *f* from $V(G)$ to $\{0, 1, 2, \ldots, (n+1)\}$ is defined by

$$f(u_i) = \frac{i(i+1)}{2} (0 \le i < 3)$$

$$f(u_3) = 4$$

$$f(u_4) = 9$$

for $4 < i \le n+1$

$$f(u_i) = \frac{i(i+1)}{2} + j(j = 0, 1, 2, \ldots, n-4)$$

and the induced function f^* from $E(G)$ to the set of all natural numbers is defined by

$$f^*(u_0, u_1) = 1$$

$$f^*(u_1, u_2) = 2$$

$$f^*(u_2, u_0) = 3$$

$$f^*(u_0, u_3) = 4$$

and for $3 < i \leq n+1$

$$f^*(u_j u_i) = i + 1 (j = 3, 4, \ldots, n).$$

Hence we see that the induced edge labels which are obtained by the difference of the labels of the end vertices are the first $n+2$ natural numbers. Hence natural difference labeling is proved for Tadpole graph $T(3, n)$. See Figure 1. □

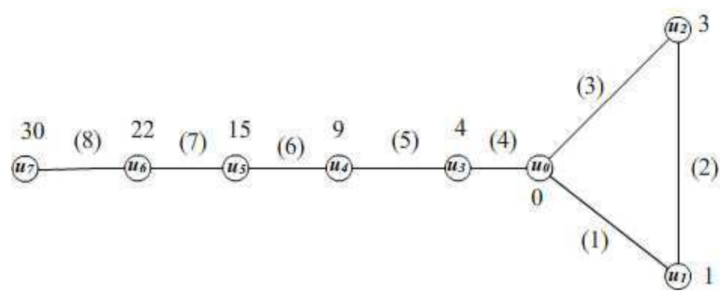

Figure 1. Natural Difference Labeling of Tadpole graph $T(3, 6)$.

Theorem 2. *Cycle* $C_n (n > 3)$ *admits natural difference labeling upto* $n-1$ *edges.*

Proof. Let the graph G be the cycle C_n. Let $|V(G)| = n$ and $|E(G)| = n$. Let $u_0, u_1, \ldots, u_{n-1}$ be the vertices of C_n arranged in the clockwise direction. The mapping f from $V(G)$ to $\{0, 1, 2, \ldots, n-1\}$ is defined by

$$f(u_i) = \frac{i(i+1)}{2} (0 \leq i \leq n-1)$$

and the induced function f^* from $E(G)$ to the set of natural numbers is defined by

$$f^*(u_i, u_i + 1) = i + 1(0 \leq i \leq n - 2).$$

Hence the edge labels obtained by the difference of the labels of the end vertices upto $(n - 1)$ edges are the first $(n - 1)$ natural numbers. The nth edge has the induced labeling given by

$$f^*(u_{n-1}u_0) = \frac{n(n-1)}{2}$$

Hence C_n admits natural difference labeling upto $(n - 1)$ edges. See Figure 2. □

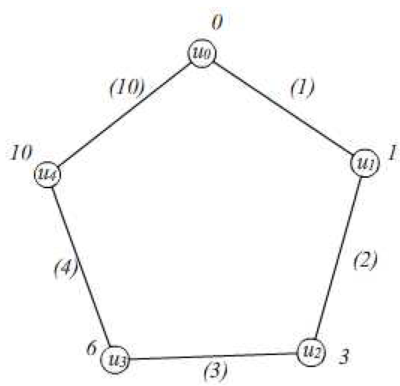

Figure 2. Natural Difference Labeling for C_5 is admissible upto 4 edges.

Remark: C_n does not admit natural difference labeling. In view of Theorem 2, C_n admits natural difference labeling upto $(n - 1)$ edges.

Theorem 3. *Ladder graph* L_n *admits natural difference labeling upto* $(3n - i)$ *edges where* $i = 3, 4, 5, \ldots$ *along the sides of the ladder graph. Edges along the middle of the ladder graph* L_n *have the induced labeling given by*

$$f^*(u_{k+4}u_{j+1}) = (m + 3)(2j + 1).$$

Proof. Let the graph G be the ladder graph L_n. The graph consists of $2n$ vertices and $3n - 2$ edges. We define a mapping f from $V(G)$ to

$\{0, 1, \ldots, 2n-1\}$ by $f(u_i) = \frac{i(i+1)}{2}(0 \le i \le 2n-1)$ and the induced function f^* from $E(G)$ to the set of all natural numbers is defined by

$$f^*(u_i u_{i+1}) = i + 1(0 \le i \le 2n-2)$$

and for $n \ge 2$

$$f^*(u_{2n-1}u_0) = (2k+1)(k+1)$$

$k = 1, 2, \ldots$ Hence we see that the Ladder graph L_n admits natural difference labeling upto $3n - i$ edges where $i = 3, 4, 5, \ldots$ along the sides of the ladder graph. For $n > 2$, edges along the middle of the ladder graph has the induced labeling given by

$$f^*(u_{k+4}u_{j+1}) = (m+3)(2j+1) \tag{1}$$

For $m = i(i = 1, 2, \ldots) j$ takes the values from 1 to i in the decreasing order in the L. H. S of (1) and in the increasing order in the right hand side of (1) and k takes values from m to $m + t(t = 0, 1, 2, \ldots)$. See Figure 3,4. □

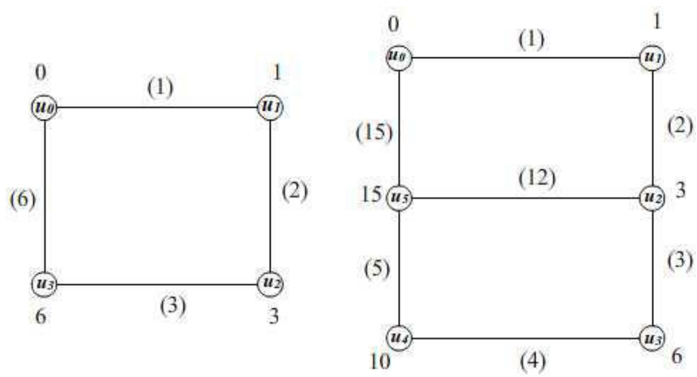

Figure 3. Ladder graph L_2 and L_3.

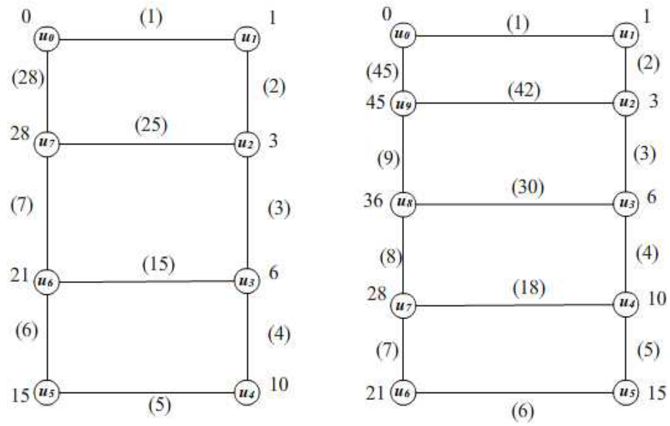

Figure 4. Ladder graph L_4 and L_5.

Remark: L_n does not admit natural difference labeling. In view of Theorem $3 L_n$ admits natural difference labeling upto $3n - i$ edges where $i = 3, 4, 5, \ldots$ along the sides of the ladder graph.

3. Conclusion

Tadpole graph $T(3, n)$ admits natural difference labeling is proved. Cycle graph C_n admits natural difference labeling upto $n - 1$ edges is proved. Ladder graph admits natural difference labeling upto $3n - i$ edges along the sides of the ladder graph and the induced labeling along the middle of the ladder graph is also shown. Natural Difference Labeling for other graphs is under investigation.

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