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MAJORITY VERTEX (EDGE) COVER OF A GRAPH

N. VIJAYASEETHA, R. SUBRAMANI, B. APSANA BEGAM and P. PRIYA

Department of Mathematics
Dhanalakshmi Srinivasan College of Arts
and Science (W) Autonomous, Perambalur, India
E-mail: sithanarayanan14@gmail.com

Abstract

Majority vertex cover and majority edge cover are defined in this paper. Majority vertex covering number and majority edge covering number are found for some standard graphs. Its relation with other parameters is determined.

Introduction

A vertex and an edge are said to cover each other if they are incident. A vertex cover of G is a set of vertices that cover all the edges of G. The smallest number of edges in any vertex cover of G is called the vertex covering number of G and is denoted by $\alpha_0(G)$. A set of edges which cover all the vertices of G is called a edge cover of G. The smallest number of edges in any edge cover of G is called the edge covering number of G and is denoted by $\alpha_1(G)$.

1. Majority Vertex Cover $\alpha_M(G)$

Definition 1.1. A set of vertices S which covers at least half of the edges is a majority vertex cover of G. The majority vertex covering number $\alpha_M(G)$ of G is the minimum number of vertices in a majority vertex cover.

Observation 1.2.

1. $\alpha_0(G) = 1 \to \alpha_M(G) = \alpha_0(G)$

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2. $\alpha_M(G) \leq \alpha_0(G)$ and there exists a graph G such that $\alpha_M(G) < \alpha_0(G)$.

For example, $\alpha_M(C_7) = 2$ and $\alpha_0(C_7) = 4$.

Majority Vertex Covering Number for some standard graphs

Proposition 1.3. *Let* $G = K_{1, p-1}$, $p \geq 2$ *vertices. Then* $\alpha_M(G) = 1$.

Proof. Let $\{u\}$ be the full degree vertex of g. Since $q = p\text{-}1 \left\lceil \frac{q}{2} \right\rceil = \left\lceil \frac{p-1}{2} \right\rceil$. Then $\{u\}$ covers $(p\text{-}1)$ edges and it covers $> \left\lceil \frac{q}{2} \right\rceil$ edges. Therefore $\{u\}$ is a majority vertex cover of G. Hence $\alpha_M(K_{1, p-1}) = 1$.

Proposition 1.4. *For* $G = P_p$, *a path on* $p \geq 2$ *vertices. Then* $\alpha_M(G) = \left\lceil \frac{p-1}{4} \right\rceil$.

Proof. Let $\{v_1, v_2, v_3, \ldots, v_p\}$ be the set of vertices of P_p with $d(v_1) = d(v_p) = 1$ and $d(v_i) = 2$ for all $i \neq 1, p$. Let $D = \{v_1, v_2, v_3, \ldots, v_{\left\lceil \frac{p-1}{4} \right\rceil}\}$.

Claim: Number of edges covered by D is at least $\left\lceil \frac{q}{4} \right\rceil = \left\lceil \frac{p-1}{4} \right\rceil$. Since each vertex of g covers exactly two edges, number of edges covered by D is $2\left\lceil \frac{q}{4} \right\rceil = 2\left\lceil \frac{p-1}{4} \right\rceil$. By fact $(2)\, 2\left\lceil \frac{q}{4} \right\rceil \geq \left\lceil \frac{q}{2} \right\rceil$. Therefore $\alpha_M(G) \leq |D| = \left\lceil \frac{p-1}{4} \right\rceil$. Let $S \subseteq V(G)$ be a set with cardinality $< \left\lceil \frac{p-1}{4} \right\rceil$. (i.e.), $|S| \leq \left\lceil \frac{p-1}{4} \right\rceil - 1$. Number of edges covered by S is $\leq 2(\left\lceil \frac{q}{4} \right\rceil - 1) = 2(\left\lceil \frac{q}{4} \right\rceil - 1) < \left\lceil \frac{q}{2} \right\rceil$. Therefore $\alpha_M(G) = \left\lceil \frac{p-1}{4} \right\rceil$. Hence $\alpha_M(G) = \left\lceil \frac{p-1}{4} \right\rceil$.

Proposition 1.5. *For* $G = C_p$, *cycle on* $p \geq 3$ *vertices, then* $\alpha_M(G) = \left\lceil \frac{p}{4} \right\rceil$.

Definition 1.6 [4]**.** A set S of vertices of G is called a majority neighborhood set if $\cup_{u \in S} \langle N(u) \rangle$ has at least $\left\lceil \frac{q}{2} \right\rceil$ edges. The minimum cardinality of a majority neighborhood set is called the majority neighborhood number of G and is denoted by $n_M(G)$.

Theorem 1.7 [4]**.** *for a graph* $G = k_{m,n}$, $n_M(G) = \min\left\{\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right\}$.

Proposition 1.8 [4]**.** *For any graph* G, $\alpha_M(G) \geq n_M(G)$.

Proposition 1.9. *Let* $G = k_{m,n}$, *Then* $\alpha_M(G) = \min\left\{\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right\}$.

Proof. Let G be a complete bipartite graph with $V_1(G) = m$, $V_2(G) = n$. Let D be an independent set of vertices of cardinality $\left\lceil \frac{m}{2} \right\rceil$.

Case (i) Let $m = n$. Then $p = 2m$ and $q = mn$.

Claim: D covers at least $\left\lceil \frac{q}{2} \right\rceil$ edges. Number of edges covered by D is $q_D = n\left\lceil \frac{m}{2} \right\rceil$. Where $m = 2r$. Then $q_D = nr = \left\lceil \frac{q}{2} \right\rceil$. When $m = 2r + 1$. Then $q_D = n(r+1) = n\left(\frac{m-1}{2}\right) + n > \left\lceil \frac{q}{2} \right\rceil$.

Therefore, $n\left\lceil \frac{m}{2} \right\rceil \geq \left\lceil \frac{q}{2} \right\rceil \ldots (1)$

Therefore number of edges covered by D is $\geq \left\lceil \frac{q}{2} \right\rceil$. Hence $\alpha_M(G) \leq |D| = \min\left\{\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right\}$.

Case (ii) Let $m \neq n$. Without loss of generality, assume that $m < n$.

Let $S = \{u_1, u_2, u_3, u_{\left\lceil \frac{m}{2} \right\rceil}\} \subseteq V_1(G)$. Now, number of edges in $\langle N[S] \rangle$ is

$= n\left\lceil \frac{m}{2} \right\rceil \geq \left\lceil \frac{nm}{2} \right\rceil$ by (1).

Therefore number of edges covered by S is $\geq \left\lceil \frac{q}{2} \right\rceil$.

Hence $\alpha_M(G) \geq |S| = \left\lceil \frac{m}{2} \right\rceil = \min\left\{\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right\}$. By theorem 1.7 and proposition 1.8.

Since $\alpha_M(G) \geq n_M(G)$ and $n_M(G) = \min\left\{\left\lceil \frac{m}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil\right\}$.

Proposition 1.10. *Let* $G = W_p$, $p \geq 5$ *be a wheel graph with p vertices. Then* $\alpha_M(G) = 1$.

Proof. Let $V(G) = \{u, v_1, v_2, v_3, \ldots, v_{p-1}\}$ be the vertices of G and $\{u\}$ be the full degree vertex of W_p. Then $\{u\}$ covers $(p-1)$ edges. Since $q = 2p - 2p = \frac{q+2}{2} - 1 = \frac{q}{2}$. Hence $\{u\}$ covers $\frac{q}{2}$ edges. This implies that $\{u\}$ is a majority vertex cover of G. Hence $\alpha_M(G) = 1$.

Proposition 1.11. *Let G be a regular bipartite graph with p vertices. Then* $\alpha_M(G) = \left\lceil \frac{p}{4} \right\rceil$.

Proof. Let G be a regular bipartite graph with bipartition $(V_1(G), V_2(G))$.

Then $|V_1(G)| = |V_2(G)| = n$. Now, $p = 2n$ and $q = nk$. Let S be an independent set of vertices of cardinality $\left\lceil \frac{p}{4} \right\rceil$. Then S covers $k\left\lceil \frac{p}{4} \right\rceil$ edges.

Claim: Number of edges covered by S is $qs \geq k\left\lceil \frac{p}{4} \right\rceil = \left\lceil \frac{q}{2} \right\rceil$.

$k\left\lceil \frac{p}{4} \right\rceil = k\left\lceil \frac{2n}{2} \right\rceil = k\left\lceil \frac{n}{2} \right\rceil$ and $\left\lceil \frac{q}{2} \right\rceil = \left\lceil \frac{nk}{2} \right\rceil$. If $n = 2m$, then $k\left\lceil \frac{n}{2} \right\rceil = km$ and $\left\lceil \frac{nk}{2} \right\rceil = km$

(i.e.), $k\left\lceil \frac{p}{4} \right\rceil = km = \left\lceil \frac{q}{2} \right\rceil \ldots (1)$

If $n - 2m + 1$, then $= k\left\lceil \frac{n}{2} \right\rceil = k(m+1) = mk + k.$

Therefore $k\left\lceil \frac{p}{4} \right\rceil = k\left\lceil \frac{n}{2} \right\rceil = mk + k \ldots (2)$

And $\left\lceil \frac{q}{2} \right\rceil = \left\lceil \frac{nk}{2} \right\rceil = \left\lceil \frac{2mk + k}{2} \right\rceil = [mk + k/2]$. Therefore

$$\left\lceil \frac{nk}{2} \right\rceil \begin{cases} < mk + k \text{ if } k > 1 \\ = mk + k \text{ if } k = 1 \end{cases}$$

In both cases, $k\left\lceil \frac{p}{4} \right\rceil = k\left\lceil \frac{n}{2} \right\rceil \geq \left\lceil \frac{nk}{2} \right\rceil = \left\lceil \frac{q}{2} \right\rceil$. Then number of edges covered by S is $\geq \left\lceil \frac{q}{2} \right\rceil$. This implies that S is a majority vertex cover of G. $\alpha_M(G) | S | = \left\lceil \frac{p}{4} \right\rceil$

Let $D \subseteq V(G)$ with $| D | \leq \left\lceil \frac{p}{4} \right\rceil$. Therefore $| D | \leq \left\lceil \frac{p}{4} \right\rceil - 1$. Suppose $n = 2m$. Then number of edges covered by $D = k\left(\left\lceil \frac{p}{4} \right\rceil - 1\right) = k\left\lceil \frac{p}{4} \right\rceil - k = km - k$ by the result (1) and $\left\lceil \frac{q}{2} \right\rceil = \left\lceil \frac{nk}{2} \right\rceil = mk$. Therefore number of edges covered by D is $< \left\lceil \frac{q}{2} \right\rceil = mk$. Suppose $n = 2m + 1$. Then number of edges covered by $D = k\left(\left\lceil \frac{p}{4} \right\rceil - 1\right) = k\left\lceil \frac{p}{4} \right\rceil - k = km + k - k = mk$ by the result (2) and $\left\lceil \frac{q}{4} \right\rceil = \left\lceil \frac{nk}{2} \right\rceil \leq mk + k$ if $k \geq 1$ by the result (3). That is, $k\left(\left\lceil \frac{q}{4} \right\rceil - 1\right) = mk \leq mk + k = \left\lceil \frac{q}{2} \right\rceil$. Therefore the number of edges covered by D is $< \left\lceil \frac{q}{2} \right\rceil$. Hence $\alpha_M(G) \geq \left\lceil \frac{p}{4} \right\rceil$. Thus, $\alpha_M(G) \geq \left\lceil \frac{p}{4} \right\rceil$.

Proposition 1.12. *For a double star* $D_{r,s}\alpha_M(D_{r,s}) = 1$.

Proof. Let u and v be the canters of G with r, s pendent vertices respectively in its neighborhood. Now, $P = r + s + 2$ and $q = r + s + 1; d(u)r + 1$ and $d(v) = s + 1$. Without loss of generality, assume that $d(v) = s + 1$. Then $\{u\}$ covers at least $(r + 1)$ edges. Since $\left\lceil \frac{q}{2} \right\rceil = \left\lceil \frac{r+s+1}{2} \right\rceil$, $r + 1 \geq \left\lceil \frac{q}{2} \right\rceil$ or $s \geq \left\lceil \frac{q}{2} \right\rceil$. i.e., $r \geq \left\lceil \frac{p}{4} \right\rceil$. Then $r + 1 \left\lceil \frac{q}{4} \right\rceil$. In both cases, $\{u\}$ covers at least $(r + 1) \geq \left\lceil \frac{q}{2} \right\rceil$ edges. $\{u\}$ is a majority vertex cover of G. Hence $\alpha_M(G) = 1$.

2. Majority Edge Cover

Definition 2.1. Let G be a graph with p vertices and q edges. A set of edges of S of $E(G)$ which covers at least half of the vertices of G is a *majority edge cover* of a graph G. A *majority edge covering number* $\alpha_{M1}(G)$ of G is the minimum number of edges in a majority edge cover.

Remark 2.2. If S is a majority edge cover of a graph G, then every superset $S' \supseteq S$ is also a majority edge cover of a set.

Observation 2.3.

1. For any graph $G = K_{1,p}$, with $p \geq 2$ vertices, $\alpha_{M1}(G) = \left\lceil \frac{p-2}{2} \right\rceil$

2. Let G be a regular bipartite graph with p vertices. Then $\alpha_{M1}(G) = \left\lceil \frac{p}{4} \right\rceil$

3. For a path, cycle, complete graph, wheel, fan with p vertices, Then $\alpha_{M1}(G) = \left\lceil \frac{p}{2} \right\rceil$.

Theorem 2.4. *For any graph G with p vertices,* $\left\lceil \frac{p}{4} \right\rceil \leq \alpha_{M1}(G) \leq \left\lceil \frac{p-2}{2} \right\rceil$, *the bounds is sharp.*

Example 2.5.

1. Let $G = C_p$, $\alpha_{M1}(G) = \left\lceil \frac{p}{4} \right\rceil$.

2. Let $G = K_{1, p-1}$, $p \geq 2$ vertices. Then $\alpha_M(G) = \left\lceil \frac{p-2}{2} \right\rceil$.

3. Let $G = D_{r, s}$ be a double star, Then $\alpha_{M1}(G) \neq \left\lceil \frac{p}{4} \right\rceil$ but $\alpha_{M1}(G) > \left\lceil \frac{p}{4} \right\rceil$.

Hence $\left\lceil \frac{p}{4} \right\rceil \leq \alpha_{M1}(G) \leq \left\lceil \frac{p-2}{2} \right\rceil$.

Remark 2.6. If the graph is connected with only $2 \leq p \leq 4$ vertices then $\alpha_M(G) = 1$.

Proposition 2.7. *Let* $\alpha_M(G)$ *be the majority vertex covering number of G. Then*

$$\alpha_M(G) \leq \alpha_{M1}(G).$$

Proof. Each edge covers exactly two vertices only. But each vertices covers two or more edges.

Case (i) Suppose the vertex covers exactly two edges then $\alpha_M(G) = \alpha_{M1}(G)$.

Case (ii) Suppose the vertex covers more than two edges then $\alpha_M(G) < \alpha_{M1}(G)$.

Remark 2.7. Let G be a disconnected graph with p vertices and $q = \left\lceil \frac{p}{4} \right\rceil$ edges. Then we can define majority edge cover for any graph.

3. Relationship between $\gamma_M(G)$, $\alpha_M(G)$ and $\alpha_{M1}(G)$

Proposition 3.1. *Let* $\alpha_1(G)$ *and* $\alpha_M(G)$ *be the edge cover and majority edge cover respectively. Then* $\alpha_1(G) > \alpha_M(G)$.

Proof. Since every edge cover exactly two vertices. Then edge cover is always greater than the majority edge cover of a graph G.

Proposition 3.2. *For a connected graph G. Then* $\gamma_M(G) \leq \alpha_M(G)$.

Proof. Since every majority vertex cover is a majority dominating set, its number satisfies the inequality.

Proposition 3.3. *Let* $\gamma_M(G), \alpha_M(G)$ *and* $\alpha_{M1}(G)$ *be the majority dominating set, majority vertex and majority edge cover respectively.*

Then

$$\gamma_M(G) \leq \alpha_M(G) \leq \alpha_{M1}(G).$$

Proof. Since by the above Proposition we get the result $\gamma_M(G) \leq \alpha_M(G) \ldots (1)$

Since any vertex may be cover one or more edges of a graph but every edge cover exactly two vertices only.

Then

$$\alpha_M(G) \leq \alpha_{M1}(G) \ldots (2)$$

From (1) and (2) we get

$$\gamma_M(G) \leq \alpha_M(G) \leq \alpha_{M1}(G).$$

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