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FUZZY MEASURES ON TRIBES

C. G. MAGADUM and M. S. BAPAT

Department of Mathematics
Smt. Kasturbai Walchand College
Sangli 416416, (M.S.), India
E-mails: magdchet2001@gmail.com

Department of Mathematics
Chintamanrao College of Commerce
Sangli, 416415, (M.S.), India
E-mails: msbapt@gmail.com

Abstract

A study of various kinds of σ-algebra is the main objective of measure theory and measure spaces. We will study σ-algebras of fuzzy sets called as tribes based on different t-norms. This paper presents Yager's tribe based on Yager's t-norm and also presents fuzzy measure on various tribes.

1. Introduction

A study of various types of measures and measure spaces is a well-developed branch in classical theory. Measure theory is useful in dealing with different concepts in mathematical sciences, physical sciences and engineering fields etc. Therefore, the study of measure spaces of fuzzy sets is obvious. Measures are defined on algebras or σ-algebras of sets. Hence, it is natural to study σ-algebra of fuzzy sets where union and intersections of fuzzy sets are defined with the help of different types of t-norms and t-conorms. Any algebra of fuzzy set is called as clan while σ-algebra of fuzzy set is called as tribes. Tribes are nothing but generalized concept of σ-algebra on fuzzy set. The real valuations defined on fuzzy tribes are called as measure on tribes.

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An axiomatic approach to a broad class of fuzzy measures is the concept of triangular norm (t-norm for short) [1]. Fuzzy measures are set functions defined on collection of fuzzy sets which are monotonic with respect to fuzzy set inclusion. Triangular norms and conorms are semi-groups of the unit interval which have been thoroughly studied in the literature of functional equations. The proposed class includes probability measures, Zadeh's possibility measures and the dual notion of necessity measures [2]. The natural generalization of a measure space is given by fuzzy tribes with triangular norms which were introduced by Butnariu and Klement [3]. T_∞-valuations were introduced by Klement [4] and intensively studied by Butnariu and Klement [3]. D. Butnariu studied the properties of triangular norms and fuzzy sets including the concepts of disjointness, emphasizing the special role of Frank t-norms T_λ [5] [6].

An algebraic generalization of the notion of tribe was proposed by M. Navara [7] [8] [9]. E. P. Klement and M. Navara proposed a complete characterization of tribes with respect to Łukasiewicz t-norm [10]. E. P. Klement and M. Navara introduced T-admissible functions on T-tribes as "all possible" fuzzy extension of Boolean function [11]. Radko Mesiar discussed a several generalization of a measurable space and presented some results and open problems [12] [13]. Level cuts of tribes are obtained which are σ-algebras and characterizations of level cuts of tribes are introduced [14]. Some classifications of representable picture t-norms are introduced by B. C. Cuong [15]. G. Barbieri and H. Weber had developed topological approach to study T_∞-valuation of fuzzy sets [16]. Measures with respect Łukasiewicz tribes and its properties are studied by D. Butnariu and E. P. Klement [17].

The present paper defines Yager's tribe based on Yager t-norm. Also, we have studied measures on tribes and introduced different combination of examples which form fuzzy tribe as well as obtained measure space by using support measure and counting measure.

2. Tribes of Fuzzy Sets

By classical theory, probability can be defined only on appropriate collection of sets, closed under the necessary operations (Union, intersection and complement etc.). In fuzzy set theory the binary operations are t-norm and t-conorm.

Let X be a non-empty set, $\sim$ be the standard fuzzy negation, $i : [0, 1]^2 \to [0, 1]$ be a triangular norm (t-norm). For any two fuzzy sets A_1, A_2 we have $i(A_1, A_2) : X \to [0, 1]$ such that $i(A_1, A_2)(x) = i(A_1(x), A_2(x))$. Then $i(A_1, A_2)$ is also a fuzzy set. Therefore, $i(A_1, A_2, A_3) = i(i(A_1, A_2), A_3)$ is a fuzzy set and $i(A_1, A_2, \ldots, A_n)$ is also a fuzzy set. We define its dual operation (t-conorm) $u : [0, 1]^2 \to [0, 1]$ such that $u(A, B) = (i(A^{\sim}, B^{\sim}))^{\sim}$.

Similarly for any two fuzzy sets A_1, A_2 we have $u(A_1, A_2) : X \to [0, 1]$ such that $u(A_1, A_2)(x) = u(A_1(x), A_2(x))$. Then $u(A_1, A_2)$ is also a fuzzy set. Therefore, $u(A_1, A_2, A_3) = u(u(A_1, A_2), A_3)$ is a fuzzy set and $u(A_1, A_2, \ldots, A_n)$ is also a fuzzy set. Also these three operations i, u, and $\sim$ form a De-Morgan triplet. We define its dual operation (t-norm) $i : [0, 1]^2 \to [0, 1]$ such that $i(A, B) = (u(A^{\sim}, B^{\sim}))^{\sim}$.

In fuzzy set theory, various t-norms are used namely minimum t-norm, Łukasiewicz t-norm, Frank t-norm, Yager's t-norm etc. The tribes are obtained with respect to these t-norms. Some basic definition of tribes and some properties of tribes are considered below.

Definition 2.1. A collection of fuzzy subsets of X is called as tribe if it is σ-algebra of fuzzy subsets [14].

Definition 2.2. A i-clan on X is a collection $\tau \subseteq [0, 1]^X$ such that

(i) $X \in \tau$,

(ii) $A \in \tau \Rightarrow A^{\sim} \in \tau$,

(iii) $A, B \in \tau \Rightarrow i(A, B) \in \tau$.

A i-clan τ is called a tribe if it satisfies the stronger condition

$$\{A_n\}_{n \in N} \in \tau \Rightarrow i(A_1, A_2, \ldots, A_n) \in \tau.$$

Due to (ii) condition in the definition of i-clan we also use u for i (since due to complement) [14].

Example 2.1. Let $X = \{a\}$ and $f_\alpha : X \to [0, 1]$ such that $f_\alpha(a) = \alpha$, $\alpha \in [0, 1]$ then $\tau = (f_\alpha)_{\alpha \in [0, 1]}$ is a fuzzy tribe under minimum t-norm [14].

Proposition 2.1. *A set of characteristic function defined on X forms a* σ-*algebra of fuzzy sets* [14].

Proposition 2.2. *Each algebra of crisp subsets of X is a i-clan with respect to an intersection t-norm* [14].

Proposition 2.3. *Every tribe is i-clan but not each i-clan is a tribe* [14].

Proposition 2.4. *A i-clan is algebra of fuzzy sets and tribe is a* σ–*algebra of fuzzy sets* [14].

3. Łukasiewicz Tribes and Yager Tribes

Here, we consider tribes using Łukasiewicz t-norm and Yager's t-norm.

3.1 Tribes using Łukasiewicz *t*-norm. Let X be a (non-empty) universal set. Let $[0, 1]^X$ i.e. collection of all fuzzy sets of X with their membership functions. Łukasiewicz tribes are collection of fuzzy sets which are closed under

- Standard fuzzy complement: If $A \in [0, 1]^X$ then $A^{\sim} \in [0, 1]^X$, $A^{\sim}(x) = 1 - A(x)$
- Łukasiewicz t-norm: Let $i_L : [0, 1] \times [0, 1] \to [0, 1]$ such that

$$i_L(a, b) = \max(0, a + b - 1)$$

- Łukasiewicz t-conorm: Let $u_L : [0, 1] \times [0, 1] \to [0, 1]$ such that

$$u_L(a, b) = \min(a + b, 1).$$

Definition 3.1.1. A Łukasiewicz tribe is a collection of fuzzy subsets of X, $\tau \subseteq [0, 1]^X$ such that

1. $1 \in \tau$ i.e. $1 = X : X \to [0, 1]$, $X(x) = 1 \; \forall x \in X$,

2. $A \in \tau \Rightarrow A^{\sim} \in \tau$,

3. $\{A_n\}_{n\in\mathbb{N}} \in \tau \Rightarrow i_L(A_1, A_2, \ldots, A_n)_{n\in N} \in \tau$.

If a Łukasiewicz tribe contains only crisp sets, its elements correspond to sets forming a σ-algebra [14].

Example 3.1.1. Let $X = \{a, b, c\}$. Let $\tau = \{0, 1\}^X = \{\phi, X, \chi_\alpha, \chi_b, \chi_c, \chi_{ab}, \chi_{ac}, \chi_{bc}\}$ then τ is Łukasiewicz tribe where $i(A, B) = \max(A + B - 1, 0)$, $A^{\sim} = 1 - A$. [14]

Example 3.1.2. Let $X = \{1, 2, \ldots, n-1\}$ for $n \in \mathbb{N}$. Let $r \in X$ and the fuzzy subset as $f_{r/n}(x) = \frac{r}{n}$. Then $\tau = \{\phi, f_{1/n}, f_{2/n}, \ldots, f_{(n-1)/n}, X\}$ is i-clan with respect to minimum t-norm and Łukasiewicz t-norm [14].

Example 3.1.3. Let $X = \{a, b, c\}$ and $\tau = \{f : X \to [0, 1]\}$, $\alpha \in [0, 1]$. Define, $f_\alpha(a) = 1$ and 0 if $x \neq a$, $f_{b_\alpha}(b) = \alpha$ and 0 if $x \neq b$, $f_{c_{1-\alpha}}(c) = 1 - \alpha$ and 0 if $x \neq c$. Therefore,

$$\tau = \{\phi, X, f_a, f_{b_\alpha}, f_{c_{1-\alpha}}, f_{ab_\alpha}, f_{b_\alpha c_{1-\alpha}}, f_{ac_{1-\alpha}} f_{ab_\alpha c_{1-\alpha}}, f_{bc}, f_{ab_{1-\alpha}c},$$

$$f_{abc_\alpha}, f_{b_{1-\alpha}c}, f_{ab_{1-\alpha}c_\alpha}, f_{bc_\alpha}, f_{b_{1-\alpha}c_\alpha}, f_{b_\alpha c_\alpha}, f_{ab_\alpha c\alpha}, f_{ab_{1-\alpha}c_{1-\alpha}}, f_{b_{1-\alpha}c_{1-\alpha}}\},$$

then τ is a Łukasiewicz tribe.

3.2. Tribes using Yager *t*-norm. Let X be a (non-empty) universal set. Let $[0, 1]^X$ i.e. collection of all fuzzy sets of X with their membership functions and $\lambda > 0$. Yager tribes are collection of fuzzy sets which are closed under

- Standard fuzzy complement: If $A \in [0, 1]^X$ then

$$A^{\sim} \in [0, 1]^X,\ A^{\sim}(x) = 1 - A(x)$$

- Yager t-norm: Let $i_Y^\lambda(a, b) : [0, 1] \times [0, 1] \to [0, 1]$ such that

$$i_Y^\lambda(a, b) = 1 - \min\{1, [(1-a)^\lambda + (1-b)^\lambda]^{1/\lambda}\}$$

- Yager t-conorm: Let $u_Y^\lambda(a, b) : [0, 1] \times [0, 1] \to [0, 1]$ such that

$$u_Y^{\lambda}(a, b) = \min\{1, [(a)^{\lambda} + (b)^{\lambda}]^{1/\lambda}\}.$$

Definition 3.2.2. A Yager tribe is a collection of fuzzy subsets of X, $\tau \subseteq [0, 1]^X$ such that

1. $1 \in \tau$ i.e. $1 = X : X \to [0, 1]$, $X(x) = 1\ \forall x \in X$,

2. $A \in \tau \Rightarrow A^{\sim} \in \tau$,

3. $\{A_n\}_{n \in \mathbb{N}} \in \tau \Rightarrow i_Y^{\lambda}(A_1, A_2, \ldots, A_n)_{n \in \mathbb{N}} \in \tau$.

Example 3.2.1. Let $X = \{a, b, c\}$. Let $\tau = \{0, 1\}^X = \{\phi, X, \chi_a, \chi_b, \chi_c, \chi_{ab}, \chi_{ac}, \chi_{bc}\}$. Then τ is Yager tribe where Yager t-norm is $i_Y^{\lambda}(a, b) = 1 - \min\{1, [(1-a)^{\lambda} + (1-b)^{\lambda}]^{1/\lambda}\}$ and $A^{\sim} = 1 - A$.

Solution: Here, (a) $\phi, X \in \tau$,

(b) Let $\chi_a, \chi_b \in \tau$ then consider

$$\begin{aligned} i_Y^{\lambda}(\chi_a, \chi_b)(a) &= 1 - \min\{1, [(1 - \chi_a(a))^{\lambda} + (1 - \chi_b(a))^{\lambda}]^{1/\lambda}\} \\ &= 1 - \min\{1, [(1-1)^{\lambda} + (1-0)^{\lambda}]^{1/\lambda}\} \\ &= 1 - \min\{1, [0^{\lambda} + 1^{\lambda}]^{1/\lambda}\} \\ &= 1 - \min\{1, 1\} \\ &= 1 - 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} i_Y^{\lambda}(\chi_a, \chi_b)(b) &= 1 - \min\{1, [(1 - \chi_a(b))^{\lambda} + (1 - \chi_b(b))^{\lambda}]^{1/\lambda}\} \\ &= 1 - \min\{1, [(1-0)^{\lambda} + (1-1)^{\lambda}]^{1/\lambda}\} \\ &= 1 - \min\{1, [1^{\lambda} + 0^{\lambda}]^{1/\lambda}\} \\ &= 1 - \min\{1, 1\} \\ &= 1 - 1 \end{aligned}$$

$$= 0$$

$$i_Y^{\lambda}(\chi_a, \chi_b)(c) = 1 - \min\{1, [(1 - \chi_a(c))^{\lambda} + (1 - \chi_b(c))^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [(1 - 0)^{\lambda} + (1 - 0)^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [1^{\lambda} + 1^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, 2^{1/\lambda}\}$$

$$= 1 - 1$$

$$= 0$$

$i_Y^{\lambda}(\chi_a, \chi_b)(x) = \phi(x).$

Therefore, $i_Y^{\lambda}(\chi_a, \chi_b) = \phi \in \tau$. Let $\chi_{ac}, \chi_{bc} \in \tau$ then consider

$$i_Y^{\lambda}(\chi_{ac}, \chi_{bc})(a) = 1 - \min\{1, [(1 - \chi_{ac}(a))^{\lambda} + (1 - \chi_{bc}(a))^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [(1 - 1)^{\lambda} + (1 - 0)^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [0^{\lambda} + 1^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, 1\}$$

$$= 1 - 1$$

$$= 0$$

$$i_Y^{\lambda}(\chi_{ac}, \chi_{bc})(b) = 1 - \min\{1, [(1 - \chi_{ac}(b))^{\lambda} + (1 - \chi_{bc}(b))^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [(1 - 0)^{\lambda} + (1 - 1)^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [1^{\lambda} + 0^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, 1\}$$

$$= 1 - 1$$

$$= 0$$

$$i_Y^{\lambda}(\chi_{ac}, \chi_{bc})(c) = 1 - \min\{1, [(1-\chi_{ac}(c))^{\lambda} + (1-\chi_{bc}(c))^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [(1-1)^{\lambda} + (1-1)^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, [0^{\lambda} + 0^{\lambda}]^{1/\lambda}\}$$

$$= 1 - \min\{1, 0\}$$

$$= 1 - 0$$

$$= 1$$

$i_Y^{\lambda}(\chi_{ac}, \chi_{bc})(x) = \chi_c(x)$.

Therefore, $i_Y^{\lambda}(\chi_{ac}, \chi_{bc}) = \chi_c \in \tau$

(c) Complement,

$\chi_a^{\sim}(a) = 1 - \chi_a(a) = 1 - 1 = 0$

$\chi_a^{\sim}(b) = 1 - \chi_a(b) = 1 - 0 = 1$

$\chi_a^{\sim}(c) = 1 - \chi_a(c) = 1 - 0 = 1$

$$\chi_a^{\sim}(x) = \begin{cases} 1 & x = b, c \\ 0 & \text{otherwise} \end{cases}$$

$$= \chi_{bc}(x)$$

$$\chi_a^{\sim} = \chi_{bc}.$$

Similarly,

$$\chi_{bc}^{\sim} = \chi_a, \chi_{ac}^{\sim} = \chi_b, \chi_b^{\sim} = \chi_{ac}, \phi^{\sim} = X, X^{\sim} = \phi$$

$$X(x) = 1, \phi(x) = 0.$$

Thus τ is a Yager tribe.

4. Measure on Tribes

Definition 4.1. A fuzzy measure μ on Θ is a function $\mu : 2^{\Theta} \to [0, 1]$

satisfying the following axioms

1. $\mu(\phi) = 0,\ \mu(\Theta) = 1$ (Boundary Condition)

2. $\theta_1 \subseteq \theta_2 \subseteq \Theta \Rightarrow \mu(\theta_1) \subseteq \mu(\theta_2)$ (Monotonicity)

The main characteristic of fuzzy measure is non-additivity, so fuzzy measures are also called as non-additive measure.

Definition 4.2. Let $(\tau, i, u, \sim)$ be a tribe on some nonempty set X, where $(i, u, \sim)$ is a dual triple. Let $m : \tau \to [-\infty, \infty]$ be a function such that

- $m(\phi) = 0$
- $m(u(A, B)) + m(i(A, B)) = m(A) + m(B).$

Then, m is called as i-valuation.

Definition 4.3. Let $(\tau, i, u, \sim)$ be a tribe on some nonempty set X, where $(i, u, \sim)$ is a dual triple. Let $m : \tau \to [-\infty, \infty]$ be a function such that

- $m(\phi) = 0$
- $A, B \in \tau$ such that $i(A, B) = \phi \Rightarrow m(u(A, B)) = m(A) + m(B).$

Then, m is called as i-additive.

Definition 4.4. Let $(\tau, i, u, \sim)$ be a tribe on some nonempty set X, where $(i, u, \sim)$ is a dual triple. Let $m : \tau \to [-\infty, \infty]$ be a function such that

- $m(\phi) = 0$
- $m(u(A, B)) + m(i(A, B)) = m(A) + m(B)$ for all $A, B \in \tau$
- $\{A_n\}_{n \in N} \in \tau \Rightarrow m(u_{n \in N}(A_n)) = \lim_{M \to \infty} m(u_{n \leq M}(A_n))$
- $\{A_n\}_{n \in N} \in \tau \Rightarrow m(i_{n \in N}(A_n)) = \lim_{M \to \infty} m(i_{n \leq M}(A_n)).$

Then m is called as i-measure.

Theorem 4.1. *Let τ is a both i-clan and i_M-clan on nonempty set X, where i is a t-norm then each i-valuation on τ is an i_M-valuation on τ (i_M is minimum t-norm).*

Proof. Let τ is a both i-clan and i_M-clan on a nonempty set X, i.e. elements of τ are fuzzy subsets of X. Let $\theta_1, \theta_2 \in \tau$ then we have $i_M(\theta_1, \theta_2) \in \tau$ and $u_M(\theta_1, \theta_2) \in \tau$.

Again $\theta_1, \theta_2 \in \tau \Rightarrow i(\theta_1, \theta_2) \in \tau$ and $u(\theta_1, \theta_2) \in \tau$. Let $\theta_1(x) \leq \theta_2(x)$ then $i_M(\theta_1, \theta_2)(x) = \theta_1(x) \Rightarrow i_M(\theta_1, \theta_2) = \theta_1$ and $u_M(\theta_1, \theta_2)(x) = \theta_2(x) \Rightarrow u_M(\theta_1, \theta_2) = \theta_2$. Let μ is a i-valuation on τ.

$\therefore\ \mu(i(\theta_1, \theta_2)) + \mu(u(\theta_1, \theta_2)) = \mu(\theta_1) + \mu(\theta_2).$

Consider,

$\mu(i_M(\theta_1, \theta_2)) + \mu(u_M(\theta_1, \theta_2))$

$= \mu(i(i_M(\theta_1, \theta_2), u_M(\theta_1, \theta_2))) + \mu(u(i_M(\theta_1, \theta_2), u_M(\theta_1, \theta_2))).$

$= \mu((i(\theta_1, \theta_2)) + \mu((u(\theta_1, \theta_2))$

$= \mu(\theta_1) + \mu(\theta_2).$

Hence, each i-valuation on τ is an i_M- valuation on τ.

Based on fuzzy tribe we construct some examples which are tribes and defined support measure and counting measure to obtain measure space.

Example 4.1. Let $X = \{a, b, c\}$ and $\tau = \{f : X \to [0, 1]\}$, $\alpha \in [0, 1]$.

Then define

$$f(x) = \begin{cases} 1 & x = a \\ \alpha & x = b \\ 1 - \alpha & x = c \\ 0 & \text{otherwise} \end{cases}$$

Here,

$$f_a(x) = \begin{cases} 1 & x = a \\ 0 & \text{otherwise} \end{cases}$$

$$f_{b_\alpha}(x) = \begin{cases} \alpha & x = b \\ 0 & \text{otherwise} \end{cases}$$

$$f_{c_{1-\alpha}}(x) = \begin{cases} 1 - \alpha & x = c \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\tau = \{\phi, X, f_a, f_{b_\alpha}, f_{c_{1-\alpha}}, f_{ab_\alpha}, f_{b_\alpha c_{1-\alpha}}, f_{ac_{1-\alpha}}, f_{ab_\alpha c_{1-\alpha}},$$

$$f_{bc}, f_{ab_{1-\alpha}c}, f_{abc_\alpha}, f_{b_{1-\alpha}c},$$

$$f_{ab_{1-\alpha}c_\alpha}, f_{bc_\alpha}, f_{b_{1-\alpha}c_\alpha}, f_{b_\alpha c_\alpha}, f_{ab_\alpha c_\alpha}, f_{ab_{1-\alpha}c_{1-\alpha}}, f_{b_{1-\alpha}c_{1-\alpha}}\}.$$

Here $(\tau, i, \sim)$ is a tribe of fuzzy subset of X. (i is the minimum t-norm i.e. standard fuzzy intersection and ~ is the fuzzy complement). Then (X, τ, m_s) and (X, τ, m_c) are measure spaces (where m_s and m_c are support and counting measure respectively).

1. $m_s : \tau \to \mathbb{R}$ such that $m_s(f) = |\operatorname{supp}(f)|$

$\operatorname{supp}(f) = \{x \in X / f(x) > 0\} \subseteq X$ and

$\operatorname{supp}(f_\alpha) = \{x \in X / f_a(x) > 0\} = \{a\}$.

Therefore, $|\operatorname{supp}(f_\alpha)| = 1$.

Now consider, $\operatorname{supp}(f_{ac_{1-\alpha}}) = \{x \in X : f_{ac_{1-\alpha}}(x) > 0\} = \{a, c\}$. Therefore, $|\operatorname{supp}(f_{ac_{1-\alpha}})| = 2$

$m_s(\phi) = |\operatorname{supp}(\phi)| = \{x \in X : \phi(x) > 0\} = |\phi| = 0.$

Now consider,

$$m_s(i(f_a, f_{ac_{1-\alpha}})) + m_s(u(f_a, f_{ac_{1-\alpha}})) = m_s(f_a) + m_s(f_{ac_{1-\alpha}}) = 1 + 2 = 3$$

$$= m_s(f_a) + m_s(f_{ac_{1-\alpha}}).$$

Consider $f_{b_\alpha c_{1-\alpha}}, f_{ab_{1-\alpha}c_\alpha}$.

Now $m_s(f_{b_\alpha c_{1-\alpha}}) = |\operatorname{supp}(f_{b_\alpha c_{1-\alpha}})| = |\{b, c\}| = 2$

$$m_s(f_{ab_{1-\alpha}c_\alpha}) = |\operatorname{supp}(f_{ab_{1-\alpha}c_\alpha})| = |\{a, b, c\}| = 3.$$

Now

$$i(f_{b_\alpha c_{1-\alpha}}, f_{ab_{1-\alpha}c_\alpha}) = \begin{cases} f_{b_\alpha c_\alpha} & \text{if } \alpha \le 1/2 \\ f_{b_{1-\alpha}c_{1-\alpha}} & \text{if } \alpha > 1/2 \end{cases}$$

and

$$u(f_{b_\alpha c_{1-\alpha}}, f_{ab_{1-\alpha}c_\alpha}) = \begin{cases} f_{ab_{1-\alpha}c_{1-\alpha}} & \text{if } \alpha \leq 1/2 \\ f_{ab_\alpha c_\alpha} & \text{if } \alpha > 1/2 \end{cases}$$

$$m_s(i(f_{b_\alpha c_{1-\alpha}}, f_{ab_{1-\alpha}c_\alpha})) = 2 \text{ and } m_s(u(f_{b_\alpha c_{1-\alpha}}, f_{ab_{1-\alpha}c_\alpha})) = 3.$$

Thus, $m_s(i(f_{b_\alpha c_{1-\alpha}}, f_{ab_{1-\alpha}c_\alpha})) + m_s(u(f_{b_\alpha c_{1-\alpha}}, f_{ab_{1-\alpha}c_\alpha})) = 2 + 3 = 5$. Hence, (X, τ, m_s) is a fuzzy measure space.

2. Define $m_c : \tau \to \mathbb{R}$ such that $m_c(f) = \int_X fdm$, where $m =$ counting measure on σ-algebra of subset of X.

Here, $f = \sum_{j=1}^{n} a_j \chi_{A_j}$ and $\int_X fdm = \sum_{j=1}^{n} a_j m(A_j)$, Where $a_j = f(x_j)$ for $j = 1, 2, 3$. $m(A_j) = m(\{x_j : f(x_j) = a_j\})$. Now consider,

$$f_{b_\alpha c_{1-\alpha}} = \alpha\chi_b + (1-\alpha)\chi_c$$

$$\therefore m_c(f_{b_\alpha c_{1-\alpha}}) = \int_X f_{b_\alpha c_{1-\alpha}} dm = \sum_{j=1}^{n} a_j m(A_j) = \alpha \cdot m(\{b\}) + (1-\alpha)m(\{c\})$$

$$= \alpha \cdot 1 + (1-\alpha) \cdot 1 = \alpha + 1 - \alpha = 1.$$

$$m_c(f_{ab_{1-\alpha}c_\alpha}) = m(\{a\}) + (1-\alpha)m(\{b\}) + (\alpha)m(\{c\}) = 1 + 1 - \alpha + \alpha = 2$$

$$m_c\{f_{ab_\alpha}\} = \int_X f_{ab_\alpha} dm = 1 \cdot m(\{a\}) + \alpha \cdot m(\{b\}) = 1 + \alpha$$

$$m_c\{f_{ac_{1-\alpha}}\} = 1 \cdot m(\{a\}) + (1-\alpha)m(\{c\}) = 1 + 1 - \alpha = 2 - \alpha$$

$$m_c\{f_{ab_\alpha}\} + m_c\{f_{ac_{1-\alpha}}\} = 1 + \alpha + 2 - \alpha = 3.$$

Now,

$$m_c(i(f_{ab_\alpha}, f_{ac_{1-\alpha}})) = m_c(f_a) = 1$$

$$m_c(u(f_{ab_\alpha}, f_{ac_{1-\alpha}})) = m_c(f_{ab_\alpha c_{1-\alpha}}) = 1 \cdot m(\{a\}) + \alpha \cdot m(\{b\}) + (1-\alpha)m(\{c\})$$

$$= 1 + \alpha + 1 - \alpha = 2.$$

Hence,

$$m_c(i(f_{ab_\alpha}, f_{ac_{1-\alpha}})) + m_c(u(f_{ab_\alpha}, f_{ac_{1-\alpha}})) = m_c(f_a) + m_c(f_{ac_{1-\alpha}}).$$

Hence, (X, τ, m_c) is a fuzzy measure space.

Example 4.2. Let $X = \{a, b, c\}$ and $\tau = \{f : X \to [0, 1]\}$. Let $\{t_n\}_{n=1}^{\infty}$ is a monotone sequence of real number in $[0, 1]$.

Then define $f(x) = \begin{cases} 1 & x = b, c \\ t_n & x = a \\ 0 & \text{otherwise.} \end{cases}$ Here, $f_{a_{t_n}}(x) = \begin{cases} t_n & x = a \\ 0 & otherwise. \end{cases}$

Therefore,

$$\tau = \{\phi, X, f_{at_n}, f_{a(1-t_n)}, f_{bc}, f_{at_nbc}, f_{a(1-t_n)}bc\}_{n=1}^{\infty}.$$

Here $(\tau, i, \sim)$ is a tribe of fuzzy subset of X, where i is the minimum t-norm i.e. standard fuzzy intersection and ~ is the fuzzy complement.

Define $m_c : \tau \to \mathbb{R}$ such that $m_c(f) = \int_X f dm$, where $m =$ counting measure on σ-algebra of subset of X.

Here, $f = \sum_{j=1}^{n} a_j \chi_{A_j}$ and $\int_X f dm = \sum_{j=1}^{n} a_j m(A_j)$, Where $a_j = f(x_j)$ for $j = 1, 2, 3$.

$$m(A_j) = m(\{x_j : f(x_j) = a_j\}).$$

Consider, $f_{a_{1/2}bc} = \left(\frac{1}{2}\right)\chi_a + (1) \cdot \chi_b + (1) \cdot \chi_c$

$$\therefore m_c(f_{a_{1/2}bc}) = \int_X f_{a_{1/2}bc} dm = \frac{1}{2} \cdot m(\{a\}) + m(\{b\}) + m(\{c\}) = \frac{1}{2} + 1 + 1 = \frac{5}{2}$$

$$m_c(f_{a_{1/3}bc}) = \int_X f_{a_{1/3}bc} dm = \frac{1}{3} \cdot m(\{a\}) + m(\{b\}) + m(\{c\}) = \frac{1}{3} + 1 + 1 = \frac{7}{3}.$$

Here, $i(f_{a_{1/2}bc}, f_{a_{1/3}bc}) = f_{a_{1/3}bc}$ and $u(f_{a_{1/2}bc}, f_{a_{1/3}bc}) = f_{a_{1/2}bc}$.

Hence,

$$m_c(i(f_{a_{1/2}bc},\ f_{a_{1/3}bc})) + m_c(u(f_{a_{1/2}bc},\ f_{a_{1/3}bc})) = m_c(f_{a_{1/2}bc}) + m_c(f_{a_{1/3}bc})$$

$$= \frac{5}{2} + \frac{7}{3} = \frac{29}{6}.$$

Therefore m_c is a valuation on τ.

Now consider,

$$(f_{a_{t_n}})_{n=1}^{\infty} = \sum_{n=1}^{\infty} t_n \chi_a$$

$$\therefore\ m_c((f_{a_{t_n}})_{n=1}^{\infty}) = \left(\int_X f_{a_{t_n}}\, dm\right)_{n=1}^{\infty} = \sum_{n=1}^{\infty} t_n m(\{a\})$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \cdot (1) = 1 + \frac{1}{2} + \frac{1}{3} + \dots \cdot \to \infty.$$

Now consider,

$$\{u(f_{a_{t_n}})_{n=1}^{\infty}\}(a) = u(f_{a_1}(a),\ f_{a_{1/2}}(a),\ \dots) = \max(1,\ 1/2,\ 1/3,\ \dots) = 1$$

$$\{u(f_{a_{t_n}})_{n=1}^{\infty}\}(b) = u(f_{a_1}(b),\ f_{a_{1/2}}(b),\ \dots) = \max(0,\ 0,\ 0,\ \dots) = 0$$

$$\{u(f_{a_{t_n}})_{n=1}^{\infty}\}(c) = u(f_{a_1}(c),\ f_{a_{1/2}}(c),\ \dots) = \max(0,\ 0,\ 0,\ \dots) = 0.$$

Hence, $\{u(f_{a_{t_n}})_{n=1}^{\infty}\}(x) = f_{a_1}(x)$. Therefore, $m_c\{u(f_{a_{t_n}})_{n=1}^{\infty}\} = m_c(f_{a_1}) = 1.$

Again, $\lim\limits_{M \to \infty} m_c\{u(f_{a_{t_n}})_{n=1}^{M}\} = m_c(f_{a_1}) = 1.$

Hence, $m_c\{u(f_{a_{t_n}})_{n=1}^{\infty}\} = \lim_{M \to \infty} m_c\{u(f_{a_{t_n}})_{n=1}^{M}\}$.

Similarly, we have $m_c\{i(f_{a_{t_n}})_{n=1}^{\infty}\} = \lim_{M \to \infty} m_c\{i(f_{a_{t_n}})_{n=1}^{M}\}$.

Again, $m_c\{u(f_{a_{t_n}}bc)_{n=1}^{\infty}\} = \lim_{M \to \infty} m_c\{u(f_{a_{t_n}}bc)_{n=1}^{M}\} = m_c(f_{a_1bc}) = 3$

$$m_c\{i(f_{a_{t_n}}bc)_{n=1}^{\infty}\} = \lim_{M\to\infty} m_c\{i(f_{a_{t_n}}bc)_{n=1}^{M}\} = m_c(f_{a_1 bc}) = 3.$$

Thus, (X, τ, m_c) is an infinite measure space.

5. Concluding Remarks

In the present paper we have studied tribes using Yager's t-norm and measure on tribes. Some new examples are constructed which form tribes and defined support measure and counting measure on it to form measure space.

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