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SOME RESULTS ON DOMINATION NUMBER OF EFFICIENTLY DOMINATABLE NEIGHBOURHOOD CONTRACTED GRAPHS

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Abstract

A dominating set S of a graph G is an efficient dominating set (EDS) if $|N[v] \cap S| = 1$, for all $v \in V(G)$. A graph is efficiently dominatable if it has an EDS. Not all graphs are efficiently dominatable. The neighbourhood contracted graph G_v of G, with respect to the vertex v, is the graph with vertex set $V - N(v)$, where two vertices $u, w \in V - N(v)$ are adjacent in G_v such that one of the following conditions hold: (i) $w = v$ and u is adjacent to any vertex of $N(v)$ in G. (ii) $u, w \notin N[v]$ and u, w are adjacent in G. In this paper, we study a few efficiently dominatable neighbourhood contracted graphs. In particular, we consider efficiently dominatable graphs whose domination number is $\frac{n}{2}$ and derive characterization for its neighbourhood contracted graph to be efficiently dominatable.

1. Introduction

In this paper, graphs considered here are finite, simple, undirected and connected, unless otherwise mentioned. For terminologies and notations not defined here, the reader is referred to [6, 18].

Let G be a graph with vertex set V and edge set E. Let $u, v \in V(G)$ and $S \subseteq V(G)$. With these notations, we define the basic terminologies as below:

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$N(u) = \{v \in V(G) : uv \in E(G)\}$ and $N[u] = N(u) \cup \{u\}$. The eccentricity of u in G, denoted by $ecc_G(u)$, is the distance between u and a vertex farthest from u. The maximum and the minimum of the eccentricities of all vertices in G are referred to as the diameter $(diam(G))$ and radius $(rad(G))$ of G respectively. A vertex v with $ecc(v) = rad(G)$ is a central vertex. A set S is a 2-packing if for each pair of vertices $u, v \in S$, $N[u] \cap N[v] = \emptyset$.

Definition 1.1 [6]. A dominating set $S \subseteq V(G)$ is an efficient dominating set (EDS) of G if $|N[u] \cap S| = 1$, for all $u \in V(G)$.

A graph G is said to be efficiently dominatable if G has an EDS. Those graphs that have an efficient dominating set include Path P_n, for all n, Complete graph K_n, for all n and Complete bipartite graph $K_{m,n}$, if and only if $m = 1$ or $n = 1$. Not all graphs are efficiently dominatable. For example, the cycle C_4 is not efficiently dominatable.

The concept of neighbourhood contracted graph was introduced by S. S. Kamath and studied in [8].

Definition 1.2 [8]. Let G be a graph and let v be any vertex of G. Then the neighbourhood contracted graph G_v of G, with respect to the vertex v, is the graph with vertex set $V - N(v)$, where two vertices $u, w \in V - N(v)$ are adjacent in G_v such that one of the following conditions hold:

1. $w = v$ and u is adjacent to any vertex of $N(v)$ in G.

2. $u, w \notin N[v]$ and u, w are adjacent in G.

In the literature, the terminology "efficient domination" was introduced by Bange et al. [1]. They have proved that, "if a graph G has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of G. In particular, all efficient dominating sets have the same cardinality".

The study of efficient domination in graphs and in special classes of graphs are widely studied in the literature, both theoretically and from algorithmic perspectives. Efficient domination has been studied on general graphs [1, 14, 15] and on some special classes of graphs like Co-comparability graphs [4], Permutation graphs and Trapezoid graphs [7], Serpinski graphs [10], Cubic Vertex-transitive graphs [11]. The study on efficient domination is in progress using eigenvalues by [3]. Structure of squares and efficient domination in graph classes has been studied by [9]. Super-efficient graphs are studied by [13]. In [12], some examples and results on efficient domination partitionable graph are presented. The critical concepts with respect to efficient domination, like vertex removal, edge removal/addition, were introduced and studied in [14, 16].

Throughout this paper, we use the notation $\mathcal{E}$ to denote the class of efficiently dominatable graphs which was introduced by Thilak and studied in [14].

Efficient domination stands unique among other variants of domination because of its three properties: Domination, independence and 2-packing. Efficient domination has its wide applications in communication networks, mobile ad hoc networks, coding theory, fault tolerance analysis, wireless sensor networks.

This paper initiates the concept of efficient domination in neighbourhood contracted graphs. A few efficiently dominatable neighbourhood contracted graphs are identified and properties are studied.

2. Efficient Domination and Neighbourhood Contraction in Graphs

Not all graphs are efficiently dominatable. Thus, for any vertex $v \in V(G)$, G_v may or may not be efficient. In G_v the $N[v]$ vertices are contracted to form a new vertex. Throughout this paper, we use the notation v^* to denote the vertex obtained by contracting $N[v]$ vertices in G_v. In other words, we use v^* notation to represent the vertex v in G_v.

The following is evident from the definition of neighbourhood contracted graphs: For any vertex $v \in V(G)$,

1. $|V(G_v)| = |V(G)| - |N(v)|$.

2. $|E(G_v)| = |E(G)| - |\langle N(v) \rangle|$.

Observation 2.1.

1. If v is a pendant vertex in G, then $G_v \cong G - v$. Thus, $G_v \in \mathcal{E}$ if and only if $G - v \in \mathcal{E}$.

2. Let T be a tree of diameter three and let a and b be its two central vertices. Let p be the number of pendant vertices adjacent to vertex a and q be the pendant vertices adjacent to vertex b. Then, $G_a \cong K_{q+1}$ and $G_b \cong K_{p+1}$. Thus, $G_a \in \mathcal{E}$ and $G_b \in \mathcal{E}$.

3. [8] $(P_n)_v = \begin{cases} P_{n-1} & \text{if } \deg(v) = 1 \\ P_{n-1} & \text{if } \deg(v) = 2 \end{cases}$

Thus, $(P_n) \in \mathcal{E}$, for all $v \in V(G)$.

4. [8] $(K_m)_v \in K_1$, for all $v \in V(G)$.

Thus, $(K_m)_v \in \mathcal{E}$, for all $v \in V(G)$.

5. [8] $(K_{1,m})_v = \begin{cases} K_1 & \text{if } v \text{ is the central vertex} \\ K_{1,m-1} & \text{if } v \text{ is the Pendant vertex} \end{cases}$

Thus, $(K_{1,m})_v \in \mathcal{E}$, for all $v \in V(G)$.

6. [8] $(C_n)_v \in C_{n-2}$.

We know that $C_{n-2} \in \mathcal{E}$ if and only if $n \equiv 2 (\bmod 3)$. Thus, it follows that if C_n is efficiently dominatable, then $(C_n)_v$ is not efficiently dominatable and vice-versa.

Proposition 2.1. *Let* $G \in \mathcal{E}, \gamma(G) \geq 2$ *and* $G_u \in \mathcal{E}$, *for any* $u \in V(G)$. *Then,* $\gamma(G_u) < \gamma(G)$.

Proof. Let S be an EDS of G. For any $u \in V(G)$, let $G_u \in \mathcal{E}$.

Case (i). Let $u \in S$.

Then, there exist a vertex $v \in S$ such that $d_G(u, v) = 3$. In G_u, $d_{G_u}(u^*, v) = 2$.

Case (ii). Let $u \notin S$.

Let $u \in N(w)$ for some $w \in S$. Also, there exists a vertex $v \in S$ such that $d_G(w, v) = 3$.

Thus, in G_u, $d_{G_u}(w, v) = 1$.

Thus, in all the cases, $\gamma(G_u) < \gamma(G)$.

Observation 2.2 [8]. G_v is a trivial graph if and only if $\deg(v) = n - 1$ in G.

Proposition 2.2. *Let $G \in \mathcal{E}$, $\gamma(G) = k$ and S be an EDS of G. If for any $\deg_G(v) = n - \gamma(G) - 1$, then $G_v \in \mathcal{E}$.*

Proof. Let $\deg_G(v) = n - \gamma(G) - 1$, for any $v \in V(G) - S$. Then, for all $u \in S$ and $w \in N(u)$, $d_G(v, w) = 1$. Thus, $G_v \cong K_{1, k-1}$ and hence $G_u \in \mathcal{E}$ and $\gamma(G_u) = 1$.

Theorem 2.3. *Let G be a graph of diameter at most two. Then, for any $u \in V(G)$, $G_u \in \mathcal{E}$ and $\gamma(G_u) = 1$.*

Proof. Let $diam(G) = 1$. Then, $G \in \mathcal{E}$ and $\gamma(G) = 1$. That is, there exist a vertex $v \in V(G)$ such that $\deg_G(v) = n - 1$.

Case (i). $v \in V(G)$ and $\deg_G(v) = n - 1$.

Then, by Observation 2.2, it follows that $G_v \cong K_1$. Thus, $G_v \in \mathcal{E}$ and $\gamma(G) = 1$,

Case (ii). $u \in V(G)$ such that $u \neq v$

Then, for any vertex $u \in V(G)(u \neq v)$, $d_G(u, v) = 1$. Thus, in G_u, all the vertices $V - N[u]$ are adjacent to $u^*(= v)$. Thus, $ecc_{G_v}(u^*) = 1$ and hence $G_u \in \mathcal{E}$ and $\gamma(G_u) = 1$.

Suppose that $diam(G) = 2$. Then, the eccentricities of all the vertices in G are either one or two.

Case (i). Let w be a vertex, where $ecc(w) = 1$.

Then, G is efficiently dominatable and $S = \{w\}$ is an EDS of G. It follows from the Observation 2.2 that $G_w \cong K_1$. and thus $G_w \in \mathcal{E}$ with $\gamma(G_w) = 1$.

Case (ii). Let $u \in V(G)$ be such that $ecc(u) = 2$.

In G_u, all the $N_2(u)$ in G are now adjacent to u^* in G_u and hence $\deg_{G_u}(u^*) = n - 1$. Then, $G_u \in \mathcal{E}$ with $S = \{u^*\}$ as an EDS. Hence the result.

Thus, in all the cases, for any $u \in V(G)$, G_u is efficient.

2.1 Efficiently dominatable Graphs whose domination number is $\frac{n}{2}$ (*n* even)

In this section, we consider efficiently dominatable graphs whose $\gamma(G) = \frac{n}{2}$, where n is even.

Proposition 2.4. *Let $G \in \mathcal{E}$ and S be an EDS of G. Let $\gamma(G) = \frac{n}{2}$, for n even. For any $u \in S$, $G_u \in \mathcal{E}$ if and only if $G - u \in \mathcal{E}$.*

Proof. As $\gamma(G) = \frac{n}{2}$ and n is even, $\deg_G(u) = 1$, for any $u \in S$. Then, $G_u \cong G - u$ and hence $G_u \in \mathcal{E}$ if and only if $G - u \in \mathcal{E}$. □

Theorem 2.5. *Let $G \in \mathcal{E}$ and S be its EDS. $\gamma(G) = \frac{n}{2}$, where n is even and $n \geq 2$.* Then, for any $v \in V(G) - S$, $G_v \in \mathcal{E}$ if and only if $\deg_{G_v}(u) \leq 2$ or $\deg_{G_v}(v) = \gamma(G)$.

Proof. Let $v \in V(G) - S$ be an arbitrary vertex. Let us assume that $\deg_G(u) \leq 2$ or $\deg_G(v) = \gamma(G)$. If $\deg_G(v) = 1$, then $G_v \cong K_2$. Hence $G_v \in \mathcal{E}$. Suppose that $\deg_G(v) = 2$. Then there exist $u \in S$ such that $d_G(u, v) = 2$. In G_v, $d(u, v^*) = 1$ and hence v^* is efficiently dominated by u. Thus, $G_v \in \mathcal{E}$ and $\gamma(G_v) = \gamma(G) - 1$. Suppose that $\deg_G(v) = \gamma(G)$. Then,

$d_G(v, u) = 2$, for all $u \in S$. Hence, $G_v \cong K_{1, \gamma-1}$ and thus $G_v \in \mathcal{E}$ and $\gamma(G_v) = 1$. Thus, if $\deg_G(v) \leq 2$ or $\deg_G(v) = \gamma(G)$ then $G_v \in \mathcal{E}$.

Conversely, for any $v \in V(G) - S$, let $G_v \in \mathcal{E}$. Clearly, $1 \leq \deg_G(v) \leq \gamma(G)$. Suppose that $2 < \deg_G(v) < \gamma(G)$. Since $\deg_G(v) \geq 3$, the set $S_1 = \{u \in S : d_G(u, v) = 2\}$ is non-empty. Let $S_2 = S - S_1$. Then $d_G(u, v) \geq 3$, for all $u \in S_2$. Clearly, $d_{G_v}(v^*, u) = 1$, for any $u \in S_1$. Since $\deg_G(u) = 1$, for all $u \in S$, it follows that v^* must be included in any EDS of G_v. As $d_{G_v}(v^*, u') = 1$, at least for one $u' \in S_2$, u is left undominated efficiently in G_v, a contradiction. Hence, if $G_v \in \mathcal{E}$, then $\deg_G(v) \leq 2$ or $\deg_G(v) = \gamma(G)$. □

3. Conclusions

This paper focuses on the concept of efficient domination in neighbourhood contracted graphs. A few efficiently dominatable neighbourhood contracted graphs are identified and properties are studied. In particular, we have considered efficiently dominatable graphs whose domination number is $\frac{n}{2}$ and have derived characterization for its neighbourhood contracted graph to be efficiently dominatable. This study is in its initial stage and hence can be extensively studied in terms of various graph parameters.

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