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A NOTE ON *t*-DERIVATIONS OF BH-ALGEBRAS

P. GANESAN[1] and N. KANDARAJ[2]

[1]Research Scholar, PG and Research
Department of Mathematics
Saiva Bhanu Kshatriya College
Aruppukottai-626 101, Tamil Nadu, India

[2]Associate Professor, PG and Research
Department of Mathematics
Saiva Bhanu Kshatriya College
Aruppukottai-626 101, Tamil Nadu, India
E-mail: n.kandarajsbkc1998@gmail.com

Abstract

The notion of BCK-algebra was proposed by Imai and Iseki in 1966. In the same year Iseki introduced the notion of a BCI-algebras, which is generalization of a BCK-algebra. Y. B. Jun, E. H. Roh and H. S. Kim defined the notion of BH-algebra. Motivated by some results on derivations on rings and the generalizations of BCK and BCI-algebras. In 2019, P. Ganesan and N. Kandaraj introduced the notion of derivations on BH-algebras. In this paper, we study the notion of t-derivations on BH-algebras and investigate simple, interesting and elegant results.

1. Introduction

Imai Y. and Iseki K. [7, 8] introduced the on axiom system of propositional calculi and have been extensively investigated by many researchers. Iseki K. and Tanaka S. [9] introduced the theory of BCK-algebra. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Y.B. Jun, E.H. Roh and H.S. Kim [12] introduced the notion of BH-algebras. They investigated several relations between BH-algebras and BCK-algebras. In 1957, Posner [18] introduced the notion of derivations in Prime rings theory. Also Lee P. H. and Lee T. K. [16] developed on derivations of prime rings.

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The notion of derivation in ring theory is quite old and plays an important role in algebra. Al-Shehri N.O. and Bawazeer S. M. [4] introduced the notion of derivations of BCC algebras Many Research papers have appeared on the derivations of BCI-algebras in different ways. Zhan J and Liu Y. L. [19] developed the notion of f-derivations on BCI-algebras. Muhiuddin G and Abdullah Al-roqi M [17] introduced on t-derivations of BCI-algebras. Recently, in the year 2019 Ganesan p and Kandaraj N. defined and studied the notion of derivations, Compositions of derivations and f-derivations of BH-algebras using the idea of regular derivations in BH-algebras and obtained some of its properties. The term algebra is used here to denote the algebraic structure defined on a non-empty set with a binary composition satisfying certain laws that resemble the algebra of logic but not the usual algebra.

The notion of the derivations is the same as that in ring theory and the usual algebraic theory. Motivated by a lot of Work done on derivations of BH-algebras and on derivations of other related abstract algebraic, structures such as BCI, TM, and d-algebras. In this paper we introduce the notion of t-derivations and show that if θ_t and θ'_t are t-derivations on U, then $(\theta_t \circ \theta'_t)$ is also a t-derivation on BH-algebra U and $(\theta_t \circ \theta'_t) = (\theta'_t \circ \theta_t)$. Finally we prove that $(\theta'_t * \theta_t) = (\theta_t * \theta'_t)$ where θ_t and θ'_t are t-derivations of BH-algebras.

2. Preliminaries

We review some basic definitions and properties that will be useful in our results.

Definition 2.1 [12]**.** Let X be a set X with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a BH-algebra, if it satisfies the following axioms

(1) $x * x = 0$

(2) $x * 0 = x$

(3) If $x * y = 0$ and $y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.

Define a binary relation $\leq$ on X by taking $x \leq y$ if and only if $x * y = 0$. In this case $(X, \leq)$ is a partially ordered set [ ]

Let $(X, *, 0)$ be a BH-algebra and $x \in X$. Define $x * X = \{x * y \mid y \in X\}$.

Then X is said to be edge BH-algebra if for any $x \in X$, $x * X = \{x, 0\}$

Definition 2.2 [12]. Let S be a nonempty subset of a BH-algebra X. Then S is called Sub algebra of X, if $x * y \in S$ for all $x, y \in S$.

Definition 2.3 [12]. Let X be a BH-algebra and $I(\neq 0) \subseteq X$. Then I is called a BH-ideal of X if

(1) $0 \in I$

(2) $x * y \in I$ and $y \in I \Rightarrow x \in I$ for all $x, y \in I$

In BH-algebra X for all $x \cdot y, z \in x$, the following Property hold [ ]

1. $((x * y) * (x * z)) * (z * y) = 0$

2. $(x * y) * x = 0$

3. $(x * (x * y)) = y$

Theorem 2.4 [11]. *Every BH-algebra satisfying the condition* (1) *is a BCI-algebra and satisfying the conditions* (1) *and* (2) *is a BCK-algebra.*

Theorem 2.5 [20]. *Every BH-algebra satisfying the condition*

$(x * y) * z = (x * z) * y$ *for all* $x, y, z \in X$ *is a BCH-algebra.*

For a BH-algebra X, we denote $x \wedge y$ *for* $y * (y * x)$, $\forall x, y \in X$.

3. Composition of *t*-Derivations on BH-algebras

Definition 3.1. Let U be a BH-algebra and θ_t, θ'_t be two self-maps of U. we define $\theta_t \circ \theta'_t : U \to U$ such that $(\theta_t \circ \theta'_t)(u) = \theta_t, (\theta'_t(u))$ for all $u \in U$.

Example 3.2. Let $U = \{0, a, b, c\}$ be a BH-algebra with the following Cayley table.

*	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Let $\theta_t : U \to U$ such that $\theta_t(x) = \begin{cases} c & \text{if } u = 0, a \\ 0 & \text{if } u = b, c \end{cases}$

Let $\theta'_t : U \to U$ such that $\theta'_t(x) = 0 \, \forall \, u = 0, a, b, c$

Then $(\theta_t \circ \theta'_t)(u) = \theta_t(\theta'_t(u)) \, \forall u \in U$.

Proposition 3.3. *Let U be a BH-algebra and* θ_t, θ'_t be (l, r)-*t-derivations on U. Then the composition of* θ_t *and* θ'_t *is a left-right t-derivations on U.*

Proof. Let U be a BH-algebra and θ_t, θ'_t be (l, r)-t-derivations on U.

Since θ_t, θ'_t are left-right-t-derivations on U.

$\theta_t(u * v) = (\theta_t(u) * v) \wedge (u * \theta_t(v)) \, \forall \, u, v \in U$

Therefore $\theta_t(u * v) = (\theta_t(u) * v)$

To claim: $(\theta_t \circ \theta'_t)$ is a left-right t-derivation on U.

(i.e.) $(\theta_t \circ \theta'_t)(u * v) = (\theta_t \circ \theta'_t)(u) * v$

$$
\begin{aligned}
(\theta_t \circ \theta'_t)(u * v) &= (\theta_t(\theta'_t(u * v))) \\
&= (\theta_t(\theta'_t(u) * v)) \\
&= (\theta_t(\theta'_t(u)) * v)) \\
&= (\theta_t(\theta'_t(u))) * v \\
&= (\theta_t \circ \theta'_t)(u) * v. \text{ Hence the result}
\end{aligned}
$$

Proposition 3.4. *Let U be a BH-algebra and* θ_t, θ'_t *be right-left t-derivations on U. Then the composition of* θ_t *and* θ'_t *is a right left t-derivation on U.*

Proof. Let θ_t, θ'_t be two right-left t-derivations on U.

$\theta_t(u * v) = (u * \theta_t(v)) \wedge (\theta_t(u) * v)\ \forall\, u, v \in U$

Therefore $\theta_t(u * v) = (u * \theta_t(v))$

$\theta'_t(u * v) = (u * \theta'_t(v)) \wedge (\theta'_t(u) * v)$ for all $u, v \in U$

Therefore $\theta'_t(u * v) = (u * \theta'_t(v))$

Claim: $(\theta_t \circ \theta'_t)$ is a right-left t-derivations on U.

(i.e.) $(\theta_t \circ \theta'_t)(u * v) = u * (\theta_t \circ \theta'_t)(v)$

$$\begin{aligned}(\theta_t \circ \theta'_t)(u * v) &= \theta_t(\theta'_t(u * v))\\ &= \theta_t(u * \theta'_t(v))\\ &= u * \theta_t(\theta'_t(v))\\ &= u * (\theta_t \circ \theta'_t)(v)\end{aligned}$$

$(\theta_t \circ \theta'_t)$ is a right-left t-derivation on U.

From the above two propositions we get the following theorem.

Theorem 3.5. *Let U be a BH-algebra and θ_t, θ'_t be two t-derivations on U, then the composition of θ_t and θ'_t is also a t-derivation on U.*

Theorem 3.6. *Let U be a BH-algebra and θ_t be right-left t-derivation on U. Let θ'_t be a left -right t-derivation on U. Then $(\theta_t \circ \theta'_t) = (\theta_t \circ \theta'_t)$.*

Proof. Let θ'_t be left-right t-derivation on U.

$\theta'_t(u * v) = (\theta'_t(u) * v) \wedge (u * \theta'_t(v))$ for all $u, v \in U$

Therefore $\theta'_t(u * v) = (\theta'_t(u) * v)$

Let θ_t be right-left t-derivation on U.

$\theta_t(u * v) = (u * \theta_t(v)) \wedge (\theta_t(u) * v)$ for all $u, v \in U$

Therefore $\theta_t(u * v) = (u * \theta_t(v))$

Now $(\theta_t \circ \theta'_t)(u * v) = \theta_t(\theta'_t(u * v))$

$$= \theta_t(\theta'_t(u) * v)$$

$$= \theta'_t(u) * (\theta_t(v)), \ldots 1$$

Similarly $(\theta'_t \circ \theta_t)(u * v) = \theta'_t(\theta_t(u * v))$

$$= \theta'_t(u * \theta_t(v))$$

From 1 and 2 $(\theta_t \circ \theta'_t)(u * v) = (\theta'_t \circ \theta_t)(u * v)$

Hence $(\theta_t \circ \theta'_t) = (\theta'_t \circ \theta_t)$.

Definition 3.7. Let U be a BH-algebra and θ_t, θ'_t be two self-maps on U. We define $\theta_t * \theta'_t : U \to U$ such that $(\theta_t * \theta'_t)(u) = \theta_t(u) * \theta'_t(u)$ for all $u \in U$

Example 3.8. Let $U = \{0, a, b, c\}$ be a BH-algebra with the following Cayley table

*	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	b
c	c	c	c	0

Let $\theta_t : U \to U$ such that $\theta_t(u) = c \ \forall \ u = 0, a, b, c$.

$$\theta'_t(u) = 0 \ \forall \ u = 0, a, b, c.$$

Define $(\theta_t * \theta'_t) : U \to U$ such that $(\theta_t * \theta'_t)(u) = c$ for all $u = 0, a, b, c$

Therefore $(\theta_t * \theta'_t)(u) = \theta_t(u) * \theta'_t(u)$ for all $u \in U$.

Theorem 3.9. *Let X be a BH-algebra and* θ_t, θ'_t *be t-derivations on U. Then* $(\theta_t * \theta'_t)(u) = \theta_t(u) * \theta'_t(u)$

Proof. Let θ'_t be left-right t-derivation on U.

Now $(\theta_t \circ \theta'_t), (u * v) = \theta_t(\theta'_t(u * v))$

$$= \theta_t(\theta'_t(u) * v)$$

$$= \theta'_t(u) * \theta_t(v)$$

There are $(\theta_t \circ \theta'_t)(u * v) = \theta'_t(u) * \theta_t(v)), \ldots 1$

Again $(\theta_t \circ \theta'_t)(u * v) = \theta_t(\theta'_t(u * v))$

$$= \theta_t(u * \theta'_t(v))$$

$$= \theta_t(u) * \theta'_t(v)$$

Therefore $(\theta_t \circ \theta'_t)(u * v) = \theta'_t(u) * \theta'_t(v), \ldots 2$

From equations 1 and 2, we have

$$\theta'_t(u) * \theta_t(v) = \theta_t(u) * \theta'_t(v)$$

Replacing v by u, we have

$$\theta'_t(u) * \theta_t(u) = \theta_t(u) * \theta'_t(u)$$

$$(\theta'_t * \theta'_t)(u) = (\theta_t * \theta'_t)(u) \text{ for all } u \in U.$$

Hence we have $(\theta'_t * \theta_t) = (\theta_t * \theta'_t)$.

Definition 3.10. Let $L_t\mathrm{Der}(U)$ denote the set of all left-right t-derivations of a BH-algebra U. Define the binary operation $\wedge$ on $L_t\mathrm{Der}(U)$ as given below.

For θ_t, θ'_t belongs to $L_t\mathrm{Der}(U)$. We define $(\theta_t \wedge \theta'_t)(u) = \theta_t(u) \wedge \theta'_t(u)$ for all $u \in U$.

Lemma 3.11. *If θ_t, θ'_t be left-right t-derivations of U, then $(\theta_t \wedge \theta'_t)$ is also a left right t-derivations of U.*

Proof. Let U be a BH-algebra and θ_t, θ'_t be left-right t-derivations on U.

$\theta_t(u * v) = \theta_t(u) * v$. Since (l, r) t-derivations and $\theta'_t(u * v) = \theta'_t(u) * v$

Claim: $(\theta_t \wedge \theta'_t)(u * v) = ((\theta_t \wedge \theta'_t)(u)) * v$

Now $(\theta_t \wedge \theta'_t)(u * v) = \theta_t(u * v) \wedge \theta'_t(u * v)$

$$= (\theta_t(u) * v) \wedge (\theta'_t(u) * v)$$

$$= (\theta'_t(u) * v) * ((\theta'_t(u) * v) * (\theta_t(u) * v))$$

$$= (\theta'_t(u) * v)$$

$(\theta_t \wedge \theta'_t)(u * v) = (\theta'_t(u) * v), \ldots 1$

$((\theta_t \wedge \theta'_t)(u) * v = (\theta_t(u) \wedge \theta'_t(u)) * v$

$$= (\theta'_t(u) * (\theta'_t(u) * \theta_t(u))) * v$$

$$= \theta_t(u) * v$$

$((\theta_t \wedge \theta'_t)(x)) * v = \theta_t(u) * v), \ldots 2$

Using the equations 1 and 2 we have

$$(\theta_t \wedge \theta'_t)(u * v) = (\theta_t \wedge \theta'_t)(u) * v$$

Hence $(\theta_t \wedge \theta'_t)$ is a left-right t-derivation on U.

Lemma 3.12. *The binary composition* $\wedge$ *defined on* $L_t\mathrm{Der}(U)$ *is associative.*

Proof. θ_t, θ'_t, θ''_t be left-right t-derivations on BH-algebra U.

Claim: $(\theta_t \wedge \theta'_t) \wedge \theta''_t = \theta_t \wedge (\theta'_t \wedge \theta''_t)$

Now $((\theta_t \wedge \theta'_t) \wedge \theta''_t)(u * v) = (\theta_t \wedge \theta'_t)(u * v) \wedge \theta''_t(u * v)$

$$= (\theta_t(u) * v) \wedge (\theta''_t(u) * v)$$

$$= \theta_t(u) * v, \ldots 1$$

Also $\theta_t \wedge (\theta'_t \wedge \theta''_t))(u * v) = \theta_t(u * v) \wedge (\theta'_t \wedge \theta''_t)(u * v)$

$$= \theta_t(u) * v) \wedge (\theta'_t(u) * v)$$

$$= (\theta'_t(u) * v) * ((\theta'_t(u) * v) * (\theta_t(u) * v))$$

$$= \theta_t(u) * v, \ldots 2$$

From the results 1 and 2

$$((\theta_t \wedge \theta'_t) \wedge \theta''_t)(u * v) = (\theta_t \wedge (\theta'_t \wedge \theta''_t))(u * v)$$

put $v = 0$ in the above equation, we have

$$((\theta_t \wedge \theta'_t) \wedge \theta''_t)(u) = (\theta_t \wedge (\theta'_t \wedge \theta''_t))(u)$$

Equating the operator $(\theta_t \wedge \theta'_t) \wedge \theta''_t = \theta_t \wedge (\theta'_t \wedge \theta''_t)$

Hence the lemma.

From the above two lemma we get the following theorem

Theorem 3.13. *$L_t Der(u)$ is a semi group under the binary operation $\wedge$ which is defined by $(\theta_t \wedge \theta'_t)(u) = \theta_t(u) \wedge \theta'_t(u)\ \forall\ u, v \in U$ and θ_t, θ'_t belongs to $L_t Der(u)$.*

Definition 3.14. Let $R_t Der(u)$ denote the set of all right left t-derivations on BH-algebras U. Define the operation $\wedge$ on $R_t Der(u)$ as given below. For $\theta_t, \theta'_t \in R_t \mathrm{Der}(u)$, we define $(\theta_t \wedge \theta'_t)(u) = \theta_t(u) \wedge \theta'_t(u)\ \forall\ u \in U$.

Note:

Analogously we can prove the following result $R_t Der(u)$ is a semi group under the binary operation $\wedge$ which is defined by $(\theta_t \wedge \theta'_t)(u) = \theta_t(u) \wedge \theta'_t(u)\ \forall\ u \in U$. and θ_t, θ'_t belongs to $R_t Der(u)$.

From the above two results we have

Theorem 3.15. *If $tDer(u)$ denotes the set of all t-derivations on U, then it is a semigroup under the binary operation $\wedge$ defined by $(\theta_t \wedge \theta'_t)(u) = \theta_t(u) \wedge \theta'_t(u)\ \forall\ u \in U$ and $\theta_t, \theta'_t \in tDer(u)$.*

4. Conclusion

An algebraic structure that arises from the study of algebraic formulations of propositional logic. Taking different theorems or statements of propositional logic, different algebraic structures could be obtained. The BH - Algebra is one such algebra. The derivation concept is an important and very interesting area of research in the theory of algebraic Structures in Mathematics. The deep theory has been developed for derivations through various algebras. It plays an important role in algebra, algebraic geometry and linear differential equations.

We have considered the concept of t-derivations in BH-algebras. Finally, we investigated the notion of the composition of t-derivations in BH-algebras. In our opinion these definitions and main results may be similarly extended to some other algebras such as BCI-algebras [1, 2, 10, 13], d-algebras [5, 6, 14, 15] and B-algebras [3] so forth. In future any researcher can study the notion of t-derivations in different algebraic structures which may have a lot of applications in various fields. This work is a foundation for the further study of the researcher on derivations of algebras.

The future study of derivations on BH-algebras may be the following topics should be covered.

(a) To find the generalized derivations on BH-algebras.

(b) To find the t-derivations of Q-algebras, d-algebras, B-algebras and so on so.

(c) To find more results and its applications in t-derivations on BH-algebras.

(d) To find to investigate how these concepts could be applied to the field of computers for processing information.

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