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ON $g^{*}\alpha$-HOMEOMORPHISM IN TOPOLOGICAL SPACES

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Abstract

This paper focuses on new class of homeomorphism called $g^{*}\alpha$- homeomorphism which are stronger than homeomorphism. We also introduce $g^{*}\alpha c$- homeomorphism and prove that $g^{*}\alpha c$- homeomorphism form a group under the operation of composition of maps.

1. Introduction

The notion homeomorphism plays more vital role in topology. Maki [5] et al., introduced and investigated g-homeomorphism and gc-homeomorphisms. Devi [2] et al. introduced and studied sg-homeomorphism and gs-homeomorphisms.

In this paper we introduce the new class of homeomorphisms named as $g^{*}\alpha$- homeomorphism and study their properties. Also, we introduce another form of homeomorphism namely $g^{*}\alpha c$- homeomorphism and its properties under composition of functions are analyzed.

2. Preliminaries

Definition 2.1. A map $f : (X, \tau) \to (Y, \sigma)$ is called $g^{*}\alpha$- continuous [6] if $f^{-1}(V)$ is $g^{*}\alpha$- closed in (X, τ) for every closed set V in (Y, σ).

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Definition 2.2. A map $f : (X, \tau) \to (Y, \sigma)$ is called $g^*\alpha$-irresolute [6] if $f^{-1}(V)$ is $g^*\alpha$-closed in (X, τ) for every $g^*\alpha$-closed set V in (Y, σ).

Definition 2.3. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called *g*-homeomorphism [5] if *f* and f^{-1} are *g*-continuous.

Definition 2.4. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called *gpr*-homeomorphism [3] if *f* and f^{-1} are *gpr*-continuous.

Definition 2.5. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called *gsp*-homeomorphism [2] if *f* is *gsp*-continuous and *gsp*-open.

Definition 2.6. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called *gp*-homeomorphism [3] if *f* is *gp*-continuous and *gp*-open.

Definition 2.7. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called *g*^-homeomorphism [7] if f is *g*^-continuous and *g*^-open.

3. $g^*\alpha$- Homeomorphism

We introduce the following definition.

Definition 3.1. A bijective function $f : (X, \tau) \to (Y, \sigma)$ is called $g^*\alpha$-homeomorphism if *f* is both $g^*\alpha$-continuous and $g^*\alpha$-open function.

Example 3.2. Consider $X = Y = \{1, 2, 3, 4\}$ with topologies $\tau = \{\varphi, X, \{1\}, \{1, 3\}\}$ and $\sigma = \{\varphi, Y, \{3\}, \{2, 3\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by $f(1) = 4$, $f(2) = 3$, $f(3) = 2$ and $f(4) = 1$. Here f is bijective, $g^*\alpha$-continuous and $g^*\alpha$-open. Hence *f* is $g^*\alpha$-homeomorphism.

Proposition 3.3. *Every homeomorphism is a* $g^*\alpha$- *homeomorphism.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism. Then *f* is bijective, continuous and an open map. Since every continuous map is $g^*\alpha$-continuous and every open map is $g^*\alpha$-open, *f* is $g^*\alpha$-homeomorphism.

The converse of the above theorem need not be true as it can be seen from the above example 3.2.

Proposition 3.4. *Every g-homeomorphism is a* $g^*\alpha$- *homeomorphism*.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ g-be a homeomorphism. Then f is bijective, g-continuous and an g-open map. Let V be a closed set in (Y, σ). Then $f^{-1} : (V)$ is $g^*\alpha$- closed in (X, τ). This implies that f is $g^*\alpha$- continuous. Let U be an open set in (X, τ). Then $f(U)$ is g-open in (Y, σ). Hence f is $g^*\alpha$- open function. Therefore, f is $g^*\alpha$- homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 3.5. Consider $X = Y = \{1, 2, 3\}$ with topologies $\tau = \{\varphi, X, \{1\}, \{1, 2\}\}$ and $\sigma = \{\varphi, Y, \{1\}, \{1, 3\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is $g^*\alpha$- homeomorphism but not g-homeomorphism, Since for the closed set $\{2\}$ in Y, $f^{-1}(\{2\}) = \{2\}$ is not g-closed in X.

Proposition 3.6. *Every g^-homeomorphism is a* $g^*\alpha$- *homeomorphism*.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ g^-be a homeomorphism. Then f is bijective, g^-continuous and an g^-open map. Let V be a closed set in (Y, σ). As every g^-closed set is $g^*\alpha$- closed, $f^{-1}(V)$ is $g^*\alpha$- closed in (X, τ). This implies that f is $g^*\alpha$- continuous. Let U be an open set in (X, τ). Then $f(U)$ is g^-open in (Y, σ). since every g^-open set is $g^*\alpha$- pen, $f(U)$ is $g^*\alpha$- open in (X, τ). Therefore, f is $g^*\alpha$- homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 3.7. Consider $X = Y = \{1, 2, 3, 4\}$ with topologies $\tau = \{\varphi, X, \{3\}, \{4\}, \{3, 4\}\}$ and $\sigma = \{\varphi, Y, \{1\}, \{3, 4\}, \{1, 3, 4\}\}$. Let

$f : (X, \tau) \to (Y, \sigma)$ be defined as $f(1) = 2, f(2) = 3, f(3) = 1$ and $f(4) = 4$. Here f is $g^*\alpha$- homeomorphism but not g^-homeomorphism, since for the closed set $f^{-1}(\{1, 2\}) = \{1, 3\}$ is not g^-closed in X.

Theorem 3.8. *Every* $g^*\alpha$- *homeomorphism is a gsp-homeomorphism.*

Every $g^*\alpha$- *homeomorphism is a gp-homeomorphism.*

Every $g^*\alpha$- *homeomorphism is a gpr-homeomorphism.*

Proof. Proof follows from the fact every $g^*\alpha$- continuous function is *gsp*, *gp*, *gpr*-continuous function and every $g^*\alpha$- open function is (*gsp*-open, *gp*-open, *gpr*-open). Hence *f* is *gsp*, *gp*, *gpr*-homeomorphism.

The reverse implications are not true which can be seen from the following example.

Example 3.9. Consider $X = Y = \{1, 2, 3, 4\}$ with topologies $\tau = \{\varphi, X, \{3, 4\}\}$ and $\sigma = \{\varphi, Y, \{1\}, \{2\}, \{1, 2\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(1) = 3, f(2) = 4, f(3) = 1$ and $f(4) = 2$. Here f is *gsp*-homeomorphism but not $g^*\alpha$- homeomorphism. Since for the closed set $\{1\}$ in (Y, σ), $f^{-1}(\{1\}) = \{3\}$ is *gsp*-closed but not $g^*\alpha$- closed in (X, τ).

Example 3.10. Consider $X = Y = \{1, 2, 3, 4\}$ with topologies $\tau = \{\varphi, X, \{3, 4\}\}$ and $\sigma = \{\varphi, Y, \{1, 2, 3\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is *gp*-homeomorphism but not $g^*\alpha$- homeomorphism. Since for the closed set $\{4\}$ in (Y, σ), $f^{-1}(\{4\}) = \{4\}$ is is *gp*-closed but not $g^*\alpha$- closed in (X, τ).

Example 3.11. Consider $X = Y = \{1, 2, 3, 4\}$ with topologies $\tau = \{\varphi, X, \{3, 4\}, \{2, 3, 4\}\}$ and $\sigma = \{\varphi, Y, \{1, 3\}, \{4\}, \{1, 3, 4\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined as $f(1) = 4, f(2) = 3, f(3) = 1$ and $f(4) = 2$. Then f is *gpr*-homeomorphism but not $g^*\alpha$- homeomorphism. Since for the image of open set $\{3, 4\}$ in (X, τ), $f(\{3, 4\}) = \{1, 2\}$ is not $g^*\alpha$- open in (Y, σ).

Remark 3.12. The composition of two $g^*\alpha$- homeomorphisms need not be $g^*\alpha$- homeomorphism. This can be proved by the following example.

Example 3.13. Consider $X = Y = \{1, 2, 3, 4\}$ with topologies $\tau = \{\varphi, X, \{3\}, \{1, 3, 4\}\}$ and $\sigma = \{\varphi, Y, \{1\}, \{1, 2, 4\}\}$ and $\eta = \{\varphi, Z, \{1, 3, 4\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(1) = 3, f(2) = 2, f(3) = 1, f(4) = 4$ and $g : (Y, \sigma) \to (Z, \eta)$ by $g(1) = 4, g(2) = 3, g(3) = 2$. Here f and g are $g^*\alpha$- homeomorphisms. But their composition $g \circ f : (X, \tau) \to (Z, \eta)$ is not $g^*\alpha$- homeomorphism as the open set $\{1, 3, 4\}$ in (X, τ) is not $g^*\alpha$- open in (Z, η).

Theorem 3.14. *Assume that* $f : X \to Y$ *is bijective and* $g^*\alpha$- *continuous. Then the below facts are true.*

(i) *f is* $g^*\alpha$- *open*

(ii) *f is* $g^*\alpha$- *homeomorphism*

(iii) *f is* $g^*\alpha$- *closed.*

Proof. (a) $\Rightarrow$ (b)

Let f be a $g^*\alpha$- open map. By hypothesis, f is bijective and $g^*\alpha$- continuous. Hence *f* is $g^*\alpha$- homeomorphism.

(b) $\Rightarrow$ (c)

Let *f* be a $g^*\alpha$- homeomorphism. Let *V* be a closed set in (X, τ). Then $X \setminus V$ is open in (X, τ). Since *f* is $g^*\alpha$- open, $f(X \setminus V) = Y \setminus f(V)$ which is $g^*\alpha$- open in (Y, σ). This implies that $f(V)$ is $g^*\alpha$- closed in (Y, σ). Hence *f* is $g^*\alpha$- closed.

(c) $\Rightarrow$ (a)

Let *V* be open in (X, τ). Then V^c is closed in (X, τ). By (c), $f(V)$ is $g^*\alpha$- closed in (Y, σ). But $f(V^c) = (f(V))^c$. This implies that $(f(V))^c$ is $g^*\alpha$- closed in (Y, σ) and so $f(V)$ is $g^*\alpha$- open in (Y, σ). This proves (a).

4. $g^*\alpha c$- Homeomorphism

Definition 4.1. A bijective function $f:(X,\tau)\to(Y,\sigma)$ is called $g^*\alpha c$-homeomorphism if f is $g^*\alpha$-irresolute and its inverse f^{-1} is also $g^*\alpha$-irresolute.

Theorem 4.2. *Every $g^*\alpha c$-homeomorphism is a $g^*\alpha$-homeomorphism.*

Proof. Let $f:(X,\tau)\to(Y,\sigma)$ be $g^*\alpha c$-homeomorphism. Since every $g^*\alpha$-irresolute map is $g^*\alpha$-continuous, by the hypothesis f is $g^*\alpha$-homeomorphism. The converse is not true.

Example 4.3. Consider $X=Y=\{1,2,3\}$ with topologies $\tau=\{\varphi, X, \{1,3\}\}$ and $\sigma=\{\varphi, Y, \{2\}\}$. Define $f:(X,\tau)\to(Y,\sigma)$ by $f(1)=2$, $f(2)=3$, $f(3)=1$. Here f is $g^*\alpha$-homeomorphism but not $g^*\alpha c$-homeomorphism, since $\{1,2\}$ is $g^*\alpha$-closed in (Y,σ) but $f^{-1}(\{1,2\})=\{1,3\}$ is not $g^*\alpha$-closed in (X,τ).

Theorem 4.4. *Every $g^*\alpha c$-homeomorphism is a gsp-homeomorphism (resp gp-homeomorphism and gpr-homeomorphism).*

Proof. We know that each $g^*\alpha c$-homeomorphism is a $g^*\alpha$-homeomorphism and each $g^*\alpha$-homeomorphism is a *gsp*-homeomorphism (resp. *gp*-homeomorphism and *gpr*-homeomorphism).

Converses need not be true in general.

Example 4.5. Let $X=Y=\{1,2,3,4\}$ with $\tau=\{\varphi, X, \{1,4\}\}$ and $\sigma=\{\varphi, Y, \{1,3,4\}\}$.

Define $f:(X,\tau)\to(Y,\sigma)$ by $f(1)=2$, $f(2)=3$, $f(3)=4$, $f(4)=1$. Here f is *gsp*-homeomorphism but not $g^*\alpha c$-homeomorphism, since for the $g^*\alpha$-closed set $\{2\}$ in (Y,σ), $f^{-1}(\{2\})=\{1\}$ is *gsp*-closed but not $g^*\alpha$-closed in (X,τ).

Example 4.6. Let $X = Y = \{1, 2, 3, 4\}$ with $\tau = \{\varphi, X, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\sigma = \{\varphi, Y, \{3\}, \{1, 3, 4\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ be identity mapping. Here f is gp-homeomorphism but not $g^*\alpha c$- homeomorphism, since for the $g^*\alpha$- closed set $\{1\}$ in (Y, σ), $f^{-1}(\{1\}) = \{1\}$ is gp-closed but not $g^*\alpha$- closed in (X, τ).

Example 4.7. Let $X = Y = \{1, 2, 3\}$ with $\tau = \{\varphi, X, \{1\}, \{1, 2\}, \{1, 3\}\}$ and $\sigma = \{\varphi, Y, \{1\}, \{1, 3\}\}$. Define $f : (X, \tau) \to (Y, \sigma)$ be identity mapping. Here f is gpr-homeomorphism but not $g^*\alpha c$- homeomorphism, since for the $g^*\alpha$- closed set $\{1, 2\}$ in (Y, σ), $f^{-1}(\{1, 2\}) = \{1, 2\}$ is gp-closed but not $g^*\alpha$- closed in (X, τ).

Theorem 4.8. *The composition of two $g^*\alpha c$- homeomorphism is a $g^*\alpha c$- homeomorphism.*

Proof. Let V be a $g^*\alpha$- open set in (Z, η). Now $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) = f^{-1}(U)$ where $U = g^{-1}(V)$. By hypothesis, U is $g^*\alpha$- open in (Y, σ) and by hypothesis $f^{-1}(U)$ is $g^*\alpha$- open in (X, τ). Therefore $g \circ f$ is $g^*\alpha$- irresolute. Also for an $g^*\alpha$- open set S in (X, τ), we have $(g \circ f)(S) = g(f(S)) = g(T)$ where $T = f(S)$. By hypothesis $f(S)$ is $g^*\alpha$- open in (Y, σ) and again by hypothesis $g(f(S))$ is $g^*\alpha$- open in (Z, η) (ie) $(g \circ f)(S)$ is $g^*\alpha$- open in (Z, η) and therefore $(g \circ f)^{-1}(V)$ $g^*\alpha$- irresolute. Hence $g \circ f$ is $g^*\alpha c$- homeomorphism.

Theorem 4.9. *The set $g^*\alpha\text{-}h(X, \tau)$ is a group under composition of functions.*

Proof. Define a binary operation $* : g^*\alpha c\text{-}h(X, \tau) \times g^*\alpha c\text{-}h(X, \tau) \to g^*\alpha c\text{-}h(X, \tau)$ by $f * g = g \circ f$ for all $f, g \in g^*\alpha c\text{-}h(X, \tau)$ and $\circ$ is the usual operation of composition of functions. Then by Theorem 4.8,

$g \circ f \in g^{*}\alpha c\text{-}h(X, \tau)$. As the composition of functions is associative and the identity function $I : (X, \tau) \to (X, \tau)$ belonging to $g^{*}\alpha c\text{-}h(X, \tau)$ serves as the identity. If $f \in g^{*}\alpha c\text{-h}(X, \tau)$, then $f^{-1} \in g^{*}\alpha c\text{-}h(X, \tau)$ such that $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exists for each element of $g^{*}\alpha c\text{-}h(X, \tau)$. Therefore $g^{*}\alpha c\text{-}h(X, \tau)$ is a group under composition of functions.

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