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LOCATING GEO SPECTRUM AND GEO ENERGY OF MORE GRAPHS

K. PALANI, M. LALITHA KUMARI and G. SUGANYA

[1]Associate Professor
[2]Research Scholar, Reg. No. 20212012092007
[3]Assistant Professor
PG and Research, Department of Mathematics
A.P.C. Mahalaxmi College for Women
Thoothukudi, TN, India
Affiliated to Manonmaniam Sundaranar University
Abishekapatti, Tirunelveli- 627012, TN, India
E-mail: palani@apcmcollege.ac.in
lalithasat32@gmail.com
gsuganyaveni@gmail.com

Abstract

Let $G = (V, E)$ be a (p, q) simple connected graph. Gutmanbeholded the absicth of energy of graph G relevant to the adjacency matrix of G. Later, the study was extended to many new energy terms. Geo energy of a graph G is the sum of absolute values of the spectrum of its Geo matrix. The scope of this study is to inspect the geo energy of certain classes of graphs through graph theory and utilizations. In this paper we investigate the Geo spectrum and Geo energy of few classes of graphs.

1. Introduction

Graphs considered in this paper are finite and connected. A vasty study of utilization on graph energy was pursued by Gutman and Balakrishnan [1, 4]. The absicth of graph energy was debuted by Gutman [3] in 1978 as the sum of absolute values of spectrum of the adjacency matrix of G. The absicth of geodetic was proposed by F. Harary [6] in 1993. Let $u, v \in V(G)$. Then $d(u, v)$ denotes the number of edges in the shortest path from u to v. The shortest path of $u - v$ is known as geodesic path. A set $S \subseteq V(G)$ is a

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geodetic set if $I[S] = V(G)$, where $I[S]$ is the closed interval $I[u, v]$ consists of all vertices lying in a $u - v$ geodesic of G, i.e., $I[S] = \bigcup_{u, v \in S} I[u, v]$. The geodetic number of G is the minimum cardinality among all the geo sets and is denoted by 'g'. Path union of a graph G is defined by joining two copies G by a path. Globe graph $Gl(p)$ is that two isolated vertices are joined by 'p' paths of length two. For graph theoretic parlance one may refer Harary [5]. Energy of a graph G is relevant to the adjacency matrix of G, accordingly, energy of a graph is the sum of absolute values of eigenvalues of its adjacency matrix. Two graphs are said to be Geo-equienergetic if they attain same geo energy. Later, the study was extended to many new energy terms. Geo energy of a graph G is the sum of absolute values of the spectrum of its Geo matrix. In this paper we scrutinize the Geo spectrum and Geo energy of few classes of graphs.

2. Preliminaries

Definition 2.1. Let $G = (V, E)$ be a (p, q) simple graph. Let $S \subseteq V(G)$ be the minimum geodetic set of G and $|S| = g$. Then the geo matrix of G corresponding to S is a square matrix G_S of order p and is defined as

$$G_S = \begin{cases} 1 & \text{if } v_i \sim v_j \ i \neq j \\ 1 & \text{if } i = j \text{ and } v_i \in S \\ 0 & \text{otherwise} \end{cases}$$

where the symbol '~' denotes the adjacency of a vertex of G. The characteristic polynomial of G_S is denoted by $f_p(G_S, \lambda) = \det(G_S - \lambda I)$. The geo spectrum of the graph G is the eigenvalues of the matrix G_S. Let $\lambda_1, \lambda_2, \ldots, \lambda_p$ be the spectrum of G_S. Then the geo energy GE of G corresponding to S is defined as $GE_S(G) = \sum_{i=1}^{p} |\lambda_i|$.

Remark 2.2. Though all the geo sets are of same cardinality, the geo energy $GE(G)$ need not be the same for all geo set.

Remark 2.3. If G has a unique minimum geodetic set, then $GE_S(G)$ can be denoted as $GE(G)$.

Lemma 2.4 [10]**.** *Let* $B = \begin{bmatrix} B_0 & B_1 \\ B_1 & B_0 \end{bmatrix}$ *be a symmetric* 2×2 *block matrix with* B_0 *and* B_1 *are square matrices of same order. Then spectrum of B is the union of spectra of* $B_0 + B_1$ *and* $B_0 - B_1$.

Lemma 2.5 [10]**.** *If an* $n \times n$ *matrix A is partitioned as* $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ *and* A_{11}, A_{22} *are square matrices. If* A_{11} *is non-singular then,* $\det(A) = \det(A_{11})\det(A_{22} - A_{21}A_{11}^{-1}A_{12})$. *Also if* A_{22} *is non singular then* $\det(A) = \det(A_{22})\det(A_{11} - A_{12}A_{22}^{-1}A_{21})$.

3. Geo Energy of Some Graphs

Theorem 3.1. *For the Globe graph* $Gl(p)$ *with* $p \geq 2$,

$$GE(Gl(p)) = p - 1 + \sqrt{8p + 1}.$$

Proof. Let $V(Gl(p)) = \{u_1, u_2, v_1, \ldots, v_p\}$, where $d(u_1) = d(u_2) = p$ and $|V(Gl(p))| = p + 2$. Then $S = \{v_1, \ldots, v_p\}$ is the unique geo set of $Gl(p)$. Therefore the corresponding unique geo matrix is of the form $G_S(Gl(p)) = \begin{pmatrix} O_p & J_{2\times p} \\ J_{p\times 2} & I_p \end{pmatrix}$.

Then the corresponding characteristic polynomial is,

$$f(Gl(p), \lambda) = \det(G_S(Gl(p)) - \lambda I_p)$$

$$f(Gl(p), \lambda) = \lambda(\lambda - 1)^{p-1}(\lambda^2 - \lambda - 2p)$$

Therefore the geo spectrum of $Gl(p)$ is,

$$Spec_S(Gl(p)) = \begin{Bmatrix} 0 & 1 & \frac{1 + \sqrt{8p+1}}{2} & \frac{1 - \sqrt{8p+1}}{2} \\ 1 & p - 1 & 1 & 1 \end{Bmatrix}$$

Hence,

$$GE(Gl(p)) = 1_{(p-1\ times)} + \left|\frac{1+\sqrt{8p+1}}{2}\right| + \left|\frac{1-\sqrt{8p+1}}{2}\right|$$

$$GE(Gl(p)) = p - 1 + \sqrt{8p+1}.$$

Theorem 3.2. *For any complete graph K_p with $p \geq 3$, $GE(K_p) = p$.*

Proof. Let $V(K_p) = \{v_1, v_2, \ldots, v_p\}$. Then the unique geo set is $S = \{v_1, v_2, \ldots, v_p\}$. Therefore the geo-matrix is,

$$G_S(K_p) = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

And the corresponding characteristic polynomial is

$$f(K_p, \lambda) = \det(G_S(K_p) - \lambda I_p)$$

$$= \begin{vmatrix} 1-\lambda & 1 & \ldots & 1 \\ 1 & 1-\lambda & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1-\lambda \end{vmatrix}$$

$$f(K_p, \lambda) = (-1)^p (\lambda)^{p-1} (\lambda - p)$$

Hence the geo spectrum of K_p is $\begin{Bmatrix} 0 & p \\ p-1 & 1 \end{Bmatrix}$.

Thus the geo energy is

$$GE(K_p) = 0_{(p-1)times} + p$$

$$= p$$

$$GE(K_p) = p.$$

Theorem 3.3. *For a complete bipartite graph $K_{p,q}$ with $p, q \geq 2$,*

$$GE(K_{p,q}) = \begin{cases} q - 1 + \sqrt{1 + 4pq} & if \;\; p \geq q \\ p - 1 + \sqrt{1 + 4pq} & if \;\; p < q \end{cases}.$$

Proof. Let $V(K_{p,q}) = \{v_1, \ldots, v_p, v_1', \ldots, v_q'\}$. Then the unique geo set is $S = \min\{\{v_1, \ldots, v_p, \{v_1', \ldots, v_d'\}\}\}$.

(i) Suppose $p = q$, then the $G_S(K_{p,q})$ matrix is

$$G_S(K_{p,q}) = \begin{pmatrix} 1 & 0 & 0 & \ldots & 01 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 0 & \ldots & 01 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 1 & \ldots & 01 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots\vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 11 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 10 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 10 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & \ldots & 10 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots\vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 10 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

This is of the form $\begin{pmatrix} I_p & G_S(K_p) \\ G_S(K_p) & O_p \end{pmatrix}$, where O_p is the zero matrix. And the corresponding characteristic polynomial is

$$f(K_{p,q}, \lambda) = \det(G_S(K_{p,q}) - \lambda I)$$

$$= (-1)^{p+q} \lambda^{q-1} (\lambda - 1)^{p-1} (\lambda^2 - \lambda - pq)$$

(ii) Suppose $p < q$, then the geo matrix is of the form $\begin{pmatrix} O_q & J_{p \times q} \\ J_{q \times p} & I_p \end{pmatrix}$, where $J_{p \times q}$ has all the entries as 1. And the corresponding characteristic polynomial is,

$$f(K_{p,q}, \lambda) = \det(G_S(K_{p,q}) - \lambda I)$$

$$= (-1)^{p+q} \lambda^{q-1} (\lambda - 1)^{p-1} (\lambda^2 - \lambda - pq)$$

For the both cases (i), (ii) the characteristic polynomial remains same. Hence,

$$Spec_S(K_{p,q}) = \begin{Bmatrix} 0 & 1 & \frac{1+\sqrt{1+4pq}}{2} & \frac{1-\sqrt{1+4pq}}{2} \\ p-1 & q-1 & 1 & 1 \end{Bmatrix}$$

$$GE(K_{p,q}) = q - 1 + \sqrt{1+4pq}$$

(iii) Suppose $p < q$, then the $G_S(K_{p,q})$ matrix is of the form $\begin{pmatrix} I_p & J_{p\times q} \\ J_{q\times p} & O_q \end{pmatrix}$.

Therefore the corresponding characteristic polynomial is,

$$f(K_{p,q}, \lambda) = \det(G_S(K_{p,q}) - \lambda I)$$

$$= (-1)^{p+q}\lambda^{q-1}(\lambda-1)^{p-1}(\lambda^2 - \lambda - pq)$$

Therefore,

$$Spec_S(K_{p,q}) = \begin{Bmatrix} 0 & 1 & \frac{1+\sqrt{1+4pq}}{2} & \frac{1-\sqrt{1+4pq}}{2} \\ q-1 & p-1 & 1 & 1 \end{Bmatrix}$$

$$GE(K_{p,q}) = p - 1 + \sqrt{1+4pq}.$$

Theorem 3.4. *For a complement of star graph* $\overline{K_{1,p-1}}$ *with* $p \geq 3$, $GE(\overline{K_{1,p-1}}) = p$.

Proof. Let $V(\overline{K_{1,p-1}}) = \{v_0, v_1, \ldots, v_{p-1}\}$, where v_0 is the vertex of degree 0 and $d(v_i) = p-2$ for $1 \leq p-1$. Then the unique geo set is $S = \{v_0, v_1, \ldots, b_{p-1}\}$. Therefore

$$G_S(\overline{K_{1,p-1}}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

$G_S(\overline{K_{1,\, p-1}})$ is of the form $\begin{pmatrix} 1 & 0_{1\times p-1} \\ 0_{p-1\times 1} & G_S(K_p) \end{pmatrix}$

$$\det G_S(\overline{K_{1,\, p-1}}) = \begin{vmatrix} 1-\lambda & 0_{1\times p-1} \\ 0_{p-1\times 1} & G_S(K_{p-1}) - \lambda I_{p-1} \end{vmatrix}$$

Then by lemma 2.5,

$$\det(G_S(\overline{K_{1,\, p-1}}) = \det(1-\lambda)(\det(G_S(K_{p-1}) - \lambda I_{p-1}) - 0_{1\times p-1}(1-\lambda)^{-1} 0_{p-1\times 1})$$

$$= \lambda^{p-2}(\lambda - 1)(\lambda - (p-1)).$$

Hence the geo spectrum of $\overline{K_{1, p-1}}$ is $Spec_S = \begin{Bmatrix} 0 & 1 & p-1 \\ p-2 & 1 & 1 \end{Bmatrix}$.

Thus the geo- energy is $GE(\overline{K_{1,\, p-1}}) = 0_{(p-2) times} + 1 + p - 1$

$$GE(\overline{K_{1,\, p-1}}) = p.$$

Proposition 3.5. *Any complete graph* K_p *and complement of star graph* $\overline{K_{1, p-1}}$ *attain geo equienergetic. But they are non-isomorphic and not a co-spectral graph.*

Theorem 3.6. *For the path union of star graph* $P_1 \cup K_{1,\, p-1}$ *with* $p \geq 2$,

$$GE(P_1 \cup K_{1,\, p-1}) = 2(p - 2 + \sqrt{2} + \sqrt{p-1}).$$

Proof. Let $V(P_1 \cup K_{1, p-1}) = \{v_0, v_1, \ldots, v_{p-1}\} \cup \{u_0, u_1, \ldots, u_{p-1}\}$ where $d(v_0) = d(u_0) = p-1$. Then set of all pendent vertices of $P_1 \cup K_{1, p-1}$ is the unique geodetic set S.

Therefore $S = \{v_1, \ldots, v_{p-1}\} \cup \{u_1, \ldots, u_{p-1}\}$. Then the geo matrix of $P_1 \cup K_{1, p-1} G_S$ is of the form $\begin{pmatrix} B_0 & B_1 \\ B_1 & B_0 \end{pmatrix}$.

Then by lemma 2.4 the spectrum of $P_1 \cup K_{1, p-1}$ is the union of spectrum of $B_0 + B_1$ and $B_0 - B_1$.

Consider $B_0 + B_1 = \begin{pmatrix} 1 & J_{1+p-1} \\ J_{p-1\times 1} & I_{p-1} \end{pmatrix}$. Then $|(B_0 + B_1) - \lambda I| = (-1)^p (\lambda - 1)^{p-2} (\lambda^2 - 2\lambda - (p-2))$. Therefore the geo spectrum of $B_0 + B_1$ is,

$$Spec_S(B_0 + B_1) = \begin{Bmatrix} 0 & 1 & \frac{1+\sqrt{p-1}}{2} & \frac{1-\sqrt{p-1}}{2} \\ p & p-2 & 1 & 1 \end{Bmatrix}$$

Now consider $B_0 - B_1 = \begin{pmatrix} -1 & J_{1+p-1} \\ J_{p-1\times 1} & I_{p-1} \end{pmatrix}$. Then $|(B_0 - B_1) - \lambda I| = (-1)^p (\lambda - 1)^{p-2} (\lambda^2 - p)$. Therefore the geo spectrum of $B_0 - B_1$ is,

$$Spec_S(B_0 - B_1) = \begin{Bmatrix} 1 & \sqrt{p} & -\sqrt{p} \\ p-2 & 1 & 1 \end{Bmatrix}$$

Then the geo spectrum of $P_1 \cup K_{1,\, p-1}$ is,

$$Spec_S(P_1 \cup K_{1,\, p-1}) = \begin{Bmatrix} 0 & 1 & \frac{1+\sqrt{p-1}}{2} & \frac{1-\sqrt{p-1}}{2} & \sqrt{p} & -\sqrt{p} \\ p & 2p-4 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

Hence,

$$GE(P_1 \cup K_{1,\, p-1}) = 1_{(2p-4 times)} + |\sqrt{p}| + |-\sqrt{p}| + \left|\frac{1+\sqrt{p-1}}{2}\right| + \left|\frac{1-\sqrt{p-1}}{2}\right|.$$

$$GE(P_1 \cup K_{1,\, p-1}) = 2(p - 2 + \sqrt{p} + \sqrt{p-1}).$$

Theorem 3.7. *For the path union of complete graph $P_1 \cup K_p$ with $p \geq 2$,*

$$GE(P_1 \cup K_p) = p + \sqrt{p^2 - 4(p-1)}.$$

Proof. Let $V(P_1 \cup K_p) = \{v_1, \ldots, v_p, v_1', \ldots, v_p'\}$. Then the unique geo set is $S = \{v_1, \ldots, v_{p-1}, v_1', \ldots, v_{p-1}'\}$. Therefore

$$G_S(P_1 \cup K_p) = \begin{pmatrix} 1 & 1 & 1 & \dots & 10 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 10 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 10 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots\vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 00 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 01 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 01 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 01 & 1 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots\vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 11 & 1 & 1 & \dots & 0 \end{pmatrix}$$

And the corresponding characteristic polynomial is

$$f(P_1 \cup K_p, \lambda) = \det(G_S(P_1 \cup K_p) - \lambda I)$$

$$f(P_1 \cup K_p, \lambda) = \lambda^{2p-3}(\lambda - 2)(\lambda^2 - (p-2)\lambda - 2(p-1))$$

$$= \lambda^{2p-3}(\lambda - p)\left(\lambda - \frac{(p-2) \pm \sqrt{p^2 - 4(p-1)}}{2}\right)$$

Hence the geo spectrum of $P_1 \cup K_p$ is

$$\begin{Bmatrix} 0 & p & \frac{(p-2) + \sqrt{p^2 - 4(p-1)}}{2} & \frac{(p-2) - \sqrt{p^2 - 4(p-1)}}{2} \\ 2p-3 & 1 & 1 & 1 \end{Bmatrix}$$

Thus the geo-energy is

$$GE(P_1 \cup K_p) = 0_{(2p-3)times} + p + \left|\frac{(p-2) + \sqrt{p^2 - 4(p-1)}}{2}\right|$$

$$+ \left|\frac{(p-2) - \sqrt{p^2 - 4(p-1)}}{2}\right|$$

$$= p + \sqrt{p^2 - 4(p-1)}$$

$$GE(P_1 \cup K_p) = 4p - 1 + \sqrt{8p+1} + \sqrt{8p+9}.$$

Theorem 3.8. *For the path union of Friendship graph* $P_1 \cup F_p$,

$$GE(P_1 \cup F_p) = 4p - 1 + \sqrt{8p+1} + \sqrt{8p+9}.$$

Proof. Let $V(P_1 \cup F_p) = \{v_0, v_1, \ldots, v_{2p}, v_0', v_1', \ldots, v_{2p}'\}$. Then the unique geo set of $P_1 \cup F_p$ is $S = V(P_1 \cup F_p) - \{v_0, v_0'\}$. Therefore

$$G_S(P_1 \cup F_p) = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where $A = \begin{pmatrix} 0 & J_{1\times 2p} \\ J_{2p\times 1} & J_{2\times 2} \end{pmatrix}_{2p+1}$ and $B = \begin{pmatrix} 0 & J_{1\times 2p} \\ J_{2p\times 1} & J_{2\times 2} \end{pmatrix}_{2p+1}$

And the corresponding characteristic polynomial is

$$f(P_1 \cup F_p, \lambda) = \det(G_S(P_1 \cup F_p) - \lambda I)$$

$$f(P_1 \cup F_p, \lambda) = \lambda^{2p}(\lambda - 2)^{2p-2}(\lambda^2 - 3\lambda - 2(p-1))(\lambda^2 - \lambda - 2(p+1))$$

Hence the geo spectrum of $P_1 \cup F_p$ is

$$\begin{Bmatrix} 0 & 2 & \frac{3+\sqrt{8p+1}}{2} & \frac{3-\sqrt{8p+1}}{2} & \frac{1+\sqrt{8p+1}}{2} & \frac{1-\sqrt{8p+9}}{2} \\ 2p & 2p-2 & 1 & 1 & 1 & 1 \end{Bmatrix}$$

Thus the geo-energy of $P_1 \cup F_p$ is

$$GE(P_1 \cup F_p) = 2_{(2p-2)times} + \left|\frac{3+\sqrt{8p+1}}{2}\right| + \left|\frac{3-\sqrt{8p+1}}{2}\right| + \left|\frac{1+\sqrt{8p+1}}{2}\right| + \left|\frac{1-\sqrt{8p+9}}{2}\right|$$

$$GE(P_1 \cup F_p) = 4p - 1 + \sqrt{8p+1} + \sqrt{8p+9}.$$

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