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A NOTE ON MEROMORPHIC FUNCTIONS ASSOCIATED WITH (p, q)- RUSCHEWEYH DERIVATIVE

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Abstract

In this article, a new subclass of meromorphic functions is introduced using (p, q)- Ruscheweyh derivative and we discuss the properties like convolution, closure, convex combinations and neighborhood for the functions belonging to this subclass.

1. Introduction

The quantum calculus has been broadly studied and has applications in several fields of mathematics, physics and engineering. Further, motivated and inspired by these applications, many mathematicians and physicist have developed the theory of post quantum calculus, an extension of the quantum calculus and is designated as (p, q)- calculus. The recent interest in the subject is due to the fact that the post quantum calculus has popped in such diverse fields as quantum algebra, number theory etc.

The (p, q)- derivative of a function f [3] defined as

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$$D_{p,q}(f(z)) = \frac{f(pz) - f(qz)}{(p-q)z}, \; z \neq 0, \; p \neq q, \; 0 < q < 1.$$

Let $\mathcal{M}$ be the collection of all meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{r=1}^{\infty} a_r z^r, \; z \in \mathcal{D}^* \tag{1.1}$$

which are regular in punctured open unit disc $\mathcal{D}^* = \mathcal{D} - \{0\} = \{z \in C : 0 < |z| < 1\}$. Then

$$D_{p,q}(f(z)) = -\frac{1}{pqz^2} + \sum_{r=1}^{\infty} [r]_{p,q} a_r z^{r-1},$$

where $[r]_{p,q} = \frac{p^r - q^r}{p - q}$.

Definition 1.1. Let f be as in (1.1) and $g(z) = \frac{1}{z} + \sum_{r=1}^{\infty} b_r z^r$, then the Hadamard product of f and g is defined as

$$(f * g)(z) = (g * f)(z) = \frac{1}{z} + \sum_{r=1}^{\infty} a_r b_r z^r.$$

Now we define a (p, q)- derivative operator $\mathcal{L}_{p,q}^{n,m}(f(z)) : \mathcal{M} \to \mathcal{M}$ as

$$\mathcal{L}_{p,q}^{n,m} f(z) = \mathcal{R}_{p,q}^{m} f(z) * \mu_{p,q}^{n} f(z) = \frac{(-1)^n}{z} + \sum_{r=1}^{\infty} \frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} a_r z^r,$$

$$n, m \in \mathcal{N} U \{0\}, \; 0 < q < 1 \text{ and } p \neq q. \tag{1.2}$$

Where $\mathcal{R}_{p,q}^{m}$ represent (p, q)- Ruscheweyh derivative operator and is given by

$$\mathcal{R}_{p,q}^{m}(f(z)) = \frac{1}{z} + \sum_{r=1}^{\infty} \frac{[m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} a_r z^r, \; m \in \mathcal{N}$$

and

$$[r]_{p,q}! = \begin{cases} [r]_{p,q}[r-1]_{p,q} \dots [1]_{p,q}, & r = 2, 3, \dots \\ 1, & r = 1 \end{cases}$$

we define a new derivative operator $\mu^{n}_{p,q} f(z)$ as

$$\mu^{n}_{p,q} f(z) = p^n q^n z D_{p,q}(\mathcal{S}^{n-1}_{p,q} f(z)) = \frac{(-1)^n}{z} + \sum_{r=1}^{\infty} p^n q^n [r]^n_{p,q} a_r z^r, \ (n \in \mathcal{N})$$

where $\mathcal{S}^n_{p,q}$ represent (p, q)- Salagean differential operator and is given by

$$\mathcal{S}^n_{p,q} f(z) = \frac{(-1)^n}{p^n q^n z} + \sum_{r=1}^{\infty} [r]^n_{p,q} a_r z^r.$$

Therefore

$$D_{p,q}(\mathcal{L}^{n,m}_{p,q} f(z)) = \frac{(-1)^{n+1}}{pqz^2} + \sum_{r=1}^{\infty} \frac{p^n q^n [r]^{n+1}_{p,q} [m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} a_r z^{r-1}, z \in \mathcal{D}^*. \quad (1.3)$$

Remark 1.2.

(1) When $n = 0$, $p = 1$ and $q \to 1^-$ in (1.2), we get Ruscheweyh derivative of $f(z)$ [5].

(2) When $n = 0$, $p = 1$, then $\mathcal{L}^{n,m}_{p,q}$ is reduced to the class $\mathcal{L}^{\mu}_{q}$ introduced by Bakhtiar Ahmad and Muhammad Arif [2].

(3) When $m = 0$, $p = 1$ and $q \to 1^-$ in (1.2), we obtain Salagean derivative operator [6].

Now, we define a subclass $\mathcal{M}^*_{p,q}(\beta, n, m)$ of regular meromorphic function on $\mathcal{D}^*$ as follows.

Definition 1.3. Let $f \in \mathcal{M}$ as in (1.1) $n, m \in \mathcal{N} U\{0\}$ and $\beta \in [0, 1)$ is said to belong to the class $\mathcal{M}^*_{p,q}(\beta, n, m)$ of meromorphic starlike of order β, if it obeys the inequality

$$\Re\left\{\frac{-pqzD_{p,q}(\mathcal{L}_{p,q}^{n,m}f(z))}{\mathcal{L}_{p,q}^{n,m}f(z)}\right\} \geq \beta,\ z \in \mathcal{D}^*.$$

Theorem 1.4. *Let* $f \in \mathcal{M}$ *as in* (1.1); $n, m \in \mathcal{N}U\{0\}$ *and* $\beta \in [0, 1)$, *then* $f \in \mathcal{M}_{p,q}^*(\beta, n, m)$ *if and only if*

$$\sum_{r=1}^{\infty}(pq[r]_{p,q} + \beta)\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}|a_r| \leq 1 - \beta. \tag{1.4}$$

Proof. Suppose that $f \in \mathcal{M}_{p,q}^*(\beta, n, m)$ then by the definition (1.3), we have

$$\Re\left\{\frac{-pqzD_{p,q}(\mathcal{L}_{p,q}^{n,m}f(z))}{\mathcal{L}_{p,q}^{n,m}f(z)}\right\} \geq 0,\ z \in \mathcal{D}^*.$$

which is equivalent to

$$\Re\left\{\frac{1 - \sum_{r=1}^{\infty}\frac{p^{n+1}q^{n+1}[r]_{p,q}^{n+1}[m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}a_r z^{r+1}}{1 + \sum_{r=1}^{\infty}\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}a_r z^{r+1}}\right\} \geq \beta,$$

Letting $z \to 1^-$, we get

$$1 - \sum_{r=1}^{\infty}\frac{p^{n+1}q^{n+1}[r]_{p,q}^{n+1}[m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}a_r \geq \beta + \beta\sum_{r=1}^{\infty}\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}a_r$$

on simplification we get,

$$\sum_{r=1}^{\infty}(pq[r]_{p,q} + \beta)\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}|a_r| \leq 1 - \beta.$$

Conversely, suppose that $f \in \mathcal{M}$ as in (1.1) and the inequality (1.4) holds for all $z \in D^*$ we will prove that $f \in \mathcal{M}_{p,q}^*(\beta, n, m)$ for $0 \leq \beta < 1$, $n \in \mathcal{N}U\{0\}$, i.e., we need to prove that

$$\Re\left\{\frac{-pqzD_{p,q}(\mathcal{L}_{p,q}^{n,m}f(z))}{\mathcal{L}_{p,q}^{n,m}f(z)}\right\} \geq \beta.$$

using the condition that $\Re(w) \geq \beta$ if $|1-\beta+\omega| \geq |1+\beta-\omega|$, it is enough to prove that $|\chi(z)|-|\psi(z)| \geq 0$, where $\chi(z) = (1-\beta)(\mathcal{L}_{p,q}^{n,m}f(z) - pqzD_{p,q}(\mathcal{L}_{p,q}^{n,m}f(z))$ and $\psi(z) = (1+\beta)(\mathcal{L}_{p,q}^{n,m}f(z) + pqzD_{p,q}(\mathcal{L}_{p,q}^{n,m}f(z))$

$|\chi(z)|-|\psi(z)|$

$$= \left| \frac{(2-\beta)}{z} - \sum_{r=1}^{\infty}(pq[r]_{p,q} - 1 + \beta)\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!} a_r z^r \right|$$

$$- \left| \frac{\beta}{z} + \sum_{r=1}^{\infty}(pq[r]_{p,q} + 1 + \beta)\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!} a_r z^r \right|$$

$$\geq \left| \frac{(2-2\beta)}{z} - \sum_{r=1}^{\infty}2(pq[r]_{p,q} + \beta)\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!} a_r z^r \right|$$

$$\geq \frac{2(1-\beta)}{|z|}\left(1 - \sum_{r=1}^{\infty}\frac{(pq[r]_{p,q}+\beta)}{1-\beta}\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}|a_r||z^{r+1}|\right)$$

$$\geq \frac{2(1-\beta)}{|z|}\left(1 - \sum_{r=1}^{\infty}\frac{(pq[r]_{p,q}+\beta)}{1-\beta}\frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!}|a_r|\right)$$

≥ 0, by (1.4).

This completes the proof. □

Corollary 1.5. *If* $f(z) \in \mathcal{M}_{p,q}^{*}(\beta, n, m)$, *then*

$$\sum_{r=1}^{\infty} a_r \leq \frac{[2]_{p,q}(1-\beta)}{(pq+\beta)p^n q^n [m+1]_{p,q}[m+2]_{p,q}}, \ \beta \in [0, 1).$$

Theorem 1.6. *If* $f_j(z) \in \mathcal{M}_{p,q}^{*}(\beta, n, m)$, $\forall j = 1, 2, \ldots, \lambda$, *where*

$$f_j(z) = \frac{1}{z} + \sum_{r=1}^{\infty} a_{r,j} z^r, \; a_{r,j} \geq 0$$

then

$$\alpha_j(z) = \frac{1}{z} + \sum_{r=1}^{\infty} b_r z^r \in \mathcal{M}_{p,q}^*(\beta, n, m), \; b_r \geq 0 \text{ and } b_r = \frac{1}{\lambda} \sum_{j=1}^{\lambda} a_{r,j}.$$

Proof. By the Theorem (1.4), $\alpha_j(z) \in \mathcal{M}_{p,q}^*(\beta, n, m)$, if and only if

$$\sum_{r=1}^{\infty} (pq[r]_{p,q} + \beta) \frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} | b_r | \leq 1 - \beta$$

consider

$$\sum_{r=1}^{\infty} (pq[r]_{p,q} + \beta) \frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} | b_r |$$

$$= \sum_{r=1}^{\infty} (pq[r]_{p,q} + \beta) \frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} \frac{1}{\lambda} \sum_{j=1}^{\lambda} | a_{r,j} |$$

$$= \frac{1}{\lambda} \sum_{j=1}^{\lambda} \sum_{r=1}^{\infty} (pq[r]_{p,q} + \beta) \frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} | a_{r,j} |$$

since $f_j(z) \in \mathcal{M}_{p,q}^*(\beta, n, m)$, $\forall j = 1, 2, \lambda$, then by the Theorem (1.4), we have

$$\sum_{r=1}^{\infty} (pq[r]_{p,q} + \beta) \frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}! [r+1]_{p,q}!} | b_r | \leq \frac{1}{\lambda} \sum_{j=1}^{\lambda} (1-\beta) = (1-\beta)$$

which implies $\alpha_j(z) \in \mathcal{M}_{p,q}^*(\beta, n, m)$ and completes the proof. □

Theorem 1.7. *If* $f \in \mathcal{M}$ *be given by* (1.1), *then* $f(z) \in \mathcal{M}_{p,q}^*(\beta, n, m)$, *if and only if*

$$f(z) = \sum_{r=0}^{\infty} \omega_r f_r(z),$$

where $f_0(z) = \frac{1}{z}$, $f_r(z) = \frac{1}{z} + \left(\frac{[m]_{p,q}!\,[r+1]_{p,q}!\,(1-\beta)}{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!\,(pq[r]_{p,q}+\beta)}\right) z^r,$

$r = 1, 2, \ldots;\ \omega_r \in [0, 1]$ *and* $\sum_{r=0}^{\infty} \omega_r = 1.$

Proof. Given

$$f(z) = \sum_{r=0}^{\infty} \omega_r f_r = \omega_0 f_0(z) + \sum_{r=1}^{\infty} \omega_r f_r(z)$$

$$= \frac{w_0}{z} + \sum_{r=1}^{\infty} \omega_r \left(\frac{1}{z} + \left(\frac{[m]_{p,q}!\,[r+1]_{p,q}!\,(1-\beta)}{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!(pq[r]_{p,q}+\beta)}\right)\right) z^r$$

by theorem (1.1),

$$\sum_{r=1}^{\infty} (pq[r]_{p,q}+\beta) \frac{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}!\,[r+1]_{p,q}!}$$

$$\left(\frac{[m]_{p,q}!\,[r+1]_{p,q}!\,(1-\beta)}{p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!(pq[r]_{p,q}+\beta)}\right)\omega_r$$

$$= (1-\beta)\sum_{r=1}^{\infty} \omega_r = (1-\beta)(1-\omega_0) \le 1-\beta,$$

therefore $f(z) \in \mathcal{M}_{p,q}^{*}(\beta, n, m)$.

Conversely, assume that $f(z) \in \mathcal{M}_{p,q}^{*}(\beta, n, m)$, put

$$\omega_r = \frac{(pq[r]_{p,q}+\beta) p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}!\,[r+1]_{p,q}!\,(1-\beta)} a_r;\ \omega_r \in [0, 1]$$

then

$$f(z) = \frac{1}{z} + \sum_{r=1}^{\infty} a_r z^r = \frac{1}{z} + \sum_{r=1}^{\infty} \frac{[m]_{p,q}!\,[r+1]_{p,q}!\,(1-\beta)}{(pq[r]_{p,q}+\beta) p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!} \omega_r z^r$$

$$= \frac{\omega_0}{z} + \sum_{r=1}^{\infty}\left(\frac{1}{z} + \frac{[m]_{p,q}!\,[r+1]_{p,q}!\,(1-\beta)}{(pq[r]_{p,q}+\beta)p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!} z^r\right)\omega_r$$

$$= \sum_{r=0}^{\infty} \omega_r f_r(z). \qquad \square$$

Theorem 1.8. *If* $f(z) = \frac{1}{z} + \sum_{r=1}^{\infty} a_r z^r$ *and* $h(z) = \frac{1}{z} + \sum_{r=1}^{\infty} c_r z^r$ *are in the class* $\mathcal{M}^*_{p,q}(\beta, n, m)$, *then* $(f * h)(z)$ *is also belong to the class* $\mathcal{M}^*_{p,q}(\rho, n, m)$, *where*

$$\rho = \frac{(pq[r]_{p,q}+\beta)^2 p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}! - pq[r]_{p,q}(1-\beta)^2 [m]_{p,q}![r+1]_{p,q}!}{(1-\beta)^2 [m]_{p,q}![r+1]_{p,q}! + (pq[r]_{p,q}+\beta)^2 p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}.$$

Proof. As f and g are in $\mathcal{M}^*_{p,q}(\beta, n, m)$, we have

$$\sum_{r=1}^{\infty} \frac{(pq[r]_{p,q}+\beta)p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}!\,[r+1]_{p,q}!(1-\beta)} a_r \leq 1$$

and

$$\sum_{r=1}^{\infty} \frac{(pq[r]_{p,q}+\beta)p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{[m]_{p,q}![r+1]_{p,q}!(1-\beta)} c_r \leq 1.$$

By the Cauchy-Schwarz inequality, we get

$$\sum_{r=1}^{\infty} \frac{(pq[r]_{p,q}+\beta)p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{(1-\beta)[m]_{p,q}!\,[r+1]_{p,q}!} \sqrt{a_r c_r} \leq 1 \qquad (1.5)$$

to prove that $f(z) * g(z) \in \mathcal{M}^*_{p,q}(\rho, n, m)$, we need to determine the greatest ρ so that,

$$\sum_{r=1}^{\infty} \frac{(pq[r]_{p,q}+\rho)p^n q^n [r]_{p,q}^n [m+r+1]_{p,q}!}{(1-\rho)[m]_{p,q}!\,[r+1]_{p,q}!} a_r c_r \leq 1$$

so, it is sufficient to prove that

$$\sum_{r=1}^{\infty}\frac{(pq[r]_{p,q}+\rho)p^nq^n[r]_{p,q}^n[m+r+1]_{p,q}!}{(1-\rho)[m]_{p,q}![r+1]_{p,q}!}a_rc_r$$

$$\leq\sum_{r=1}^{\infty}\frac{(pq[r]_{p,q}+\beta)p^nq^n[r]_{p,q}^n[m+r+1]_{p,q}!}{(1-\beta)[m]_{p,q}![r+1]_{p,q}!}\sqrt{a_rc_r}\leq 1$$

$$\Rightarrow \sqrt{a_rc_r}\leq\sum_{r=1}^{\infty}\frac{(1-\rho)(pq[r]_{p,q}+\beta)}{(1-\beta)(pq[r]_{p,q}+\rho)}$$

from (1.5);

$$\sqrt{a_rc_r}\leq\frac{(1-\beta)[m]_{p,q}![r+1]_{p,q}!}{(pq[r]_{p,q}+\beta)p^nq^n[r]_{p,q}^n[m+r+1]_{p,q}!}.$$

Hence, it suffices to show that

$$\frac{(1-\beta)[m]_{p,q}![r+1]_{p,q}!}{(pq[r]_{p,q}+\beta)p^nq^n[r]_{p,q}^n[m+r+1]_{p,q}!}\leq\frac{(1-\rho)(pq[r]_{p,q}+\beta)}{(1-\beta)(pq[r]_{p,q}+\rho)}$$

$$\Rightarrow\rho\leq\frac{(pq[r]_{p,q}+\beta)^2p^nq^n[r]_{p,q}^n[m+r+1]_{p,q}!-pq[r]_{p,q}(1-\beta)^2[m]_{p,q}![r+1]_{p,q}!}{(1-\beta)^2[m]_{p,q}![r+1]_{p,q}!+(pq[r]_{p,q}+\beta)^2p^nq^n[r]_{p,q}^n[m+r+1]_{p,q}!}.$$

Theorem 1.9. *Let* $f(z)$ *be given by* (1.1) *and* $h(z)=\frac{1}{z}+\sum_{r=1}^{\infty}c_rz^r$ *are both belongs to the class* $\mathcal{M}_{p,q}^*(\beta, n, m)$, *then*

$$\gamma(z)=\frac{1}{z}+\sum_{r=1}^{\infty}(a_e^2+c_r^2)z^r\in\mathcal{M}_{p,q}^*(\xi, n, m),$$

where

$$\xi=1-\frac{2(1-\beta)^2[2]_{p,q}!(1+pq)}{(pq+\beta)^2p^nq^n[m+1]_{p,q}[m+2]_{p,q}+2(1-\beta)^2[2]_{p,q}!}.$$

Proof. To prove that $\gamma(z)\in\mathcal{M}_{p,q}^*(\xi, n, m)$, we need to determine the greatest ξ, so that

$$\sum_{r=1}^{\infty} \frac{(pq[r]_{p,q} + \xi) p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!}{(1 - \xi)[m]_{p,q}! [r + 1]_{p,q}!} (a_r^2 + c_r^2) \leq 1$$

As $f(z), h(z) \in \mathcal{M}_{p,q}^*(\xi, n, m)$, we have

$$\sum_{r=1}^{\infty} \left\{ \frac{(pq[r]_{p,q} + \beta) p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!}{(1 - \beta)[m]_{p,q}! [r + 1]_{p,q}!} \right\}^2 a_r^2 \leq 1 \tag{1.6}$$

$$\sum_{r=1}^{\infty} \left\{ \frac{(pq[r]_{p,q} + \beta) p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!}{(1 - \beta)[m]_{p,q}! [r + 1]_{p,q}!} \right\}^2 c_r^2 \leq 1 \tag{1.7}$$

summing (1.6) and (1.7), we obtain

$$\sum_{r=1}^{\infty} \frac{1}{2} \left\{ \frac{(pq[r]_{p,q} + \beta) p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!}{(1 - \beta)[m]_{p,q}! [r + 1]_{p,q}!} \right\}^2 (a_r^2 + c_r^2) \leq 1$$

but $\gamma(z) \in \mathcal{M}_{p,q}^*(\xi, n, m)$, if and only if

$$\sum_{r=1}^{\infty} \frac{(pq[r]_{p,q} + \xi) p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!}{(1 - \xi)[m]_{p,q}! [r + 1]_{p,q}!} (a_r^2 + c_r^2) \leq 1$$

which is possible only if

$$\frac{(pq[r]_{p,q} + \xi) p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!}{(1 - \xi)[m]_{p,q}! [r + 1]_{p,q}!}$$

$$\leq \frac{1}{2} \left\{ \frac{(pq[r]_{p,q} + \beta) p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!}{(1 - \beta)[m]_{p,q}! [r + 1]_{p,q}!} \right\}^2$$

$$\frac{1 - \xi}{pq[r]_{p,q} + \xi} \geq \frac{2(1 - \beta)^2 [m]_{p,q}! [r + 1]_{p,q}!}{(pq[r]_{p,q} + \beta)^2 p^n q^n [r]_{p,q}^n [m + r + 1]_{p,q}!} = \eta(r)$$

where $\eta(r)$ is a non-increasing function of attains a maximum at $r = 1$ and the maximum value $\eta(1)=\dfrac{2(1-\beta)^2\,[2]_{p,q}!}{(pq+\beta)^2\,p^nq^n[m+1]_{p,\,q}[m+2]_{p,q}}$ hence

$$\frac{1-\xi}{pq+\xi}\geq\frac{2(1-\beta)^2\,[2]_{p,q}!}{(pq+\beta)^2\,p^nq^n[m+1]_{p,\,q}[m+2]_{p,q}}$$

on simplification, we get

$$\xi\leq 1-\frac{2(1-\beta)^2\,[2]_{p,q}!\,(1+pq)}{(pq+\beta)^2\,p^nq^n[m+1]_{p,q}[m+2]_{p,q}+2(1-\beta)^2[2]_{p,q}!}.$$

□

Definition 1.10. Let $f\in\mathcal{M}$, then the ρ-neighbourhood of *f* is defined as

$$N_\rho(f)=\left\{g\in\mathcal{M};\ g(z)=\frac{1}{z}+\sum_{r=1}^{\infty}g_rz^r\ \ and\ \sum_{r=1}^{\infty}r|\,a_r-g_r\,|\leq\rho,\ \rho\in[0,1)\right\}.$$

Definition 1.11. Let $f\in\mathcal{M}$, then f is said to belong to the class $\mathcal{M}^{\epsilon*}_{p,\,q}(\beta,\,n,\,m)$, if $\exists\,g\in\mathcal{M}^{\epsilon*}_{p,\,q}(\beta,\,n,\,m)$, satisfying

$$\left|\ \frac{f(z)}{g(z)}-1\ \right|<1-\epsilon,\ z\in\mathcal{D}^*,\ \epsilon\in[0,1).$$

Theorem 1.12. *Let* $g(z)\in\mathcal{M}^*_{p,q}(\beta,\,n,\,m)$, *and*

$$\epsilon=1-\frac{\rho p^nq^n(pq+\beta)[m+1]_{p,q}[m+2]_{p,q}}{(pq+\beta)p^nq^n[m+1]_{p,q}[m+2]_{p,q}-[2]_{p,q}(1-\beta)}$$

then $N_p(g)\subset\mathcal{M}^{\epsilon*}_{p,q}(\beta,\,n,\,m)$.

Proof. Let $f(z)\in N_\rho(g)$, then by the definition (1.10); $\sum_{r=1}^{\infty}r|\,a_r-g_r\,|\leq\rho$ which implies that $\sum_{r=1}^{\infty}|\,a_r-g_r\,|\leq\rho$.

As $g(z)\in\mathcal{M}^*_{p,q}(\beta,\,n,\,m)$, by the corollary (1.5),

$$\sum_{r=1}^{\infty} g_r \leq \frac{[2]_{p,q}(1-\beta)}{p^n q^n (pq+\beta)[m+1]_{p,q}[m+2]_{p,q}}.$$

Hence

$$\left| \frac{f(z)}{g(z)} - 1 \right| < \frac{\sum_{r=1}^{\infty} | a_r - g_r |}{1 - \sum_{r=1}^{\infty} g_r}$$

$$\leq \frac{\rho p^n q^n (pq+\beta)[m+1]_{p,q}[m+2]_{p,q}}{(pq+\beta) p^n q^n [m+1]_{p,q}[m+2]_{p,q} - [2]_{p,q}(1-\beta)} = 1 - \epsilon.$$

Thus, by the definition (1.11); $f \in \mathcal{M}^*_{p,q}(\beta, n, m)$. □

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