Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 183-193

NORDHAUS-GADDUM TYPE RELATIONS FOR SOME TOPOLOGICAL INDICES

J. V. KUREETHARA, B. K. MAJHI, P. MAHALANK and I. N. CANGUL

Department of Mathematics
Christ University, Bangalore
560029, India
E-mail: frjoseph@christuniversity.in

Department of Mathematics
School of Applied Sciences
Centurion University of Technology
and Management, Odisha, India
E-mail: bhairabakumar.majhi@cutm.ac.in

Department of Mathematics
SDM College of Engineering
and Technology Dharwad
580002, Karnataka, India

Bursa Uludag University
Department of Mathematics
16059 Bursa, Turkey
E-mail: cangul@uludag.edu.tr

Abstract

A Nordhau-Gaddum type relation is the one associated with some parameter of a graph and its complement. Nordhaus-Gaddum type relations have been studied since 1956. The most famous relation is the upper bound for the sum of the chromatic numbers of a graph and its complement. In this work, the Nordhaus-Gaddum type difference relation for the first Zagreb index, the Nordhaus-Gaddum type sum relation with the help of Cangul index, the Nordhaus-Gaddum type sum relation for the second Zagreb index and total irregularity index are studied. Further, Nordhaus-Gaddum type relations for omega invariant are also found out.

2020 Mathematics Subject Classification: 05C92, 05C07, 05C30, 05C38, 05A20, 05C10, 92E10.

Keywords: Mathematical Chemistry, Cangul Index, Nordhaus-Gaddum Type Inequality, Omega Invariant, Irregularity Index.

Received March 3, 2020; Accepted December 22, 2020

1. Introduction

A good amount of research in graph theory deals with graph invariants. Vertex degrees, edge degrees, radius, diameter, eccentricity, independence number, girth, chromatic number, domination number etc. are some of the most frequently used graph invariants. There are also topological graph indices with several applications in chemistry, physics, pharmacology, and electronic engineering and network sciences. In 1956, Nordhaus and Gaddum used a clever method of associating a graph with its complement. Let G be a graph. Let its complement be $\overline{G}$. Both the graphs $\overline{G}$ and G are simple undirected graphs having the same vertex set. However, G has an edge if and only if $\overline{G}$ does not have it. Nordhaus and Gaddum obtained relations for the chromatic number of G [9]. Such a relation helps to obtain information on one of G and $\overline{G}$ by using the known information on the other. Since then, many authors used the same method for many other graph parameters in over 300 papers. In [3], Aouchiche and Hansen gave a beautiful survey of all Nordhaus-Gaddum type results up to 2013.

Several authors obtained Nordhaus-Gaddum type identities or inequalities for some topological graph indices. Let $M_1(G) = \sum_{v \in V(G)} d_G^2 u$ and $M_2(G) = \sum_{uv \in E(G)} d_G u d_G v$. Here, $M_1(G)$ is the first Zagreb index and $M_2(G)$ is the second Zagreb index. They were defined by Gutman and Trinajstic in [8]. In [4], such relations for first and second Zagreb indices are obtained. In [12], reciprocal relations for some molecular topological indices are obtained. In [4], as exact statements for $M_1(G) + M_1(\overline{G})$ and $M_2(G) + M_2(\overline{G})$ could not be given, some upper bounds were obtained. In this paper, we supply exact statement for $M_1(\overline{G}) + M_1(G)$ and also for Above the first G, a bar is required. By means of a new function

$$C(G) = \sum_{v_i v_j \in V(G)} d_G i d_G j.$$

Let $F(G)$ be the forgotten index of the graph G. It is also known as the third Zagreb Index. It is defined as $F(G) = \sum_{v \in V(G)} d_G^3 v.$ We provide in this paper, a nice relation for $F(G) + F(\overline{G})$ in terms of m, n and $M_1(G)$. For irregularity indices Bell and total irregularity, it is easy to see that $B(\overline{G}) = B(G)$ and $Irr_t(\overline{G}) = Irr_t(G)$ where

$$B(G) = \sum_{u \in V(G)} (d_G v - \frac{2m}{n})^2$$

and

$$Irr_t(\overline{G}) = \sum_{u \in V(G)} | d_G v - \frac{2m}{n} |,$$

See [1, 2, 7, 11]. Finally in this paper, we obtain three Nordhaus-Gaddum type relations for $\Omega(G) + \Omega(\overline{G}), \Omega(G) - \Omega(\overline{G})$ and $\Omega(G) \cdot \Omega(\overline{G})$ where Ω is a recently defined topological graph invariant having many practical uses, see [5, 6].

2. Main Results

Let the order and size of a graph G be n and m, respectively. For convenience, we shall denote the order and size of its complement above G, n and m, there are bars required. For a vertex v of a graph G, let $d_G v$ denote the degree of v in G. Let $\Delta(G)$ be the maximum degree and $\delta(G)$ be the minimum degree of a graph G. Let $\Delta'(G)$ be the second maximum degree.

In [4], upper bounds for the Nordhaus-Gaddum type sum formulae $M_1(G) + M_1(\overline{G})$ and $M_2(G) + M_2(\overline{G})$ were obtained as

$$M_1(G) + M_1(\overline{G}) \leq \frac{(n(n-2) - 2m + \delta + 1)^2 + (2m - \Delta)^2}{n-1} + \Delta^2 + ((n-1) - \delta)^2$$
$$+ \frac{1}{4}(n-1)((\Delta - \delta)^2 + (\Delta' - \delta)^2)$$

and

$$M_2(G)+M_2(\overline{G}) \leq \frac{n(n-1)^3}{2}+2m^2-3m(n-1)^2$$

$$+(n-\frac{3}{2})(\frac{(2m-\Delta)^2}{n-1}+\Delta^2+\frac{(n-1)(\Delta'-\delta)^2}{4}).$$

The first inequality turns to be equality if and only if G is regular or P_3.

When G and $\overline{G}$ are both connected, Zhou and Trinajsti'c improved the upper bound for the first Zagreb index as

$$M_1(G)+M_1(\overline{G}) \leq n^3-4n^2+3n+8.$$

We now obtain an exact formula for the difference $M_1(\overline{G})-M_1(G)$.

$$M_1(\overline{G})=\sum\nolimits_{v\in V(G)} d_{\overline{G}}^2 v$$

$$=\sum_{v\in V(G)}(n-1-d_G v)^2$$

$$=(n-1)^2\sum_{v\in V(G)}1-2(n-1)\sum_{v\in V(G)}d_G v+\sum_{v\in V(G)}d_G^2 v$$

$$=n(n-1)^2-2(n-1)\cdot 2m+M_1(G).$$

Hence we proved

Theorem 1. $M_1(\overline{G})-M_1(G)=(n-1)(n^2-n-4m)$.

For a simple connected graph G, we define the Cangul index denoted by $C(G)$ as,

$$C(G)=\sum_{v_i, v_j\in V(G)} d_G i d_G j.$$

$C(G)$ can have other expressions such as,

$$C(G) = \sum_{v_i, v_j \in V(G)} d_G i d_G j$$

$$= \sum_{i=1, a_i \geq 2} \binom{a_i}{2} d_G^2 i + \sum_{l \leq i \leq j \leq k} \binom{a_i}{1} \binom{a_i}{1} d_G i d_G j$$

$$= \sum_{v_i v_j \in E(G) \cup E(\overline{G})} d_G i d_G j$$

with $D = \{d_1^{(a_1)}, d_{d_2}^{(a_2)}, \ldots, d_k^{(a_k)}\}$ as the degree sequence of G.

Next, we shall obtain an exact formula for the sum $M_2(G) + M_2(\overline{G})$ by means of the index $C(G)$.

Theorem 2.

$$M_2(G) + M_2(\overline{G}) = \frac{n(n-1)^3}{2} - 3m(n-1)^2 + (n-1)M_1(G) + C(G)$$

where $C(G)$ *is the Cangul index and* $D = \{d_1^{(a_1)}, d_d^{(a_1)}, \ldots, d_k^{(a_k)}\}$ *is the degree sequence of* G.

Proof. $M_2(G) + M_2(\overline{G}) = \sum_{v_i v_j \in (G)} d_G i d_G j + \sum_{v_i v_j \in (\overline{G})} d_{\overline{G}} i d_{\overline{G}} j$

$$= \sum_{v_i v_j \in E(G)} d_G i d_G j$$

$$+ \sum_{v_i v_j \in E(\overline{G})} (n - 1 - d_G i)(n - 1 - d_{Gj})$$

$$= \sum_{v_i v_j \in E(G)} d_G i d_G j$$

$$+ \sum_{v_i v_j \in E(\overline{G})} [(n-1)^2 + d_G i d_G j - (n-1)(d_G i + d_g j)]$$

$$-(n-1)\sum_{v_i \in V(G)} d_i(n-1-d_i)$$

$$= \sum_{v_i v_j \in E(G) \cup E(\overline{G})} d_G i d_G j + \left(\frac{n(n-1)}{2} - m\right)(n-1)^2$$

$$-(n-1)[(n-1)\cdot 2m - M_1(G)]$$

$$= \frac{n(n-1)^3}{2} - 3m(n-1)^2 + (n-1)M_1(G) + C(G).$$

Note that we have given the additive Nordhaus-Gaddum type for the second Zagreb index $M_2(G)$ in terms of the index $C(G)$. Hence we need to investigate $C(G)$ a little bit further. First let

$$C_1(G) = \sum_{i=1,\, a_i \leq 2}^{k} \binom{a_i}{2} d_G^2 i$$

and

$$C_2(G) = \sum_{1 \leq i \leq j \leq k} \binom{a_i}{1}\binom{a_j}{1} d_G i d_G j.$$

Hence $C(G) = C_1(G) + C_2(G)$.

Lemma 3. *For a path graph P_n with $n \geq 4$, we have*

$$C_1(P_n) = (n-2)^2 + (n-3)^2,$$

$$C_2(P_n) = 4n - 8$$

and hence

$$C(P_n) = (n-1)^2 + (n-2)^2.$$

Proof. As the degree sequence $D(P_n)$ of P_n is

$$D(P_n) = \{1^{(2)}, 2^{(n-2)}\},$$

we have

$$C_1(P_n) = \binom{2}{2}1^2 + \binom{n-2}{1}2^2$$

$$= (n-2)^2 + (n-3)^2.$$

Similarly

$$C_2(P_n) = \binom{2}{1} = \binom{n-2}{1}1 \cdot 2$$

$$= 4n - 8.$$

Lemma 4. *Let* C_n *be the cyclic graph with* $n \geq 3$. *Then*

$$C_1(C_n) = 2n^2 - 2n,$$

$$C_2(C_n) = 0$$

and hence

$$C(C_n) = 2n^2 - 2n.$$

Proof. $C_1(C_n) = \binom{n}{2} \cdot 2^2 = 2n^2 - 2n$ and $C_2(C_n) = 0$ as the degree sequence of C_n is $\{2^{(n)}\}$.

The following three results can be proven similarly.

Lemma 5. *For* $n \geq 4$, *let* S_n *be the star on n vertices*. *Then*

$$C_1(S_n) = \frac{(n-1)(n-2)}{2}$$

$$C_2(S_n) = (n-1)^2$$

and hence

$$C(S_n) = \frac{(n-1)(3n-4)}{2}.$$

Lemma 6. *For* $n \geq 3$, *let* K_n *be the complete graph on n vertices. Then*

$$C_1(K_n) = \frac{n(n-1)^3}{2}.$$

$$C_2(K_n) = 0$$

and hence

$$C(K_n) = \frac{n(n-1)^3}{2}.$$

Lemma 7. *For* $r \geq s \geq 1$, *let* $T_{r,s}$ *be the tadpole graph obtained by adding a path* P_s *to a vertex on a cycle* C_r. *having* $r + s$ *vertices. Then*

$$C_1(T_{r,s}) = 2(r+s-2)(r+s-3)$$

$$C_2(T_{r,s}) = 8(r+s) - 13$$

and

$$C(T_{r,s}) = 2(r+s)(r+s-1) - 1.$$

Next we obtain the additive Nordhaus-Gaddum type relation for the forgotten index $F(G) = \sum_{v \in V(G)} d_G^3 v$.

Theorem 8. $F(G) + F(\overline{G}) = (n-1)(n(n-1)^2 - 6m(n-1) + 3M_1(G))$.

Proof.

$$\begin{aligned}
F(G) + F(\overline{G}) &= \sum_{v \in V(G)} d_G^3 v + \sum_{v \in V(\overline{G})} d_{\overline{G}}^3 v \\
&= \sum_{v \in V(\overline{G})} d_G^3 v + \sum_{v \in V(\overline{G})} (n-1-d_G v)^3 \\
&= (n-1)^3 \sum_{v \in V(\overline{G})} 1 - 3(n-1)^2 \sum_{v \in V(\overline{G})} d_G v + 3(n-1) \sum_{v \in V(\overline{G})} d_G^2 v \\
&= (n-1)^3 \cdot n - 3(n-1)^2 \cdot 2m + 3(n-1) \cdot M_1(G) \\
&= (n-1)(n(n-1)^2 - 6m(n-1) + 3M_1(G)).
\end{aligned}$$

The regularity of a graph is a well-known and useful property. If all vertex degrees in a graph are equal, then we call the graph regular. A graph which is not regular is called irregular. There are several topological indices like Albertson, sigma, Bell, total irregularity indices to determine the irregularity degree of graph. For the Bell index $B(G) = \sum_{v \in V(G)} (d_G v - \frac{2m}{n})^2$. It is well-known that $B(\overline{G}) = B(G)$. We now prove a similar result for the total irregularity index

$$Irr_t(G) = \sum_{v \in V(G)} | d_G v - \frac{2m}{n} |.$$

Then we have.

Theorem 9. *For any simple graph G, we have*

$$Irr_t(\overline{G}) = Irr_t(G).$$

Proof.

$$\begin{aligned} Irr_t(\overline{G}) &= \sum_{v \in V(\overline{G})} | d_{\overline{G}} v - \frac{2\overline{m}}{\overline{n}} | \\ &= \sum_{v \in V(G)} | n - 1 - d_G v - \frac{n(n-1) - 2m}{n} | \\ &= \sum_{v \in V(G)} | n - 1 - d_G v - n + 1 + \frac{2m}{n} | \\ &= \sum_{v \in V(G)} | d_G v - \frac{2m}{n} | \\ &= Irr_t(G). \end{aligned}$$

3. Nordhaus-Gaddum type Relations for Omega Invariant

Probably, the most popular graph invariant is the Euler characteristic, known since 18th century. Recently, a new topological graph invariant named as omega invariant has been defined by Delen and Cangul [5] by

$$\Omega(D) = 2(m - n).$$

If $D = \{1^{(a_1)}, 2^{(a_2)}, \ldots, \Delta^{(a_\Delta)}\}$ is a realizable degree sequence, then the omega invariant of D is defined by

$$\Omega(D) = \sum_{i=1}^{\Delta} (i-2)a_i.$$

For more details of the omega invariant and its properties and applications, see [5, 6].

Our aim is to give some Nordhaus-Gaddum type results for omega invariant. Actually we shall give three such relations as below:

Theorem 10. $\Omega(G) + \Omega(\overline{G}) = n(n-5)$.

Proof.

$$\begin{aligned}\Omega(G) + \Omega(\overline{G}) &= 2(m-n) + 2(\overline{m} - \overline{n}).\\ &= 2(m-n) + 2\left(\frac{n(n-1)}{2} - m - n\right)\\ &= n(n-5).\end{aligned}$$

Theorem 11. $\Omega(G) - \Omega(\overline{G}) = 4m - n^2 + n$.

Proof.

$$\begin{aligned}\Omega(G) - \Omega(\overline{G}) &= 2(m-n) - (\overline{m} - \overline{n})\\ &= 2m - 2n + 2\left(\frac{n(n-1)}{2} - m - n\right)\\ &= 4m - n^2 + n.\end{aligned}$$

Theorem 12. $\Omega(G) \cdot \Omega(\overline{G}) = -2n^3 + (2m+6)n^2 - 2mn - 4m^2$.

Proof.

$$\begin{aligned}\Omega(G) \cdot \Omega(\overline{G}) &= 2(m-2) \cdot 2(\overline{m} - \overline{n})\\ &= 4(m\overline{m} - m\overline{n} - \overline{m}n + n\overline{n})\\ &= 4\left(m\left(\frac{n(n-1)}{2} - m\right) - mn - \left(\frac{n(n-1)}{2} - m\right)n + n^2\right)\\ &= -2n^3 + (2m+6)n^2 - 2mn - 4m^2.\end{aligned}$$

4. Conclusion

Graph invariants uniquely represent graphs. With the proper definition of the complement of a graph, it is worth exploring the relation of an invariant of a graph with the same invariant of its complement. In this exploration, the relation between a graph and its complement, bounds of invariants etc. could be studied. Nordhaus-Gaddum type relations are coming under this area of study. A more generalized definition of graph complement is available in [10]. Nordhaus-Gaddum type relations associated with sum, difference, product etc. could be further explored with the generalized graph complements.

References

[1] H. Abdo, N. Cohen and D. Dimitrov, Graphs with Maximal Irregularity, Filomat 28(6) (2014), DOI: 10.2298/FIL1407315A.

[2] M. O. Albertson, The Irregularity of a Graph, Ars Comb. 46 (1997), 219-225.

[3] M. Aouchiche, P. Hansen and A Survey of Nordhaus-Gaddum Type Relations, Discrete Applied Mathematics 161(4-5) (2013), 466-546.

[4] K. C. Das, I. Gutman and B. Zhou, New upper bounds on Zagreb indices, J. Math. Chem. 46 (2009), 514-521.

[5] S. Delen and I. N. Cangul, A New Graph Invariant, Turkish Journal of Analysis and Number Theory 6(1) (2018), 30-33.

[6] S. Delen and I. N. Cangul, Extremal Problems on Components and Loops in Graphs, Acta Mathematica Sinica, English Series 35(2) (2019), 161-171.

[7] I. Gutman, M. Togan, A. Yurttas, A. S. Cevik and I. N. Cangul, Inverse problem for sigma index, MATCH Commun. Math. Comput. Chem. 79(2) (2018), 491-508.

[8] I. Gutman and N. Trinajsti'c, Graph Theory and Molecular Orbitals III, Total π-Electron Energy of Alternant Hydrocarbons, Chem. Phys. Lett. 17 (1972), 535-538.

[9] E. A. Nordhaus and J. Gaddum, On Complementary Graphs, Amer. Math. Monthly 63 (1956), 175-177.

[10] E. Sampathkumar and L. Pushpalatha, Complement of a Graph: A Generalization, Graphs Comb 14 (1998), 377-392. https://doi.org/10.1007/PL00021185

[11] M. Togan, A. Yurttas, U. Sanli, F. Celik and I. N. Cangul, Inverse problem for Bell Index, FILOMAT 34(2) (2020).

[12] B. Zhou and N. Trinajsti'c, On reciprocal molecular topological index, J. Math. Chem. 44 (2008), 235-243.