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SOME COUPLED FIXED POINT RESULTS USING INTEGRAL CONTRACTIONS WITH RATIONAL EXPRESSIONS

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Abstract

In this paper, we discuss the unique existence of a coupled coincidence point, in the turf of complex valued metric spaces, for two distinct pairs of maps via an integral contraction with rational expressions, sequentially, we establish the unique existence of a common coupled fixed point for a pair of 2-variable mappings in this context. Further, we illustrate some appropriate examples to justify the consistency of the theory.

1. Introduction

Azam et al. [1] generalized the idea of metric space into complex valued metric space and established the existence of unique common fixed point for a pair of self mappings that satisfy a rational contraction. Subsequently, Sintunavarat et al. [10] and Rouzkard et al. [7] are some others who extended the works of Azam et al. using a rational contractive condition involving control functions.

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For solving coupled system of equations, Bhaskar et al. [3] proposed the concept of coupled fixed points in 2006. In follow, Lakshmikantham [6] initiated and studied the concept of coupled common fixed point for a 2-variable map and a self map in the context of partially ordered metric spaces. Later, Kutbi et al. [5] developed the concept of common coupled fixed points for a pair of 2-variable mappings in the context of complex valued metric space.

The work's based on integral type of contractive condition in metric fixed point theory is initiated by Branciari [2]. Khojasteh et al. [4] proposed a new class of integration with respect to a cone in order to establish the unique existence of a fixed point in cone metric space. Sequentially, Zada et al. [11] extended Khojasteh et al. integration with respect to complex numbers and proved common fixed point theorems in complex valued metric spaces.

We provide some basic definitions and results in section 2 that will be important in following sections. In section 3, we establish the unique existence of coupled coincidence point for a 2-variable map and a self map that satisfy an integral contraction, in the perspective of complete asymptotically regular complex valued metric space. In the same way, we exhibit the existence of unique common coupled fixed point for a pair of 2-variable mapping. Finally, we provide examples to support our theory.

2. Preliminaries

Let $\mathbb{C}$ denote the collection of all complex numbers and $z, w \in \mathbb{C}$. Define $\precsim$ as a partial order on $\mathbb{C}$ as follows:

$z \precsim w$ if and only if $\operatorname{Re}(z) \leq \operatorname{Re}(w)$ and $\operatorname{Im}(z) \leq \operatorname{Im}(w)$.

If one of the following conditions is satisfied:

- $\operatorname{Re}(z) = \operatorname{Re}(w)$, $\operatorname{Im}(z) < \operatorname{Im}(w)$;
- $\operatorname{Re}(z) = \operatorname{Re}(w)$, $\operatorname{Im}(z) = \operatorname{Im}(w)$;
- $\operatorname{Re}(z) = \operatorname{Re}(w)$, $\operatorname{Im}(z) < \operatorname{Im}(w)$;
- $\operatorname{Re}(z) = \operatorname{Re}(w)$, $\operatorname{Im}(z) = \operatorname{Im}(w)$,

Then $z \precsim w$ is true. Specifically, we will write $z \precnsim w$ if $z \neq w$ and one of $(P1)$, $(P2)$ or $(P4)$ are satisfied; and $z \prec w$ only if $(P3)$ is satisfied.

Definition 2.1 [1]. Let M be a nonempty set. A mapping ρ from M^2 to $\mathbb{C}$ is said to be a complex valued metric if it satisfies the conditions given below:

- $0 \precsim \rho(z, w)$ for all $z, w \in M$ and $\rho(z, w) = 0$ if and only if $z = w$;
- $\rho(z, w) = \rho(w, z)$ for all $x, y \in M$;
- $\rho(z, w) \precsim \rho(z, u) + \rho(u, w)$ for all $z, w, u \in M$.

The pair (M, ρ) is called a complex valued metric space.

In complex valued metric spaces, the concepts of convergence, Cauchy, and complete can be defined similarly to how they are defined in metric spaces. Unless otherwise mentioned, M denotes the complex valued metric space (M, ρ) throughout this paper.

Lemma 2.2 [1]. *Let* $\{z_n\}$ *be a sequence in M. Then*

(i) $\{z_n\}$ *converges to z if and only if* $|\rho(z_n, z)| \to 0$ *as* $n \to \infty$.

(ii) $\{z_n\}$ *is a Cauchy sequence if and only if* $|\rho(z_n, z_{n+m})| \to 0$ *as* $n \to \infty$.

Let $\mathbb{R}_+$ denotes the collection of nonnegative real numbers and Ψ be the class of nonnegative, nondecreasing Lebesgue integrable self mapping on $\mathbb{R}_+$, that is summable on any compact subset of $\mathbb{R}_+$, and such that for each $\epsilon > 0, \int_0^{\epsilon} \psi(s) ds > 0$.

Let $\mathbb{C}_+ = \{w \in \mathbb{C} : 0 \precsim w\}$. For any $z, w \in \mathbb{C}_+$ with $z \precsim w$, define

$[z, w] = \{z(t) \in \mathbb{C}_+ : z(t) = z + t(w - z)$ for some $t \in [0, 1]\}$,

$(z, w] = \{z(t) \in \mathbb{C}_+ : z(t) = z + t(w - z)$ for some $t \in (0, 1]\}$.

A set $Q = \{z = w_0, w_1, w_2, \ldots, w_n = w\}$ is a partition of $[z, w]$ if and only if the sets $\{[w_{i-1}, w_i)\}_{i=1}^{n}$ are disjoint in pairs and their union along with W is $[z, w]$.

Let $\zeta : [z, w] \to \mathbb{C}$ be a mapping defined by $\zeta(u) = \psi_1(a) + i\psi_2(b)$, where $u = a + ib \in [z, w]$ and $\psi_1, \psi_2 \in \Psi$. For a partition Q of $[z, w]$, the lower summation is

$$L_n(\zeta, Q) = \sum_{j=0}^{n-1} (\psi_1(a_i) + i\psi_2(b_i)) | w_{i+1} - w_i |$$

and the upper summation is

$$U_n(\zeta, Q) = \sum_{j=0}^{n-1} (\psi_1(a_{i+1}) + i\psi_2(b_{i+1})) | w_{i+1} - w_i |,$$

Where $w_i = a_i + b_i$, $i = 0, 1, 2, \ldots, n$. If $\lim_{n\to\infty} L_n(\zeta, Q)$ and $\lim_{n\to\infty} U_n(\zeta, Q)$ exists and equal, then we can define $\int_z^w \zeta d_c = \lim_{n\to\infty} L_n(\zeta, Q) = \lim_{n\to\infty} U_n(\zeta, Q)$.

If $\int_z^w \zeta d_c$ exists, then we say that ζ is a complex integrable function on $[z, w]$. Also, $\mathfrak{I}([z, w], \mathbb{C})$ denotes the collection of all complex integrable functions on $[z, w]$. If ζ is complex integrable function on for all $[z, w]$ in $\mathbb{C}_+$, then we say ζ is complex integrable function on $\mathbb{C}_+$.

Lemma 2.3 [11]. *Let* $\zeta \in \mathfrak{I}([z, w], \mathbb{C})$ *and* $\{z_n\}$ *be a sequence in* $[z, w]$, *then* $\lim_{n\to\infty}, \int_0^{z_n} \zeta d_c = 0$ *if and only if* $z_n \to 0$, *as* $n \to \infty$.

Definition 2.4[4]. A complex integrable function ζ on $\mathbb{C}_+$ is called subadditive if

$$\int_0^{z+w} \zeta d_c \precsim \int_0^{z} \zeta d_c + \int_0^{w} \zeta d_c, \text{ for all } z, w \in \mathbb{C}_+.$$

Definition 2.5[8]. A sequence $\{z_n\}$ in a metric M space is said to be asymptotically regular if $\lim_{n\to\infty} \rho(z_{n+1}, z_n) = 0$.

Definition 2.6[8]**.** A complex valued metric space M is called complete asymptotically regular if every asymptotically regular sequence $\{z_n\}$ in M converges to some point in M.

Definition 2.7[6]**.** Let $\mathfrak{B}, \mathfrak{B}' : M^2 \to M$ and $g : M^2 \to M$. An element $(z, w) \in M^2$ is called

- coupled coincidence point of $\mathfrak{B}$ and g if $\mathfrak{B}(z, w) = g(z)$ and $\mathfrak{B}(w, z) = g(w)$.
- coupled common fixed point of $\mathfrak{B}$ and g if $\mathfrak{B}(z, w) = g(z) = z$ and $\mathfrak{B}(z, w) = g(w) = w$.
- coupled coincidence point of $\mathfrak{B}$ and $\mathfrak{B}'$ if $\mathfrak{B}(z, w) = \mathfrak{B}'(z, w)$ and $\mathfrak{B}(w, z) = \mathfrak{B}'(w, z)$.
- common coupled fixed point of $\mathfrak{B}$ and $\mathfrak{B}'$ if $\mathfrak{B}(z, w) = \mathfrak{B}'(z, w) = z$ and $\mathfrak{B}(w, z) = \mathfrak{B}'(w, z) = w$.

3. Coupled Coincidence Point Theorems

First let us fix some notations for our convenience. Let $\mathfrak{B}, \mathfrak{B}' : M^2 \to M$ and $g : M^2 \to M$, then we define

$$L(z, w, z', w') = \frac{\rho(z, w)}{1 + \rho(z, w)};$$

$$L_{\mathfrak{B}}(z, w, z', w') = \frac{\rho(\mathfrak{B}(z, w), \mathfrak{B}(z', w'))}{1 + \rho(\mathfrak{B}(z, w), \mathfrak{B}(z', w'))};$$

$$L_{\mathfrak{B}}^{\mathfrak{B}'}(z, w, z', w') = \frac{\rho(\mathfrak{B}(z, w), \mathfrak{B}'(z', w'))}{1 + \rho(\mathfrak{B}(z, w), \mathfrak{B}'(z', w'))};$$

$$L_{\mathfrak{B}}(z, w) = \frac{\rho(\mathfrak{B}(z, w), z)}{1 + \rho(\mathfrak{B}(z, w), z)};$$

$$L_{\mathfrak{B}'}(z, w) = \frac{\rho(\mathfrak{B}'(z, w), z)}{1 + \rho(\mathfrak{B}'(z, w), z)};$$

$$L_g(z, w) = \frac{\rho(g(z), g(w))}{1 + \rho(g(z), g(w))}.$$

Let $\mathfrak{I}_{\mathbb{C}_+}$ be the collection of all complex integrable functions on $\mathbb{C}_+$ such that for each $\epsilon \succ 0, \int_o^\epsilon \zeta d_c \succ 0.$ Throughout the section the labels are used in this meaning unless otherwise stated.

Theorem 3.1. *Let* $\mathfrak{B} : M^2 \to M$ *and* $g : M \to M$ *be two mappings such that* $\mathfrak{B}(M^2) \subseteq g(M)$, $g(M)$ *is a closed subspace of M and g is one-one. If there exist* $k \in [0, 1)$ *and* $\zeta \in \mathcal{J}_{\mathbb{C}_+}$ *satisfying the contractive condition*

$$\int_0^{L_{\mathfrak{B}}(z, w, z', w')} \zeta d_c \precsim \frac{k}{2} \int_0^{l_g(z, z') + l_g(w, w')} \zeta d_c, \tag{1}$$

for all $z, z', w, w' \in M$, *then* $\mathfrak{B}$ *and* g *have a unique coupled coincidence point.*

Proof. Let $(z_0, w_0) \in M^2$. As $\mathfrak{B}(M^2) \subseteq g(M)$, we can construct sequences $\{z_n\}$ and $\{w_n\}$ in M with $\mathfrak{B}(z_n, w_n) = g(z_{n+1})$ and $\mathfrak{B}(z_n, z_n) = g(z_{n+1})$; by contractive condition (1), we get

$$\int_0^{l_g(z_{n+1}, z_n)} \zeta d_c = \int_0^{L_{\mathfrak{B}}(z_n, w_n, z_{n-1}, w_{n-1})} \zeta d_c \precsim \frac{k}{2} \int_0^{l_g(z_n, z_{n-1}) + l_g(w_n, w_{n-1})} \zeta d_c. \tag{2}$$

Similarly, we can prove that

$$\int_0^{l_g(w_{n+1}, w_n)} \zeta d_c \precsim \frac{k}{2} \int_0^{l_g(z_n, z_{n-1}) + l_g(w_n, w_{n-1})} \zeta d_c. \tag{3}$$

Adding (2) and (3) we get

$$\int_0^{l_g(z_{n+1}, z_n) + l_g(w_{n+1}, w_n)} \zeta d_c \precsim k \int_0^{l_g(z_n, z_{n-1}) + l_g(w_n, w_{n-1})} \zeta d_c$$

$$\precsim k^2 \int_0^{l_g(z_{n-1}, z_{n-2}) + l_g(w_{n-1}, w_{n-2})} \zeta d_c$$

$$\vdots \vdots$$

$$\precsim k^n \int_0^{l_g(z_1, z_0)+l_g(w_1, w_0)} \zeta d_c.$$

Letting limit as $n \to \infty$, we have $\lim_{n\to\infty}, \int_0^{l_g(z_{n+1}, z_n)+l_g(w_{n+1}, w_n)} \zeta d_c = 0$, which implies $\lim_{n\to\infty}, \int_0^{l_g(z_{n+1}, z_n)} \zeta d_c = 0$ and $\lim_{n\to\infty} \int_0^{l_g(w_{n+1}, w_n)} \zeta d_c = 0$.

By lemma 2.3, we have both the sequences $\{l_g(z_{n+1}, z_n)\}$ and $\{l_g(w_{n+1}, w_n)\}$ in $\mathbb{C}_+$ converge to 0. Thus the real sequences $\{\| \rho(g(z_{n+1}), g(z_n)) \|\}$ and $\{\| \rho(g(w_{n+1}), g(w_n)) \|\}$ converge to 0 and hence the sequences $\{g(z_n)\}$ and $\{g(w_n)\}$ in M are asymptotically regular.

Since M is a complete asymptotically regular complex valued metric space, the sequences $\{g(z_n)\}$ and $\{g(w_n)\}$ are converges to r and s respectively. Since $g(M)$ is closed, the elements r and s are in $g(M)$. Thus there exists $z, w \in M$ such that $g(z) = r$ and $g(w) = s$. Therefore

$$\lim_{n\to\infty} \mathfrak{B}(z_n, w_n) = \lim_{n\to\infty} g(z_n) = g(z) \text{ and}$$

$$\lim_{n\to\infty} \mathfrak{B}(w_n, z_n) = \lim_{n\to\infty} g(w_n) = g(w).$$

From contractive condition (1), we have

$$\int_0^{l_{\mathfrak{B}}(z_n, w_n, z, w)} \zeta d_c \precsim \frac{k}{2} \int_0^{l_g(z_n, z)+l_g(w_n, w)} \zeta d_c.$$

Letting as limit $n \to \infty$, we get

$$\int_0^{\frac{\rho(g(z), \mathfrak{B}(z, w))}{1+\rho(g(z), \mathfrak{B}(z, w))}} \zeta d_c = 0,$$

which implies $\frac{\rho(g(z), \mathfrak{B}(z, w))}{1 + \rho(g(z), \mathfrak{B}(z, w))} = 0$ and hence $g(z) = \mathfrak{B}(z, w)$. Similarly, we can show that $g(w) = \mathfrak{B}(w, z)$. Thus (z, w) is a coupled coincidence point of $\mathfrak{B}$ and g. If there exist another coupled coincidence point (z', w') of $\mathfrak{B}$ and g, by contractive condition (1), we get

$$\int_0^{l_{\mathfrak{B}}(z,\, w,\, z',\, w')} \zeta d_c \precsim \frac{k}{2} \int_0^{l_g(z,\, z') + l_g(w,\, w')} \zeta d_c,$$

which implies

$$\int_0^{l_g(z,\, z')} \zeta d_c \precsim \frac{k}{2} \int_0^{l_g(z,\, z') + l_g(w,\, w')} \zeta d_c. \tag{4}$$

Similarly, using contractive condition (1), we get

$$\int_0^{l_g(w,\, w')} \zeta d_c \precsim \frac{k}{2} \int_0^{l_g(z,\, z') + l_g(w,\, w')} \zeta d_c. \tag{5}$$

Adding (4) and (5), we get

$$\int_0^{l_g(z,\, z') + l_g(w,\, w')} \zeta d_c \precsim k \int_0^{L_g(z,\, z') + l_g(w,\, w')} \zeta d_c.$$

Since $0 \leq k < 1$, we have $\int_0^{L_g(z,\, z') + l_g(w,\, w')} \zeta d_c = 0$, which implies $l_g(z, z') + l_g(w, w') = 0$ and hence $g(z) = g(z')$ and $g(w) = g(w')$. Since g is one-one, $z = z'$ and $w = w'$. Thus (z, w) is a unique coupled coincidence point of $\mathfrak{B}$ and g.

Remark 3.2. If we drop the hypothesis g is one-one in Theorem 3.1, then we can't guaranteed the unique existence of coupled coincidence point of $\mathfrak{B}$ and g but there exist a coupled coincidence point of $\mathfrak{B}$ and g.

Example 3.3. Let $M = \{a + ia : 0 \leq a \leq 1\}$ be a complete asymptotically regular complex valued metric space with the metric

$$\rho(z, w) = | R_e(z) - R_e(w) | + i | \mathrm{Im}(z) - \mathrm{Im}(w) |.$$

Define $\mathfrak{B} : M^2 \to M$ and $g : M \to M$ by

$$\mathfrak{B}(z, w) = \frac{a + b}{30} + i \frac{a + b}{30} \text{ and } g(z) = z,$$

where $z = a + ai$ and $w = b + bi$. If we let $\zeta(t) = 1 + i$, then the contractive condition (1) becomes

$$L_{\mathfrak{B}}(z, w, z', w') \precsim \frac{k}{2}(l_g(z, z') + l_g(w, w')),$$

where

$$L_{\mathfrak{B}}(z, w, z', w') = \frac{\left|\frac{a+b-c-d}{30}\right| + i\left|\frac{a+b-c-d}{30}\right|}{\left|\frac{a+b-c-d}{30}\right| + i\left|\frac{a+b-c-d}{30}\right|};$$

$$l_g(z, z') = \frac{|a-c| + i|a-c|}{1 + |a-c| + i|a-c|};$$

$$l_g(w, w') = \frac{|b-d| + i|b-d|}{1 + |b-d| + i|b-d|}.$$

It is clear to see that for any $k \geq \frac{1}{2}$, the above contractive condition is true for all $z, w, z', w' \in M$. Thus by Theorem 3.1, $\mathfrak{B}$ and g have a unique coupled coincidence point.

Corollary 3.4. *Let* $\mathfrak{B} : M^2 \to M$ *be a mapping. If there exist* $k \in [0, 1)$ *and* $\zeta \in \mathcal{J}_{\mathbb{C}_+}$ *satisfying the contractive condition*

$$\int_0^{L_{\mathfrak{B}}(z, w, z', w')} \zeta d_c \precsim \frac{k}{2} \int_0^{L_g(z, z') + l_g(w, w')} \zeta d_c, \tag{6}$$

for all $z, w, z', w' \in M$, *then* $\mathfrak{B}$ *has a unique coupled fixed point.*

Proof. If we let $g(z) = z$ in Theorem 3.1, then $\mathfrak{B}$ has a unique coupled fixed point.

Theorem 3.5. *Let* $\mathfrak{B}, \mathfrak{B}' : M^2 \to M$ *be two mappings such that* $\mathfrak{B}(M^2) \subseteq \mathfrak{B}'(M^2)$. *If there exist* $k \in [0, 1)$ *and* $\zeta \in \mathcal{J}_{\mathbb{C}_+}$ *satisfying the contractive condition*

$$\int_0^{L_{\mathfrak{B}}^{\mathfrak{B}'}(z, w, z', w')} \zeta d_c \precsim \frac{k}{4} \int_0^{L_{\mathfrak{B}}(z, w) + L_{\mathfrak{B}}(w, z) + L_{\mathfrak{B}'}(z', w') + L_{\mathfrak{B}}(w', z')} \zeta d_c, \tag{7}$$

for all $z, w, z', w' \in M$, *then* $\mathfrak{B}$ *and* $\mathfrak{B}'$ *have a unique common coupled fixed point.*

Proof. Let $(z_0, w_0) \in M^2$. Since $\mathfrak{B}(M^2) \subseteq \mathfrak{B}'(M^2)$, we can construct two sequences $\{z_n\}$ and $\{w_n\}$ in M such that

$$z_{2m+1} = \mathfrak{B}(z_{2m}, w_{2m}) \text{ and } w_{2m+1} = \mathfrak{B}(w_{2m}, z_{2m});$$

$$z_{2m+2} = \mathfrak{B}'(z_{2m+1}, w_{2m+1}) \text{ and } w_{2m+2} = \mathfrak{B}'(w_{2m+1}, z_{2m+1}).$$

Then by contractive condition (7), we get

$$\int_0^{L(z_{2m+1}, z_m)} \zeta d_c = \int_0^{L_{\mathfrak{B}}^{\mathfrak{B}'}(z_{2m}, w_{2m}, z'_{2m-1}, w'_{2m-1})} \zeta d_c$$

$$\precsim \frac{k}{4} \int_0^{L(z_{2m+1}, z_{2m}) + L(w_{2m+1}, w_{2m}) + L(z_{2m}, z_{2m-1}) + L(w_{2m}, w_{2m-1})} \zeta d_c. \quad (8)$$

Similarly, we can prove that

$$\int_0^{L(w_{2m+1}, w_{2m})} \zeta d_c \precsim \frac{k}{4}$$

$$\int_0^{L(z_{2m+1}, z_{2m}) + L(w_{2m+1}, w_{2m}) + L(z_{2m}, z_{2m-1}) + L(w_{2m}, w_{2m-1})} \zeta d_c. \quad (9)$$

Adding (8) and (9), we get

$$\int_0^{L(z_{2m+1}, z_{2m}) + (w_{2m+1}, w_{2m})} \zeta d_c \precsim \frac{k}{2}$$

$$\int_0^{L(z_{2m+1}, z_{2m}) + L(w_{2m+1}, w_{2m}) + L(z_{2m}, z_{2m-1}) + L(w_{2m}, w_{2m-1})} \zeta d_c,$$

which implies

$$\frac{2-k}{2} \int_0^{L(z_{2m+1}, z_{2m}) + (w_{2m+1}, w_{2m})} \zeta d_c \precsim \frac{k}{2} \int_0^{L(z_{2m}, z_{2m-1}) + L(w_{2m}, w_{2m-1})} \zeta d_c.$$

Thus we have

$$\int_0^{L(z_{2m+1}, z_{2m}) + L(w_{2m+1}, w_{2m})} \zeta d_c \precsim \frac{k}{2-k} \int_0^{L(z_{2m}, z_{2m-1}) + L(w_{2m}, w_{2m-1})} \zeta d_c$$

$$\precsim \left(\frac{k}{2-k}\right)^2 \int_0^{L(z_{2m-1}, z_{2m-2}) + L(w_{2m-1}, w_{2m-2})} \zeta d_c$$

$\vdots$

$$\precsim \left(\frac{k}{2-k}\right)^{2m} \int_0^{L(z_1, z_0)+L(w_1, w_0)} \zeta d_c.$$

Letting limit as $n \to \infty$, we get

$$\int_0^{L(z_{2m-1}, z_{2m})+L(w_{2m-1}, w_{2m})} \zeta d_c = 0,$$

which implies $L(z_{2m+1}, z_{2m}) + L(w_{2m+1}, w_{2m}) = 0$, and hence the sequences $\{\rho(z_{2m+1}, z_{2m})\}$ and $\{\rho(w_{2m+1}, w_{2m})\}$ in $\mathbb{C}_+$ converge to zero. In the same way, we can prove that the sequences $\{\rho(z_{2m}, z_{2m-1})\}$ and $\{\rho(w_{2m}, w_{2m-1})\}$ in $\mathbb{C}_+$ converge to zero. Therefore, the sequences $\{\rho(z_n, z_{n+1})\}$ and $\{\rho(w_n, w_{n+1})\}$ converge to zero. (i.e.) $\{z_n\}$ and $\{w_n\}$ are asymptotically regular sequences in M. Since M is a complete asymptotically regular complex valued metric space, $\{z_n\}$ and $\{w_n\}$ are converges to z and w in M respectively. By contractive condition (7), we get

$$\int_0^{\frac{\rho(\mathfrak{B}(z, w), z_{2m})}{1+\rho(\mathfrak{B}(z, w), z_{2m})}} \zeta d_c = \int_0^{L_{\mathfrak{B}}(z, w, z_{2m-1}, w_{2m-1})} \zeta d_c$$

$$\precsim \frac{k}{4} \int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}'}(z_{2m-1}, w_{2m-1})+L_{\mathfrak{B}'}(w_{2m-1}, z_{2m-1})} \zeta d_c.$$

Letting limit as $m \to \infty$, we get

$$\int_0^{L_{\mathfrak{B}}(z, w)} \zeta d_c \precsim \frac{k}{2} \int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(w, z)} \zeta d_c. \tag{10}$$

Similarly, we can prove that

$$\int_0^{L_{\mathfrak{B}}(w, z)} \zeta d_c \precsim \frac{k}{4} \int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(w, z)} \zeta d_c. \tag{11}$$

From (10) and (11), we have

$$\int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(w, z)} \zeta d_c \precsim \frac{k}{2} \int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(w, z)} \zeta d_c,$$

which implies

$$\int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(w, z)} \zeta d_c = 0,$$

and hence $L_{\mathfrak{B}}(z, w) + L_{\mathfrak{B}}(w, z) = 0$. Thus, $\mathfrak{B}(z, w) = z$ and $\mathfrak{B}(w, z) = w$.

Similarly we can prove that $\mathfrak{B}'(z, w) = z$ and $\mathfrak{B}'(w, z) = w$. Thus (z, w) is a common coupled fixed point of $\mathfrak{B}$ and $\mathfrak{B}'$. Suppose there exist another point $(z', w') \in M^2$ is a common coupled fixed point of $\mathfrak{B}$ and $\mathfrak{B}'$. Then by contractive condition (7), we have

$$\int_0^{L_{\mathfrak{B}}^{\mathfrak{B}'}(z, w, z', w')} \precsim \frac{k}{4} \int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(w, z)+L_{\mathfrak{B}'}(z', w')+L_{\mathfrak{B}}(w', z')} \zeta d_c,$$

which implies $\int_0^{L(z, z')} \precsim 0$, and hence $z = z'$. Similarly, we can prove that $w = w'$. That is, (z, w) is a unique common coupled fixed point of $\mathfrak{B}$ and $\mathfrak{B}'$.

By contractive condition (7), we have $\int_0^{L_{\mathfrak{B}}^{\mathfrak{B}'}(z, w, w, z)} \precsim \frac{k}{4}$ $\int_0^{L_{\mathfrak{B}}(z, w)+L_{\mathfrak{B}}(w, z)+L_{\mathfrak{B}'}(z, w)+L_{\mathfrak{B}'}(w, z)} \zeta d_c$, which implies $\int_0^{L(z, w)} \zeta d_c = 0$ and hence $z = w$. Therefore, (z, z) is a unique common coupled fixed point of $\mathfrak{B}$ and $\mathfrak{B}'$.

Example 3.6. Let $M = \{a + ia : 0 \leq x \leq 1\}$ be a complete asymptotically regular complex valued metric space with the metric

$$\rho(z, w) = | R_e(z) - R_e(w) | + i | \operatorname{Im}(z) - \operatorname{Im}(w) |.$$

Define $\mathfrak{B} : M^2 \to M$ and $\mathfrak{B}' : M^2 \to M$ by

$$\mathfrak{B}(z, w) = \frac{a + b}{6} + i \frac{a + b}{6} \text{ and } \mathfrak{B}'(z, w) = \frac{2a + 3b}{50} + i \frac{2a + 3b}{50}$$

Where $x = a + ai$ and $y = b + bi$. If we let $\zeta(t) = 1 + i$, then the contractive condition (7) becomes,

$$L_{\mathfrak{B}}^{\mathfrak{B}'}(z, w, z', w') \precsim \frac{k}{14} (L_{\mathfrak{B}}(z, w) + L_{\mathfrak{B}}(w, z) + L_{\mathfrak{B}'}(z', w') + L_{\mathfrak{B}'}(w', z')), \quad (12)$$

where

$$L_{\mathfrak{B}}^{\mathfrak{B}'}(z, w, z', w') = \frac{\left|\frac{a+b}{6} - \frac{2c+3d}{50}\right| + i\left|\frac{a+b}{6} - \frac{2c+3d}{50}\right|}{\left|\frac{a+b}{6} - \frac{2c+3d}{50}\right| + i\left|\frac{a+b}{6} - \frac{2c+3d}{50}\right|};$$

$$L_{\mathfrak{B}}(z, w) = \frac{\left|\frac{a+b}{6} - a\right| + i\left|\frac{a+b}{6} - a\right|}{1 + \left|\frac{a+b}{6} - a\right| + i\left|\frac{a+b}{6} - a\right|};$$

$$L_{\mathfrak{B}}(w, z) = \frac{\left|\frac{a+b}{6} - b\right| + i\left|\frac{a+b}{6} - b\right|}{1 + \left|\frac{a+b}{6} - b\right| + i\left|\frac{a+b}{6} - b\right|};$$

$$L_{\mathfrak{B}'}(z, w) = \frac{\left|\frac{2c+3d}{50} - c\right|}{1 + \left|\frac{2c+3d}{50} - b\right|};$$

$$L_{\mathfrak{B}'}(w, z) = \frac{\left|\frac{2c+3d}{50} - d\right|}{1 + \left|\frac{2c+3d}{50} - d\right|}.$$

If we let $k \geq \frac{3}{4}$, then the contractive condition (12) is true. By Theorem 3.5, $\mathfrak{B}$ and $\mathfrak{B}'$ have a unique common coupled fixed point.

Corollary 3.7. *Let* $\mathfrak{B} : M^2 \to M$ *is a mapping. If there exist* $k \in [0, 1)$ *and* $\zeta \in \mathcal{J}_{\mathbb{C}_+}$ *satisfying the contractive condition*

$$\int_0^{L_{\mathfrak{B}}(z, w, z', w')} \zeta d_c \precsim \frac{k}{4} \int_0^{L_{\mathfrak{B}}(z, w) + L_{\mathfrak{B}}(w, z) + L_{\mathfrak{B}}(z', w') + L_{\mathfrak{B}}(w', z')} \zeta d_c, \tag{13}$$

for all $z, w, z', w' \in M$, *then* $\mathfrak{B}$ *has a unique coupled fixed point.*

Conclusion

A sufficient condition for a 2-variable mapping and a self mapping to have a unique coupled coincidence point on a complete asymptotically regular metric space is proved. Also, a common coupled fixed point theorem is proved for a pair of 2-variable mappings. Finally, nontrivial examples are provided to justify the significance of the theory.

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