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SOME CURVATURE PROPERTIES ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract

The present paper deals with some properties of Riemannian curvature tensor, Weyl curvature tensor, m-projective curvature tensor with respect to generalized Tanaka-Webster connection in a Lorentzian Para-Sasakian manifold.

1. Introduction

The Tanaka-Webster connection was introduced by Tanno [13] as a generalization of the well-known connection defined at the end of the 1970's by Tanaka in [12] and independently by Webster in [15]. This connection coincides with the Tanaka-Webster Connection if the associated CR-structure is integrable. Tanaka-Webster connection is defined as the canonical affine connection on a non-degenerate, pseudo-Harmitian CR-manifold. for a real hypersurface in a Kähler manifold with almost contact structure (ϕ, ξ, η, g), Cho adapted Tanno's g-Tanaka-Webster connection for a non-zero real number k. In 2017, Ghosh and De [5] studied the g-Tanaka-Webster connection associated to a Kenmotsu structure. With the help of g-Tanaka-Webster connection they characterized Kenmotsu manifolds and found important curvature properties of this connection on Kenmotsu manifolds. On

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the other hand, the notion of a Lorentzian Para-Sasakian manifold was introduced by Matsumoto [6]. Mihai and Rosca [8] defined the same notion independently and they found several important results on this manifold. In addition to this, LP-Sasakian manifolds had been studied by Matsumoto and Mihai [7] and De [3] and Shaikh [11]. In 1971, Pokhariyal and Mishra [10] defined a tensor field W^* on a Riemannian manifold known as m-projective curvature tensor which is given below

$$W^*(X, Y, Z) = R(X, Y, Z) - \frac{1}{2(n-1)}\{S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX$$

$$g(X - Z)QY\} \tag{1.1}$$

where R, S and Q are the Riemannian curvature tensor of type (1, 3), the Ricci tensor of type $(0, 2)$ and the Ricci operator defined by $g(Q, XY) = S(X, Y)$ respectively.

In 2010, Chaubey and Ojha [2] studied the properties of the m-projective curvature tensor in Riemannian and Kenmotsu manifolds and they proved that the m-projective curvature tensor in an η-Einstein Kenmotsu manifold is irrotational if and only if it is locally isometric to the hyperbolic space $H^n(-1)$. Later, Devi and Singh [4] found important results of m-projective curvature tensor on Kenmotsu manifold. Ayar and Chaubey investigated the properties of the α-cosymplectic manifolds with m-projective curvature tensor. Meanwhile, they obtained some connections between different curvature tensors viz, m-projective curvature tensor Weyl-projective curvature tensor which is given as follows [9]:

$$W(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1}\{S(Y, Z)X - S(X, Z)Y\}. \tag{1.2}$$

2. Preliminary

An n-dimensional differentiable manifold M^n is called an Lorentzian Para-Sasakian manifold [6], [7] if it admits a (1, 1) tensor field ϕ, a contra variant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$\text{(i) } g(X, \xi) = \eta(X), \text{ (ii) } D_X \xi = \phi X, \tag{2.4}$$

$$(D_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$

where D denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

In a Lorentzian Para-Sasakian manifold, the following relations hold:

$$\text{(i) } \phi\xi = 0, \text{ (ii) } \eta(\phi X) = 0, \tag{2.6}$$

$$rank\phi = n - 1. \tag{2.7}$$

If we put

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.8}$$

for any vector fields X and Y, then the tensor field $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field [7]. And since the vector field η is closed in a Lorentzian Para Sasakian manifold, we have [7], [3]

$$\text{(i) } (D_X \eta)(Y) = \Phi(X, Y), \text{ (ii) } \Phi(X, \xi) = 0 \tag{2.9}$$

for any vector fields X and Y. A Lorentzian Para Sasakian manifold M^n is said to be η-Einstein if Ricci tensor S is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) \tag{2.10}$$

for any vector fields X, Y where a, b are functions on M^n. Also, in an n-dimensional Lorentzian Para-Sasakian manifold M^n with structure (ϕ, ξ, η, g) the following relations hold [7], [3]:

$$g(R(X, Y, Z), \xi) = \eta(R(X, Y, Z)) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \tag{2.11}$$

$$R(\xi, X, Y) = g(X, Y)\xi - \eta(Y)X, \tag{2.12}$$

$$R(X, Y, \xi) = \eta(Y)X - \eta(X)Y, \tag{2.13}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{2.14}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.15}$$

for any vector fields X, Y, Z. The generalized Tanaka-Webster connection [14] ∇ for a Lorentzian Para- Sasakian manifold M^n is defined by

$$\nabla_X Y = D_X Y + ((D_X \eta)Y)\xi - \eta(Y) D_X \xi + \eta(X)\phi Y, \tag{2.16}$$

for all vector fields X and Y. By virtue of (2.4) (ii), (2.8) and (2.9) (i), the equation (2.16) can be written as

$$\nabla_X Y = D_X Y + g(X, \phi Y)\xi - \eta(Y)\phi X + \eta(X)\phi Y. \tag{2.17}$$

3. Curvature Tensor of Lorentzian Para-Sasakian Manifolds with Respect to Generalized Tanaka-Webster Connection

Putting $Y = \xi$ in (2.17) and using (2.1), (2.6) (i), we have

$$\nabla_X \xi = D_X \xi + \phi X. \tag{3.1}$$

Using (2.4) (ii) in (3.1) we get $\nabla_X \xi = 2\phi X$. Now

$$(\nabla_X \eta)Y = \nabla_X \eta(Y) - \eta(\nabla_X Y). \tag{3.2}$$

From (2.17) and (3.2) we get

$$(\nabla_X \eta)Y = (D_X \eta)Y + g(X, \phi Y). \tag{3.3}$$

With the help of (2.8) and (2.9), from the above equation, it follows that $(\nabla_X \eta)Y = 2g(X, \phi Y)$. Again

$$(\nabla_X g)(Y, Z) = \nabla_X g(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z). \tag{3.4}$$

Finally using (2.17) in (3.4), yields

$$(\nabla_X g)(Y, Z) = -2\eta(X) g(Y, \phi Z). \tag{3.5}$$

Theorem 3.1. *The generalized Tanaka-Webster connection* ∇ *associated to the Levi-Civita connection is just one affine connection, which is not metric and its torsion is of the form*

$$\widetilde{T}(X, Y) = 2\{\eta(X)\phi Y - \eta(Y)\phi X\}. \tag{3.6}$$

Proof. We see in (3.5) that the generalized Tanaka-Webster connection is not metric connection.

Now the torsion tensor $\widetilde{T}$ of ∇ is given by

$$\widetilde{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X\ Y].$$

Using (2.17) in the previous relation we get (3.6).

The curvature tensor K of M^n with respect to the generalized Tanaka-Webster connection ∇ is defined by

$$K(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z.$$

Then, in a Lorentzian Para Sasakian manifold, we have

$$\begin{aligned} K(X, Y, Z) = {} & R(X, Y, Z) + 3g(Y, \phi Z)\phi X - 3g(X, \phi Z)\phi Y \\ & + 3\eta(Y)g(X, Z)\xi - 3\eta(X)g(Y, Z)\xi \\ & + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y. \end{aligned} \tag{3.7}$$

Suppose that X, Y, Z are orthogonal to ξ. Then the equation (3.7) becomes

$$K(X, Y, Z) = R(X, Y, Z) + 3g(Y, \phi Z)\phi X - 3g(X, \phi Z)\phi Y. \tag{3.8}$$

From the equation (3.8), we get

$$\widetilde{S}(Y, Z) = S(Y, Z) - 3g(Y, Z) - 3\eta(Y)\eta(Z) \tag{3.9}$$

where $\widetilde{S}$ and S are the Ricci tensors of the connections ∇ and D respectively.

Contracting Y and Z in (3.9), we obtain

$$\widetilde{r} = r - 3(n-1) \tag{3.10}$$

where $\widetilde{r}, r$ are the scalar curvatures of the connections ∇ and D respectively.

From (3.9) yields

$$\widetilde{Q}Y = QY - 3Y - 3\eta(Y)\xi \tag{3.11}$$

where $\widetilde{S}(Y, Z) = g(\widetilde{Q}Y, Z)$.

Let $'R$ and $'K$ be the curvature tensors of $(0, 4)$ type given by

$$'R(X, Y, Z, U) = g(R(X, Y, Z), U) \tag{3.12}$$

and

$$'K(X, Y, Z, U) = g(K(X, Y, Z), U). \tag{3.13}$$

Theorem 3.2. *In a Lorentzian Para-Sasakian manifold, curvature tensor with respect to the generalized Tanaka-Webster connection* ∇ *has the following properties*:

(a) $K(X, Y, Z) + K(Y, Z, X) + K(Z, X, Y) = 0,$

(b) $'K(X, Y, Z, U) + 'K(Y, X, Z, U) = 0,$

(c) $'K(X, Y, Z, U) + 'K(X, Y, U, Z) = 0,$

(d) $'K(X, Y, Z, U) - 'K(Z, U, X, Y) = 0.$ (3.14)

Proof. By using (3.8) and first Bianchi identity

$$R(X, Y, Z) + R(Y, Z, X) + R(Z, X, Y) = 0$$

with respect to Riemannian connection D, we obtain (3.14) (a).

By virtue of equations (3.8), (3.12) and (3.13) we have

$$\begin{aligned} 'K(X, Y, Z, U) &= 'R(X, Y, , Z, U) + 3g(Y, \phi Z)g(\phi X, U) \\ &\quad - 3g(X, \phi Z)g(\phi Y, U). \end{aligned} \tag{3.15}$$

Now interchanging X and Y in (3.15) and using the equation (3.8), we get (3.14) (b). Immediately we obtain the equations (3.14) (c) and (3.14) (d).

Lemma 3.1. *Let* M^n *be an n-dimensional Lorentzian Para-Sasakian manifold with the generalized Tanaka-Webster connection* ∇. *Then, we have*

$$K(X, Y, \xi) = R(X, Y, \xi), \tag{3.16}$$

$$\eta(K(X, Y, \xi)) = 0, \tag{3.17}$$

$$\widetilde{S}(X, \xi) = S(X, \xi) \tag{3.18}$$

for all $X, Y \in TM^n$.

4. Weyl Projective Curvature Tensor of Lorentzian Para-Sasakian Manifolds with Respect to Generalized Tanaka-Webster Connection

Analogous to the definition given in (1.2), the Weyl projective curvature tensor $\widetilde{W}$ of type (1, 3) in a Lorentzian Para-Sasakian manifold M^n with respect to generalized Tanaka Webster connection ∇ is given by

$$\widetilde{W}(X, Y, Z) = K(X, Y, Z) - \frac{1}{n-1}\{\widetilde{S}(Y, Z)X - \widetilde{S}(X, Z)Y\}. \tag{4.1}$$

By making use of (1.2), (3.2), (3.8) in (4.1), we have

$$\begin{aligned}\widetilde{W}(X, Y, Z) = {} & W(X, Y, Z) + 3g(Y, \phi Z)\phi X - 3g(X, \phi Z)\phi Y \\ & + \frac{1}{n-1}\{3g(Y, Z)X - 3g(X, Z)Y \\ & + 3\eta(Y)\eta(Z)X - 3\eta(X)\eta(Z)Y\}.\end{aligned} \tag{4.2}$$

From (4.2), we have, the Weyl projective curvature tensor with respect to generalized Tanaka-Webster connection ∇ satisfies the following algebraic properties

$$\widetilde{W}(X, Y, Z) + \widetilde{W}(Y, X, Z) = 0,$$

and

$$\widetilde{W}(X, Y, Z) + \widetilde{W}(Y, X, Z) + \widetilde{W}(Z, X, Y) = 0$$

for vector fields X, Y, Z on M^n.

Theorem 4.1. *An n-dimensional Lorentzian Para-Sasakian manifold is ξ-Weyl projectively at with respect to generalized Tanaka-Webster connection if and only if the manifold is also ξ-Weyl projectively flat with respect to the Riemannian connection.*

Proof. Putting $Z = \xi$ in (4.2) and using (2.1), (2.4) (i) it follows that

$$\widetilde{W}(X, Y, \xi) = W(X, Y, \xi). \tag{4.3}$$

Hence proofs the theorem.

5. *m*-Projective Curvature Tensor on Lorentzian Para-Sasakian Manifolds

Analogous to the (1.1), the m-projective curvature tensor $\widetilde{W}$ in a Lorentzian Para-Sasakian manifold M^n with respect to generalized Tanaka-Webster connection ∇.

$$\widetilde{W}^*(X, Y, Z) = K(X, Y, Z) - \frac{1}{2(n-1)}\{\widetilde{S}(Y, Z)X - \widetilde{S}(X, Z)Y + g(Y, Z)\widetilde{Q}X - g(X, Z)\widetilde{Q}Y\}. \tag{5.1}$$

Theorem 5.1. *An n-dimensional Lorentzian Para-Sasakian manifold M^n is m-projectively at with respect to generalized Tanaka-Webster connection if and only if the manifold has constant scalar curvature $n^2 + 2n - 1$.*

Proof. Let $\widetilde{W}^* = 0.$ From the equation (5.1) we have

$$K(X, Y, Z) = \frac{1}{2(n-1)}\{\widetilde{S}(Y, Z)X - \widetilde{S}(X, Z)Y + g(Y, Z)\widetilde{Q}X - g(X, Z)\widetilde{Q}Y\}. \tag{5.2}$$

With the help of (3.2), (3.3) and (3.5), the equation (5.2) becomes

$$\begin{aligned} R(X, Y, Z) = {} & -3g(Y, \phi Z)\phi X + 3g(X, \phi Z)\phi Y + \frac{1}{2(n-1)}\{S(Y, Z)X \\ & -3g(Y, Z)X - 3\eta(Y)\eta(Z)X - S(X, Z)Y \\ & +3g(X, Z)Y + 3\eta(X)\eta(Z)Y + G(Y, Z)QX \\ & -3g(Y, Z)X - 3g(Y, Z)\eta(X)\xi - g(X, Z)QY \\ & +3g(X, Z)Y + 3g(X, Z)\eta(Y)\xi\}. \end{aligned} \tag{5.3}$$

Replacing Z by ξ in (5.3) and then using (2.1), (2.4) (i), (2.6) (i) and (2.13), we obtain

$$(n+2)\{\eta(Y)X - \eta(X)Y\} = \eta(Y)QX - \eta(X)QY.$$

Again putting $Y = \xi$ in the above relation and using (2.1), we have

$$QX = (n+2)X + \eta(X)\xi$$

$\Leftrightarrow$

$$S(X, Y) = (n+2)g(X, Y) + \eta(X)\eta(Y). \tag{5.4}$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the tangent space at each point of the manifold M^n. By putting $X = Y = \{e_i\}$ in the above relation (5.4) and taking the summation over $i, 1 \leq i \leq n,$ we get

$$r = n^2 + 2n - 1.$$

This completes the proof.

6. Conclusion

Starting from generalized Tanaka-Webster connection on Lorentzian Para-Sasakian manifolds, we derived the Riemannian curvature of this connection and obtained some properties of the different curvature tensors viz, Weyl projective curvature tensor, m-projective curvature tensor.

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