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INTEGRAL SOLUTIONS OF CUBIC DIOPHANTINE EQUATION WITH FIVE UNKNOWNS SIMPLE FORM FOR COEFFICIENTS $x^3 + y^3 = 13(z + w)p^2$

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Abstract

The homogeneous cubic equation with five unknowns $x^3 + y^3 = 13(z + w)p^2$ is analyzed for its non-zero distinct integral points through employing linear transformations. A few interesting properties among the solutions and special numbers are presented.

Notation

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

$$\Pr_n = n(n+1)$$

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular cubic equations, homogeneous or non-homogeneous, have aroused the interest of numerous Mathematicians since antiquity [1, 2].

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For illustration one may refer [3-10]. This paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous cubic equation with five unknowns $x^3 + y^3 = 13(z + w)p^2$. A few interesting relations between the solutions and the special numbers are presented.

2. Method of Analysis

The cubic Diophantine equation with five unknowns studied for its non-zero distinct integer solutions is given by

$$x^3 + y^3 = 13(z + w)p^2 \tag{1}$$

Introducing the linear transformations

$$x = u + v,\ y = u - v,\ z = u + d,\ w = u - d \tag{2}$$

in (1), we get

$$u^2 + 3v^2 = 13p^2 \tag{3}$$

Now, we solve (3) through different methods and thus obtain different patterns of solutions to (1).

Pattern I

Assume $p = p(a, b) = a^2 + 3b^2$ (4)

where a and b are non-zero distinct integers.

Write 13 as $(1 + i2\sqrt{3})(1 + i2\sqrt{3})$ (5)

Using (4) and (5) in (3) and applying the method of factorization, it is written as the system of double equations as

$$u + i\sqrt{3}v = (1 + i2\sqrt{3})(a + i\sqrt{3}b)^2$$

$$u + i\sqrt{3}v = (1 + i2\sqrt{3})(a - i\sqrt{3}b)^2$$

Equating the real and imaginary parts in either of the above equation we have,

$$u = u(a,\ b) = a^2 - 3b^2 - 12ab$$

$v = v(a, b) = 2a^2 - 6b^2 + 2ab$

In view of (2), the corresponding solutions of (1) are given by

$x = x(a, b) = 3a^2 - 9b^2 - 10ab$

$y = y(a, b) = -a^2 + 3b^2 - 14ab$

$z = z(a, b, d) = a^2 - 3b^2 - 12ab + d$

$w = w(a, b, d) = a^2 - 3b^2 - 12ab - d$

$p = p(a, b) = a^2 + 3b^2$

Properties.

A few interesting properties observed are as follows:

$x(a, 1) + y(a, 1) - t_{6,a} \equiv -6 \pmod{23}$

$z(a, 1, 1) + w(a, 1, 1) - t_{6,a} \equiv -6 \pmod{24}$

$x(a, 1) + p(a, 1) - t_{10,a} \equiv -6 \pmod{7}$

$x(a, a) + z(a, a, 1) + w(a, a, 1) + t_{90,a} \equiv 0 \pmod{43}$

$y(a, a) + p(a, a) + t_{18,a} \equiv 0 \pmod{17}$

Pattern II

Rewrite (3) as $u^2 + 3v^2 = 13p^2 * 1$ (6)

Write 1 as $1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4}$ (7)

Applying a similar analysis presented as in pattern I and performing a few calculations, the corresponding non-zero distinct integral solutions of (1) are given by

$x = x(A, B) = -4A^2 + 12B^2 - 56AB$

$y = y(A, B) = -16A^2 + 48B^2 - 16AB$

$$z = z(A, B, d) = -10A^2 + 30B^2 - 36AB + d$$

$$w = w(A, B, d) = -10A^2 + 30B^2 - 36AB - d$$

$$p = p(A, B) = 4A^2 + 12B^2$$

Properties.

$$x(A, 1) + y(A, 1) + t_{42, A} \equiv 60 \pmod{91}$$

$$z(A, 1, 1) + w(A, 1, 1) - 5p(A, 1) + t_{82, A} \equiv 0 \pmod{171}$$

$$x(1, B) + y(1, B) - t_{122, B} \equiv -7 \pmod{13}$$

$$x(A, A) + p(A, A) + t_{66, A} \equiv 0 \pmod{31}$$

$$y(A, A) + p(A, A) - 32\,\mathrm{Pr}_A \equiv 0 \pmod{32}$$

Pattern III

Instead of (7) write 1 as

$$1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{7^2} \tag{8}$$

For this choice, the corresponding integer solutions are found to be

$$x = x(a, b) = -119a^2 + 357b^2 - 574ab$$

$$y = y(a, b) = -203a^2 + 609b^2 + 70ab\,]$$

$$z = z(a, b, d) = -161a^2 + 483b^2 - 252ab + d$$

$$w = w(a, b, d) = -161a^2 + 483b^2 - 252ab - d$$

$$p = p(a, b) = 49a^2 + 147b^2$$

Properties.

$$x(a, 1) + y(a, 1) + t_{646, a} \equiv 141 \pmod{825}$$

$$z(a, 1, 1) + w(a, 1, 1) + 322\,\mathrm{Pr}_a \equiv 56 \pmod{182}$$

$x(a, a) + y(a, a) - t_{282, a} \equiv 0(\mathrm{mod}\, 139)$

$x(a, a) + p(a, a) + 10t_{30, a} \equiv 0(\mathrm{mod}\, 130)$

$y(a, a) + p(a, a) - 336t_{6, a} \equiv 0(\mathrm{mod}\, 336)$

Pattern IV

Instead of (7) write 1 as

$$1 = \frac{(1 + i15\sqrt{3})(1 - i15\sqrt{3})}{26^2} \tag{9}$$

For this choice, the corresponding integer solutions are found to be

$x = x(a, b) = -1872a^2 + 5616b^2 - 7280ab$

$y = y(a, b) = -2756a^2 + 8268b^2 + 1976ab$

$z = z(a, b, d) = -2314a^2 + 6942b^2 - 2652ab + d$

$w = w(a, b, d) = -2314a^2 + 6942b^2 - 2652ab - d$

$p = p(a, b) = 676a^2 + 2028b^2$

Properties.

$x(a,1) - y(a, 1) - 221t_{10, a} \equiv -2652(\mathrm{mod}\, 8593)$

$z(a, 1, 1) - w(a, 1, 1) + p(a, 1) - 13t_{106, a} \equiv 41(\mathrm{mod}\, 663)$

$x(a, a) + z(a, a, 1) + 78t_{42, a} \equiv 1(\mathrm{mod}\, 1482)$

$y(a, a) - p(a, a) - 299t_{34, a} \equiv 0(\mathrm{mod}\, 4485)$

$y(1, b) - z(1, b, 1) - 6t_{444, b} \equiv -443(\mathrm{mod}\, 5948)$

Pattern V

Instead of (7) write 1 as

$$1 = \frac{(1 + i56\sqrt{3})(1 - i56\sqrt{3})}{97^2} \tag{10}$$

For this choice, the corresponding integer solutions are found to be

$$x = x(A, B) = -26869A^2 + 80607B^2 - 98746AB$$

$$y = y(A, B) = -38121A^2 + 114363B^2 + 31234AB$$

$$z = z(A, B, d) = -32495A^2 + 97485B^2 - 33756AB + d$$

$$w = w(A, B, d) = -32495A^2 + 97485B^2 - 33756AB - d$$

$$p = p(A, B) = 9409A^2 + 28227B^2$$

Properties.

$$x(A, 1) - y(A, 1) - 2813t_{10, A} \equiv -33756 (\mathrm{mod}\, 121541)$$

$$z(A, 1, 1) - w(A, 1, 1) + p(A, 1) - 28229$$

is a perfect square

$$p(A, 1) + x(A, 1) + 291t_{122, A} \equiv 108834 (\mathrm{mod}\, 115915)$$

$$y(A, A) + p(A, A) - 1649t_{178, A} \equiv 0 (\mathrm{mod}\, 143463)$$

$$x(1, B) + p(1, B) - 1649t_{134, B} \equiv -582 (\mathrm{mod}\, 8439)$$

Note.

Instead of (5), taking $13 = \frac{(5 + i3\sqrt{3})(5 - i3\sqrt{3})}{2^2}$ in the above patterns, we can obtain the corresponding nonzero integer solutions to (1).

Pattern VI

Rewrite (3) as $u^2 - p^2 = 3(4p^2 - v^2)$ (11)

which can be written in the ratio form as

$$\frac{u - p}{2p + v} = \frac{3(2p - v)}{u + p} = \frac{A}{B}, B \neq 0 \quad (12)$$

Solving (12) and in view of (2) the integral solutions of (1) are found to be

$$x = x(A, B) = 3A^2 - 9B^2 - 10AB$$

$$y = y(A, B) = -A^2 + 3B^2 - 14AB$$

$$z = z(A, B, d) = A^2 - 3B^2 - 12AB + d$$

$$w = w(A, B, d) = A^2 - 3B^2 - 12AB - d$$

$$p = p(A, B) = -A^2 - 3B^2$$

Properties.

$$x(A, 1) + y(A, 1) - t_{6, A} \equiv -6 \pmod{23}$$

$$z(A, 1, 1) + w(A, 1, 1) - 2\,\mathrm{Pr}_A \equiv -6 \pmod{26}$$

$$x(A, A) + p(A, A) + t_{42, A} \equiv 0 \pmod{19}$$

$$y(A, A) + p(A, A) + t_{34, A} \equiv 0 \pmod{15}$$

$$p(1, B) + x(1, B) + t_{26, B} \equiv 2 \pmod{21}$$

Note.

Instead of (12), (11) can be written in three different ways as follows:

(i) $\dfrac{u - p}{3(2p + v)} = \dfrac{(2p - v)}{u + p} = \dfrac{A}{B},\ B \neq 0$

(ii) $\dfrac{u - p}{2p - v} = \dfrac{3(2p + v)}{u + p} = \dfrac{A}{B},\ B \neq 0$

(iii) $\dfrac{u - p}{3(2p - v)} = \dfrac{(2p + v)}{u + p} = \dfrac{A}{B},\ B \neq 0$

Following the procedure similar to the above, the corresponding non-zero integral solutions to the above three cases are as follows:

Case (i).

$$x = x(A, B) = 9A^2 - 3B^2 - 10AB$$

$$y = y(A, B) = -3A^2 + B^2 - 14AB$$

$$z = z(A, B, d) = 3A^2 - B^2 - 12AB + d$$

$$w = w(A, B, d) = 3A^2 - B^2 - 12AB - d$$

$$p = p(A, B) = -3A^2 - B^2$$

Properties.

$$x(A, 1) + y(A, 1) - t_{14, A} \equiv -2 \pmod{19}$$

$$z(A, 1, d) + w(A, 1, d) - 6pr_A \equiv -2 \pmod{19}$$

$$x(A, A) + p(A, A) + t_{18, A} \equiv 0 \pmod{7}$$

$$y(A, A) + p(A, A) + t_{42, A} \equiv 0 \pmod{19}$$

$$p(1, B) + x(1, B) + y(1, B) + t_{8, B} \equiv 3 \pmod{26}$$

Case (ii).

$$x = x(A, B) = A^2 - 3B^2 + 14AB$$

$$y = y(A, B) = -3A^2 + 9B^2 + 10AB$$

$$z = z(A, B, d) = -A^2 + 3B^2 + 12AB + d$$

$$w = w(A, B, d) = -A^2 + 3B^2 + 12AB - d$$

$$p = p(A, B) = A^2 + 3B^2$$

Properties.

$x(1, B) + y(1, B) + 26$ is a Nasty number

$$z(1, B, d) + w(1, B, d) - t_{14, B} \equiv -2 \pmod{29}$$

$$x(A, A) + y(A, A) - t_{58, A} \equiv 0 \pmod{27}$$

$$x(A, A) + p(A, A)$$

is a perfect square

$y(A, A) + p(A, A) - 20pr_A \equiv 0(\bmod 20)$

Case (iii).

$x = x(A, B) = 3A^2 - B^2 + 14AB$

$y = y(A, B) = -9A^2 + 3B^2 + 10AB$

$z = z(A, B, d) = -3A^2 + B^2 + 12AB + d$

$w = w(A, B, d) = -3A^2 + B^2 + 12AB - d$

$p = p(A, B) = 3A^2 + B^2$

Properties.

$x(A, 1) + y(A, 1) + p(A, 1) + t_{8, A} \equiv 3(\bmod 22)$

$z(A, 1, d) + w(A, 1, d) + x(A, 1) + pr_A \equiv 1(\bmod 41)$

$x(1, B) + p(1, B) \equiv 6(\bmod 14)$

$y(1, B) + p(1, B) - t_{10, B} \equiv -6(\bmod 13)$

$p(A, A) + z(A, A, 1) + w(A, A, 1)$

is a Nasty number

Conclusion

Since the cubic Diophantine equations with five unknowns are rich in variety, one may search for other choices of Diophantine equations to find their corresponding integer solutions.

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