Advances and Applications in Mathematical Sciences
Volume 21, Issue 1, November 2021, Pages 377-382

CONGRUENCES ON PSEUDO-COMPLEMENTED ALMOST DISTRIBUTIVE FUZZY LATTICES

A. NASREEN SULTANA and R. KAMALI

Department of Mathematics
Vels Institute of Science Technology
and Advanced Studies (VISTAS)
Tamil Nadu, 600117, India
E-mail: a.nasreensultana@gmail.com
kamali.sbs@velsuniv.ac.in

Abstract

The concept of congruences is characterized in terms of Pseudo-Complemented Almost Distributive Fuzzy Lattices (PCADFL). This characterization is used to demonstrate an equational class of PCADFL. Also the congruence kernels in PCADFL are characterized.

1. Introduction

The notion of Pseudo-Complementation on Almost Distributive Fuzzy Lattices (PCADFL) is given by SG. Karpagavalli and A. Nasreen Sultana [5] proved that it is equationally definable on ADFL [1] by using properties of pseudo-complementation on almost distributive lattice [7]. Based on the concept of multiplicatively closed subset S of an ADL A, two special congruence relations: ϕ^S and ψ^S were introduced on ADLs by Pawar in [8]. In the case, A is a distributive lattice, the two congruence relations coincide with the congruence ψ^S studied by Speed [6]. In this paper, it is proved that the PCADFL is equationally definable in different ways and the congruence kernels in PCADFL are characterized.

2020 Mathematics Subject Classification: 06D99, 06D72, 06D15, 08A72.
Keywords: Pseudo-Complemented Almost Distributive Fuzzy Lattice (PCADFL); Congruence; Minimal prime ideal; Filter; Congruence kernel.
Received May 17, 2021; Accepted June 7, 2021

2. Preliminaries

In this section we recall certain elementary definitions and results required.

Definition 2.1 [5]**.** Let $(R, \vee, \wedge, 0)$ be an algebra of type $(2, 2, 0)$ and (R, Λ) be a fuzzy poset. A unary operation $a \to a^*$ on R. Then (R, A) is called a Pseudo-Complementation con Almost Distributive Fuzzy Lattice (PCADFL), if the following conditions are satisfied:

(1) $A(1, a \vee b) = A(a \vee b, 1) = 1$

(2) $A(0, a \wedge b) = A(a \wedge b, 0) = 1$

(3) $A(a \wedge a^*, 0) = A(0, a \wedge a^*) = 1$

(4) $A(a^* \wedge b, b) - A(b, a^* \wedge b) = 1$

(5) $A((a \vee b)^*, (a^* \wedge b^*)) = A((a^* \wedge b^*), (a \vee b)^*) = 1$

(6) $A((a^*)^*, a) = A(a(a^*)^*) = 1,$ for all $a, b \in R.$

Definition 2.2 [3]**.** An equivalence relation θ on can ADL is called a congruence relation on L if $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$ for all $(a, b), (c, d) \in \theta.$

Theorem 2.3 [3]**.** *An equivalence relation* θ *on can ADL is a congruence relation if and only if for any* $(a, b) \in \theta,\ x \in L,\ (a \vee x, b \vee x), (x \vee a, x \vee b),$ $(a \wedge x, b \wedge x), (x \wedge a, x \wedge b)$ *are all in* $\theta.$

3. Congruences on PCADFL

In this section we prove some important properties of congruences on PCADFL. A congruence θ on a PCADFL $(R, \vee, \wedge, *, 0, 1)$ is a congruence of the fuzzy lattice $(R, \vee, \wedge, 0, 1)$ also has the substitution property for the operation.

Definition 3.1. Let (R, A) be a PCADFL, with an equivalence relation θ on R is called a congruence relation on for any $a, b \in R$ and

$x, y \in \theta$ then $x \equiv y(\theta(a, b))$ if and only if $A(x \wedge a, y \wedge a) > 0$ and $A((x \vee b) \wedge (a^* \wedge b)^*, (y \vee b) \wedge (a^* \wedge b)^*) > 0$.

Theorem 3.2. *Let (R, A) be PCADFL. A subset F of R is a congruence-kernel if and only if F is a filter in R. Moreover, the smallest congruence in pseudo-complement * having a given filter F as its congruence kernel is $\psi(F)$.*

Proof of Theorem 3.2. It is sufficient to prove that, for a given filter F, $\psi(F)$ has the substitution property for the * operation. Suppose that $a, b \in R$ and $a \equiv b(\psi(F))$, that is $A(a \wedge f, b \wedge f) > 0$ for some $f \in F$. Then

$$A(a^{**} \wedge f^{**}, b^{**} \wedge f^{**}) = A((a \wedge f)^{**}, b^{**} \wedge f^{**})$$

$$- A((b \wedge f)^{**}, b^{**} \wedge f^{**})$$

$$= A((b^{**} \wedge f^{**}, b^{**} \wedge f^{**}) > 0.$$

Therefore $A(a^* \wedge f^{**}, b^* \wedge f^{**}) > 0$, and hence $A(a^* \wedge f, b^* \wedge f) > 0$. Thus the smallest congruence in pseudo-complement * having a given filter F as its congruence kernel is $\psi(F)$.

Let I is an ideal in a PCADFL, and $I_* = \{a \in R : a \geq x^*$ for any $a \in I\}$. The characterization of congruence-kernels is proved in the following theorem.

Theorem 3.3. *Let (R, A) be PCADFL. Then the following conditions are equivalent*:

(i) *I is a congruence-kernel*,

(ii) *I is an ideal of the lattice R and $a^{**} \in I$ for each $a \in I$*,

(iii) *I is an ideal of the lattice R and each minimal prime ideal belonging to I is a minimal prime ideal*,

(iv) *I is an intersection of minimal prime ideals of the lattice R.*

Proof of Theorem 3.3. (i) $\Rightarrow$ (ii): Suppose $I = \ker(\theta)$ that for some *-congruence θ. Then I is certainly a fuzzy lattice-ideal and for $a \in I$, $a \equiv 0(\theta)$, so that $a^{**} \equiv 0^{**}$ and hence (ii) holds for I. Because of $a^{**} \in I$ for any $a \in I$, $\ker(\psi(I_*)) = I$. If ϕ is a *-congruence with $\ker(\phi) = I$ then $A(a \wedge x^*, b \wedge x^*) > 0$ for $x \in I$ which implies $a \equiv b(\phi)$ since $x^* \equiv 0^* = 1$, therefore the smallest *-congruence having I as its kernel.

(ii) $\Rightarrow$ (iii): Suppose that P is a minimal prime ideal belonging to I. Let $a \in P$, therefore $a \wedge c \in I$ for some $c \in R/P$. Then $A(a^{**} \wedge c^{**}, (a \wedge c)^{**}) > 0$. So $a^{**} \in P$, since P is a prime and $c \leq c^{**}$. Therefore is a minimal prime ideal.

(iii) $\Rightarrow$ (iv): Each ideal in a distributive fuzzy lattice is the intersection of all the minimal prime ideals belonging to it.

(iv) $\Rightarrow$ (i): Moreover, the smallest congruence on $(R, \vee, \wedge, *, 0, 1)$ having I as its kernel is $\theta(I)$, where $a \equiv b(\theta(I))$ for any $a, b \in R$ if and only if $A(a \wedge x^*, b \wedge x^*) > 0$ for some $x \in I$. Thus $\theta(I) = \psi(I_*)$. Therefore, I is a congruence-kernel.

Theorem 3.4. *Let I be a given congruence-kernel in a PCADFL $(R, \vee, \wedge, *, 0, 1)$. Then the following conditions relating to an equivalence relation ϕ on R are equivalent*:

(i) ϕ *is the largest *-congruence such that* $I - \ker(\phi)$,

(ii) $\phi = R(I)$,

(iii) *For any* $a, b \in R$, $a \equiv b(\phi)$ *if and only if* $A(a^{**} \wedge x^*, b^{**} \wedge x^*) > 0$ *for some* $x \in I$,

(iv) $\phi = \theta(I) \vee R$ *in the lattice of *-congruence,*

(v) *For any* $a, b \in R$, $a \equiv b(\phi)$ *if and only if* $A(a \wedge (y \vee y^*) \wedge x^*, b \wedge (y \vee y^*) \wedge x^*) > 0$, *for some* $y \in R$ *and* $x \in I$,

(vi) ϕ *is the smallest *-congruence with* $I = \ker(\phi)$ *and such that* R/ϕ *is a boolean algebra.*

Proof of Theorem 3.4. (i) $\Rightarrow$ (ii): By Theorem 3.3, each minimal prime ideal belonging to I is a minim al prime ideal. Hence $R(I)$ is a *-congruence. Therefore $\phi = R(I)$.

(ii) $\Rightarrow$ (iii): ϕ is a *-congruence with $\ker(\phi) = I$ and hence $\phi \subseteq R(I)$. Let P be a minimal prime ideal and suppose that $a \equiv b(R(P))$ for given $a, b \in R$. Then $a, b \in P$ or $a, b \in R/P$. In the first case $a \vee b \in P$ and

$$A(a^{**} \wedge (x \vee y)^*, b^{**} \wedge x^*) = A(a^{**} \wedge (x^* \wedge y^*), b^{**} \wedge x^*)$$

$$= A(b^{**} \wedge (x \vee y)^*, b^{**} \wedge x^*) = A(b^{**} \wedge x^*, b^{**} \wedge x^*) > 0.$$

While in the second case $a \wedge b \in R/P$ so that $(x \vee y)^* \in P$. Such that

$$A(a^{**} \wedge ((x \vee y)^*)^*, a^{**} \wedge b^{**}) = A(a^{**} \wedge (x \wedge y)^{**}, a^{**} \wedge b^{**})$$

$$= A(a^{**} \wedge b^{**}, a^{**} \wedge b^{**}) = 1 > 0.$$

Therefore $a^{**} \wedge ((x \wedge y)^*)^* = b^{**} \wedge ((x \wedge y)^*)^*$. Thus for any minimal prime ideal P, $a \equiv b(R(P))$ for any $a, b \in R$ if and only if $A(a^{**} \wedge p^*, b^{**} \wedge p^*) = 1 > 0$ for some $p \in P$.

(iii) $\Rightarrow$ (iv): It is obvious.

(iv) $\Rightarrow$ (v): Because is pseudo-complemented, $R = \psi(F)$, thus

$$A(\theta(I) \vee R, \psi(I_* \vee F)) = A(\psi(I_*) \vee \psi(F), \psi(I_* \vee F))$$

$$= A(\psi(I_* \vee F), \psi(I_* \vee F)) > 0,$$

where $I_* \vee F = \{a \in R : a = y \wedge f, y \in I_*, f \in F\}$ is the join of I_* and F in the lattice filters on R.

(v) $\Rightarrow$ (vi): If ϕ is the congruence of (v) then $\ker(\phi) = I$ and $y \vee y^* \in \ker(\phi)$ for each $y \in R$, thus

$$A(a \wedge (y \vee y^*) \wedge x^*, b \wedge (y \vee y^*) \wedge x^*)$$

$$= A(((a \wedge y) \vee (a \vee y^*)) \wedge x^*, b \wedge (y \vee y^*) \wedge x^*)$$

$$= A(((b \wedge y) \vee (b \wedge y^*)) \wedge x^*, b \wedge (y \vee y^*) \wedge x^*)$$

$$= A(b \wedge (y \vee y^*) \wedge x^*, b \wedge (y \vee y^*) \wedge x^*) = 1 > 0.$$

and so R/ϕ is boolean. It is clear that ϕ is the smallest *-congruence with these properties and so (v) and (vi) are equivalent.

References

[1] G. Berhanu, Yohannes and T. Bekalu, Almost distributive fuzzy lattice, International Journal of Mathematics and its Application 5(1-C) (2017), 307-316.

[2] W. H. Cornish, Congruences on distributive pseudo-complemented lattices, Bull. Aust. Math. Soc. 8 (1973), 161-179.

[3] G. C. Rao, Almost distributive lattices, Doctoral Thesis, Dept. of Mathematics, Andhra University, Visakhapatnam, (1980).

[4] H. Lakser, Principal congruences of pseudo complemented distributive lattices, Proc. Amer. Math. Soc. 37 (1973), 32-36.

[5] SG. Karpagavalli and A. Nasreen Sultana, Pseudo-complementation on almost distributive fuzzy lattices, Journal of Critical Reviews 7(7) (2020), 758-761.

[6] T. P. Speed, Two congruences on distributive lattices, Bull. Soc. R. Sci. Liege. 38(3-4) (1969), 86-96.

[7] U. M. Swamy, G. C. Rao and G. Nanaji Rao, Pseudo-complementation on almost distributive lattices, Southeast Asian Bulletin of Mathematics 24 (2000), 95-104.

[8] Y. S. Pawar and I. A. Shaikh, Congruence relations on almost distributive lattices, Southeast Asian Bull. Math. 36 (2012), 519-527.