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# SOME RESULTS AND APPLICATIONS OF THE EXPONENTIAL TRANSFORM

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## Abstract

In this paper we have obtained some results of the exponential transform like exponential transform of partial derivatives, Initial value theorem and Final value theorem and discussed some applications of the exponential transform to evaluate definite integral, to finding solutions of partial differential equation, simultaneous differential equation and integral equation.

## 1. Introduction

Many integral transforms specially Laplace transforms is useful to find the solution of integral equations, solution initial value problem and boundary value problem. It is also useful to solve definite integrals, it is also useful to find solution of difference, differential difference equations also useful in transfer function and impulse response function of a linear system, fractional differential equation, partial differential equations.

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Recently N. S. Ambarkhane, H. A. Dhirbasi, K.L. Bondar [1] introduced an integral transform named "Exponential transform" and proved its some properties like Linearity, Shifting, Second Shifting, Change of Scale. Moreover, exponential transform of some basic functions are derived. Ambarkhane Nagnath S., Bondar Kirankumar L. [2] discussed convolution theorem and the exponential transform of derivatives and integrations of a function $f(t)$. Nagnath S. Ambarkhane [3] discussed some results and application of the inverse exponential transform. Several authors [4-16] discussed the applications of different integral transformations along with its results.

Like Laplace transforms we have some significant applications of Exponential transform to finding solution of initial value problems, evaluating some definite integrals, solution of partial differential equation, to find the solution of simultaneous differential equations and to find the solution of integral equations. The main aim of this paper is to obtain some results and applications of the exponential transform.

## 2. Preliminaries

### 2.1. Exponential transform

**Definition 2.1** [1]. Let $f(t)$ be function defined for all positive values of $t$, then

$$\bar{f}(s) = \int_0^\infty a^{-st} f(t) dt,\ a > 1.$$

Provided the integral exists is called exponential transform of $f(t)$. It is denoted by $A[f(t)]$. Thus,

$$A[f(t)] = \bar{f}(s) = \int_0^\infty a^{-st} f(t) dt,\ a > 1.$$

Here $A$ is called exponential transformation operator, the parameter $s$ is real or complex number.

**2.2. Exponential transform of some functions** [1]:

(i) $A[1] = \dfrac{1}{(s \log a)}$, $a > 1$, $\mathrm{Re}(s) > 0$.

(ii) $A[t^n] = \dfrac{n!}{(s \log a)^{n+1}}$, $a > 1$, $n \geq 0$ $\mathrm{Re}(s) > 0$.

(iii) $A[t^n] = \dfrac{\Gamma(n+1)}{(s \log a)^{n+1}}$, $n > -1$, $a > 1$, $\mathrm{Re}(s) > 0$.

(iv) $A[e^{kt}] = \dfrac{n!}{(s \log a - k)^{n+1}}$, $a > 1$, $(s \log a) > k$, $\mathrm{Re}(s) > 0$.

(vii) $A[\sinh kt] = \dfrac{n!}{(s \log a)^2 - k^2}$, $a > 1$, $(s \log a)^2 > k^2$ $\mathrm{Re}(s) > 0$.

(viii) $A[\sin kt] = \dfrac{k}{(s \log a)^2 + k^2}$, $a > 1$, $\mathrm{Re}(s) > 0$.

(ix) $A[\cos kt] = \dfrac{(s \log a)}{(s \log a)^2 + k^2}$, $a > 1$, $\mathrm{Re}(s) > 0$.

**2.3. Exponential transform of derivatives** [2]

**Theorem 2.3.1.** *If* $A[f(t)] = \bar{f}(s)$, *then* $A[f'(t)] = (s \log a).A[f(t)] - f(0)$.

**Theorem 2.3.2.** *If* $A[f(t)] = \bar{f}(s)$, *then*

$$A[f''(t)] = (s \log a)^2 A[f(t)] - (s \log a).f(0) - f'(0).$$

**Theorem 2.3.3.** *If* $A[f(t)] = \bar{f}(s)$, *then*

$$A[f'''(t)] = (s \log a)^2 A[f(t)] - (s \log a)^2 f(0) - (s \log a) f'(0) - f''(0).$$

**Theorem 2.3.4.** *If* $A[f(t)] = \bar{f}(s)$, *then*

$$A[f^n(t)] = (s \log a)^n A[f(t)] - (s \log a)^{n-1} f(0) - (s \log a)^{n-2} f'(0)$$

$$- (s \log a)^{n-3} f''(0) - \ldots - f^{n-1}(0).$$

**2.4. Exponential transform of integration [2]**

**Theorem 2.4.1.** *If* $A[f(t)] = \bar{f}(s)$, *then*

$$A\left[\int_0^t f(t)dt\right] = \frac{1}{(s \log a)} \bar{f}(s).$$

**2.5. Exponential transform of a function multiplied by** $t$ [2]

**Theorem 2.5.1.** *If* $A[f(t)] = \bar{f}(s)$, *then*

$$A[t^n . f(t)] = \left[\frac{(-1)^n}{(\log a)^n}\right] \frac{d^n}{ds^n} [\bar{f}(s)].$$

**2.6. Exponential transform of a function divided by** $t$ [2]

**Theorem 2.6.1.** *If* $A[f(t)] = \bar{f}(s)$, *then*

$$A\left[\frac{1}{t} f(t)\right] = (\log a) \int_s^{\infty} \bar{f}(s) ds.$$

**2.7. Inverse exponential transform** [3]

**Definition 2.7.1.** If the Exponential Transform of a function $f(t)$ is $\bar{f}(s)$ i.e. $A[f(t)] = \bar{f}(s)$, then $f(t)$ is called as Inverse Exponential Transform of $\bar{f}(s)$ and is written as $f(t) = A^{-1}[\bar{f}(s)]$.

$A^{-1}$ is called the Inverse Exponential Transformation operator.

**2.8. Inverse exponential transform of some functions** [3]

Using definition of inverse exponential transform we get,

[1] $A^{-1}\left[\frac{1}{s \log a}\right] = 1, \quad a > 1, (s \log a) > 0.$

[II] $A^{-1}\left[\frac{1}{(s \log a)^{n+1}}\right] = \frac{t^n}{n!}, \quad a > 1, (s \log a) > 0, n = 0, 1, 2, 3, \ldots$

[III] $A^{-1}\left[\frac{1}{(s \log a) - k}\right] = e^{kt}, \quad a > 1, (s \log a) > k.$

[IV] $A^{-1}\left[\dfrac{(s\log a)}{(s\log a)-k^2}\right] = \cosh kt, \quad a > 1,\ (s\log a)^2 > k^2.$

[V] $A^{-1}\left[\dfrac{1}{(s\log a)^2-k^2}\right] = \dfrac{1}{k}(\sinh kt), \quad a > 1,\ k > 0,\ (s\log a)^2 > k^2.$

[VI] $A^{-1}\left[\dfrac{1}{(s\log a)^2-k^2}\right] = \dfrac{1}{k}(\sin kt), \quad a > 1,\ k > 0.$

[VII] $A^{-1}\left[\dfrac{(s\log a)}{(s\log a)^2-k^2}\right] = \cos kt, \quad a > 1.$

## 3. Main Results

### 3.1. Initial value theorem

**Theorem 3.1.1.** *If* $A[f(t)] = \bar{f}(s)$, *then*

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}[(s\log a).\bar{f}(s)].$$

**Proof.** We have,

$$A[f'(t)] = (s\log a).\bar{f}(s) - f(0)$$

$$\int_0^\infty a^{-st} f'(s)dt = (s\log a).\bar{f}(s) - f(0)$$

Taking $s \to \infty$

$$\lim_{s\to\infty}\int_0^\infty a^{-st} f'(t)dt = \lim_{s\to\infty}[(s\log a).\bar{f}(s) - f(0)]$$

$$\lim_{s\to\infty}[(s\log a).\bar{f}(s)] = f(0) + \int_0^\infty [\lim_{s\to\infty} a^{-st}]f'(t)dt$$

$$\lim_{s\to\infty}[(s\log a).\bar{f}(s)] = f(0) + \int_0^\infty 0 \cdot f'(t)dt$$

$$\lim_{s\to\infty}[(s\log a).\bar{f}(s)] = f(0) + 0$$

$$\lim_{s\to\infty}[(s\log a).\bar{f}(s)] = f(0)$$

$$\lim_{s\to\infty}[(s\log a).\bar{f}(s)] = \lim_{t\to 0} f(t)$$

$$\therefore \lim_{t\to 0} f(t) = \lim_{s\to\infty}[(s\log a).\bar{f}(s)].$$

### 3.2. Final value theorem

**Theorem 3.2.1.** *If* $A[f(t)] = \bar{f}(s)$, *then* $\lim_{t\to\infty} f(t) = \lim_{s\to 0}[(s\log a).\bar{f}(s)]$.

**Proof.** We have,

$$A[f'(t)] = (s\log a).\bar{f}(s) - f(0)$$

$$\int_0^\infty a^{-st} f'(t)dt = (s\log a).\bar{f}(s) - f(0)$$

Taking $s \to 0$

$$\lim_{s\to 0}\int_0^\infty a^{-dt} f'(t)dt = \lim_{s\to 0}[(s\log a).\bar{f}(s) - f(0)]$$

$$\lim_{s\to 0}[(s\log a).\bar{f}(s)] = f(0) + \int_0^\infty [\lim_{s\to 0} a^{-st}]f'(t)dt$$

$$\lim_{s\to 0}[(s\log a).\bar{f}(s)] = f(0) + \int_0^\infty f'(t)dt$$

$$\lim_{s\to 0}[(s\log a).\bar{f}(s)] = f(0) + [f(t)]_0^\infty$$

$$\lim_{s\to 0}[(s \log a).\bar{f}(s)] = f(0) + \lim_{t\to\infty} f(t) - f(0)$$

$$\lim_{s\to 0}[(s \log a).\bar{f}(s)] = \lim_{t\to\infty} f(t)$$

$$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0}[(s \log a).\bar{f}(s)].$$

### 3.3. Exponential transform of partial derivatives

**Theorem 3.3.1.** *If* $y = y(x, t)$, *then*

(I) $A\left[\frac{\partial y}{\partial t}\right] = (s \log a).\bar{y}(x, s) - y(x, 0)$,

(II) $A\left[\frac{\partial^2 y}{\partial t^2}\right] = (s \log a)^2 \bar{y}(x, s) - (s \log a).y(x, 0) - y_t(x, 0)$,

(III) $A\left[\frac{\partial y}{\partial x}\right] = \frac{d\bar{y}}{dx}$,

(IV) $A\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d^2\bar{y}}{dx^2}$,

Where $A[y(x, t)] = \bar{y}(x, s)$, $y_t(x, s)$, $y_t(x, 0) = \left[\frac{\partial y}{\partial t}\right]_{t=0}$.

**Proof.** (I) We have,

$$A\left[\frac{\partial y}{\partial t}\right] = \int_0^\infty a^{-st}\left(\frac{\partial y}{\partial t}\right)dt$$

$$A\left[\frac{\partial y}{\partial t}\right] = \lim_{k\to\infty}\int_0^k a^{-st}\left(\frac{\partial y}{\partial t}\right)dt$$

$$A\left[\frac{\partial y}{\partial t}\right] = \lim_{k\to\infty}\left[[a^{-st}y(x, t)]_{t=0}^k + (s \log a)\int_0^k a^{-st}y(x, t)dt\right]$$

$$A\left[\frac{\partial y}{\partial t}\right] = \left[[- y(x, 0)] + (s \log a)\int_0^\infty a^{-st}ydt\right]$$

$$\therefore\ A\left[\frac{\partial y}{\partial t}\right] = (s\log a).\bar{y}(x,\ s) - y(x,\ 0).$$

(II) Let $v = \left[\frac{\partial y}{\partial t}\right] = y_t$, then

$$A\left[\frac{\partial^2 y}{\partial t^2}\right] = \left[\frac{\partial v}{\partial t}\right]$$

$$A\left[\frac{\partial^2 y}{\partial t^2}\right] = (s\log a).\bar{v}(x,\ s) - v(x,\ 0) \text{ by (I)}$$

$$A\left[\frac{\partial^2 y}{\partial t^2}\right] = (s\log a).A[v] - y,\ (x,\ 0)$$

$$A\left[\frac{\partial^2 y}{\partial t^2}\right] = (s\log a).[(s\log a).\bar{y}(x,\ s) - y(x,\ 0)] - y_t(x,\ 0) \text{ by (I)}$$

$$\therefore\ A\left[\frac{\partial^2 y}{\partial t^2}\right] = (s\log a)^2\,\bar{y}(x,\ s) - (s\log a).y(x,\ 0) - y_t(x,\ 0).$$

(III) We have,

$$A\left[\frac{\partial y}{\partial x}\right] = \int_0^\infty a^{-st}\left(\frac{\partial y}{\partial x}\right)dy$$

$$A\left[\frac{\partial y}{\partial x}\right] = \frac{d}{dx}\int_0^\infty a^{-st}\,y\,dy$$

$$A\left[\frac{\partial y}{\partial x}\right] = \frac{d}{dx}\,A[y(x,\ t)]$$

$$\therefore\ A\left[\frac{\partial y}{\partial x}\right] = \frac{d\bar{y}}{dx}.$$

(IV) We have,

$$A\left[\frac{\partial^2 y}{\partial x^2}\right] = A\left[\frac{\partial u}{\partial x}\right], \text{ where } u = \frac{\partial y}{\partial x}$$

$$A\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d}{dx} A[u]$$

$$A\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d}{dx} A\left[\frac{\partial y}{\partial x}\right]$$

$$A\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d}{dx}\left(\frac{d\overline{y}}{dx}\right) \text{ by (III)}$$

$$\therefore\ A\left[\frac{\partial^2 y}{\partial x^2}\right] = \frac{d^2\overline{y}}{dx^2}.$$

### 3.4. Application to evaluating some definite integrals

**Example 3.4.1.** Consider the definite integral $\int_0^\infty t.e^{-3t} \sin t \cdot dt$

Let $I = \int_0^\infty e^{-3t}(t \sin t)dt$

Put, $3 = (s \log a)$

$$\therefore\ I = \int_0^\infty e^{-(s \log a)t}(t \sin t)dt$$

$$\therefore\ I = \int_0^\infty a^{-st}(t \sin t)dt$$

$$\therefore\ I = \int_0^\infty e^{-(s \log a)t}(t \sin t)dt$$

$$\therefore\ I = A[t. \sin t]$$

$$\therefore\ I = -\frac{1}{(\log a)} \frac{d}{ds}\left[\frac{1}{(s \log a)^2 + 1}\right]$$

$$I = -\frac{1}{(\log a)}\left[-\frac{1}{[(s\log a)^2+1]^2}\times(2s\log a)\times(\log a)\right]$$

$$I = -\frac{2(s\log a)}{[(s\log a)^2+1]^2}$$

$$I = \frac{2\times 3}{[9+1]^2}$$

$$I = \frac{6}{[10]^2}$$

$$\therefore\ I = \frac{3}{50}$$

$$\therefore \int_0^\infty t.e^{-3t}\sin t dt = \frac{3}{50}.$$

**3.5. Application to Partial Differential Equation (PDE)**

**Example 3.5.1.** Consider the PDE,

$$\frac{\partial y}{\partial x}-\frac{\partial y}{\partial t} = 1-e^{-t},\ 0 < x < 1,\ t > 0,\ \text{given that}\ \ y(x,\ 0) = x$$

By taking exponential transform, we get

$$\frac{d\bar{y}}{dx}-[(s\log a).\bar{y}-y(x,\ 0)] = \frac{1}{(s\log a)}-\frac{1}{(s\log a+1)}$$

$$\frac{d\bar{y}}{dx}-(s\log a).\bar{y}+x = \frac{1}{(s\log a)}-\frac{1}{(s\log a+1)}$$

$$\frac{d\bar{y}}{dx}-(s\log a).\bar{y} = \frac{1}{(s\log a)}-\frac{1}{(s\log a+1)}-x$$

$$\frac{d\bar{y}}{dx}-(s\log a).\bar{y} = \frac{1}{(s\log a)(s\log a+1)}-x \quad (3.5.1)$$

It is linear equation,

$$\therefore\ I.F. = e^{-\int(s\log a)dx}$$

The solution of (3.5.1) is given by,

$$\bar{y}e^{-(s\log a)x} = c + \int\left[\frac{1}{(s\log a)(s\log a+1)} - x\right]e^{-(s\log a)x}dx$$

$$\bar{y}e^{-(s\log a)x} = c + \int\left[\frac{1}{(s\log a)(s\log a+1)}\right]e^{-(s\log a)x}dx - \int xe^{-(s\log a)x}dx$$

$$\bar{y}e^{-(s\log a)x} = c + \frac{1}{(s\log a)(s\log a+1)}\left[\frac{e^{-(s\log a)x}}{-(s\log a)}\right]$$

$$-\left[\frac{xe^{-(s\log a)x}}{-(s\log a)} - \int\frac{e^{-(s\log a)x}}{-(s\log a)}dx\right]$$

$$\bar{y}e^{-(s\log a)x} = c + \frac{1}{(s\log a)(s\log a+1)}\left[\frac{e^{-(s\log a)x}}{-(s\log a)}\right]$$

$$-\left[\frac{xe^{-(s\log a)x}}{-(s\log a)} - \int\frac{e^{-(s\log a)x}}{-(s\log a)}dx\right]$$

$$\bar{y}e^{-(s\log a)x} = c + \frac{1}{(s\log a)(s\log a+1)}\left[\frac{e^{-(s\log a)x}}{-(s\log a)}\right]$$

$$+\frac{xe^{-(s\log a)x}}{(s\log a)} + \frac{e^{-(s\log a)x}}{(s\log a)^2}$$

$$\therefore\ \bar{y} = ce^{-(s\log a)x} - \frac{1}{(s\log a)^2(s\log a+1)} + \frac{x}{(s\log a)} + \frac{1}{(s\log a)^2}$$

$$\therefore\ \bar{y} = ce^{(s\log a)x} + \frac{x}{(s\log a)} + \frac{1}{(s\log a)^2(s\log a+1)}$$

$$\therefore\ \bar{y} = ce^{(s\log a)x} + \frac{x}{(s\log a)} + \frac{1}{(s\log a)} - \frac{1}{(s\log a+1)}$$

$y$ is bounded $\Rightarrow$ $\bar{y}$ is bounded $\Rightarrow$ $\bar{y}$ is finite as $x\to\infty \Rightarrow c = 0$

$$\therefore\ \bar{y} = \frac{x}{(s\log a)} + \frac{1}{(s\log a)} - \frac{1}{(s\log a+1)}$$

By taking Inverse exponential transform, we get

$$y = x + 1 - e^{-t}.$$

**3.6. Application to integral equations**

**Example 3.6.1.** Consider the Integral equation,

$$f(t) = 1 + \int_0^t f(u)\sin(t-u)du$$

The given equation is expressed as

$$f(t) = 1 + f(t) * \sin t$$

By taking exponential transform, we get

$$\bar{f}(s) = \frac{1}{(s\log a)} + \bar{f}(s). \frac{1}{(s\log a)^2 + 1}$$

$$\bar{f}(s) - \frac{1}{(s\log a)^2 + 1}\bar{f}(s) = \frac{1}{(s\log a)}$$

$$\left[1 - \frac{1}{(s\log a)^2 + 1}\right]\bar{f}(s) = \frac{1}{(s\log a)}$$

$$\bar{f}(s) = \frac{(s\log a^2) + 1}{(s\log a)(s\log a)^2}$$

$$\bar{f}(s) = \frac{1}{(s\log a)} + \frac{1}{(s\log a)^3}$$

By taking Inverse exponential transform, we get

$$A^{-1}[\bar{f}(s)] = A^{-1}\left[\frac{1}{(s\log a)}\right] + A^{-1}\left[\frac{1}{(s\log a)^3}\right]$$

$$\therefore\ f(t) = 1 + \frac{t^2}{2}.$$

**3.7. Application to Simultaneous Differential Equations**

**Example 3.7.1.** Consider the Simultaneous Differential Equations

$$\frac{dx}{dt} + y = 0, \frac{dy}{dt} - x = 0, \text{ if } x(0) = 1, y(0) = 0$$

Given equations can be written as,

$$x' + y = 0$$

$$y' - x = 0$$

By taking exponential transform, we get

$$[(s \log a)\bar{x} - x(0)] + \bar{y} = 0$$

$$[(s \log a)\bar{y} - y(0)] - \bar{x} = 0$$

Using conditions, we get

$$(s \log a)\bar{x} + \bar{y} = 1$$

$$(s \log a)\bar{y} + \bar{x} = 0$$

Solving above equations for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x} = \frac{(s \log a)}{(s \log a)^2 + 1}, \; \bar{y} \frac{1}{(s \log a)^2 + 1}$$

By taking Inverse exponential transform, we get

$$A^{-1}[\bar{x}] = A^{-1}\left[\frac{(s \log a)}{(s \log a)^2 + 1}\right], A^{-1}[\bar{y}] = A^{-1}\left[\frac{1}{(s \log a)^2 + 1}\right]$$

$$\therefore x = \cos t, \; y = \sin t.$$

## 4. Conclusion

In this work we have obtained some results of the exponential transform like exponential transform of partial derivatives, Initial value theorem and Final value theorem and discussed some applications of the exponential transform to evaluate definite integral, to finding solutions of partial differential equation, simultaneous differential equation, and integral equation.

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