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FIXED POINT THEOREMS IN CONVEX *G*-METRIC SPACES

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Abstract

The aim of this paper is to introduce convex structure G-metric spaces and extended Mann's iteration algorithm to these spaces. By using Mann's iteration scheme, a series of fixed point results and Mann's iteration algorithm was generalized. Also, strong convergence theorems for contraction mappings in convex G-metric spaces were developed. Furthermore, the problem of T-stability of the Mann's iteration procedure for the mappings in complete convex G-metric spaces was considered.

1. Introduction

The fixed point theorems, convex metric spaces and convex structure have been well discussed in literature (Alnafei et al. [1], Hadzic [5], Hamaizia [5], Saha et al. [12], Shimizu [14]). The properties of fixed point of the b-metric spaces and E-metric spaces have been established by different authors and references (Chen et al. [3], Goswami et al. [4], Haokip and Goswami [4], Mehmood et al. [8]). The T-stable is one of the important requirements for a fixed point iteration to be valuable and applicable from a numerical point of view (Rani and Jyoti [12]).

In Mustafa and Sims [11], the authors introduced a new concept of generalized metric space called G-metric spaces. After that, many authors have proven several fixed point results in these spaces (Modi and Bhatt [9], Mustafa et al. [10]). This paper introduced the concept of the convex G-metric space by the convex structure. Moreover, Mann's iteration algorithm was

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extended to G-metric space. Additionally, by means of Mann's iteration scheme, strong convergence theorems for two types of contraction mapping in convex G-metric spaces were obtained.

2. Preliminaries

In this section, definitions of general metric, properties and the other result that are needed in the sequel are explained.

Definition 1. [9.11]**.** Assume that X is a nonempty set, and let $G : X \times X \times X \to R^{+}$ be a function satisfying the following properties:

(i) $G(u, v, w) = 0$ if $u = v = w$;

(ii) $0 < G(u, u, v), \forall u, v \in X$ with $u \neq v$;

(iii) $G(u, u, v) \leq G(u, v, w), \forall u, v, w \in X$ with $u \neq w$;

(iv) $G(u, v, w) = G(v, u, w) = G(w, v, u) = \ldots$ (all permutations of u, v, w), (symmetry in three variables);

(v) $G(u, v, w) \leq G(u, a, a) + G(a, v, w) \forall u, v, w, a \in X$ (rectangle inequality). Then the function G is called a generalized metric or a G-metric on X and the pair (G, X) is called a G-metric space.

Definition 2. [9.11]**.** Let (G, X) be a G-metric and let $\{u_n\}$ be a sequence of points of X, a point $\{u_n\}$ is said to be limit of the sequence $\{u_n\}$ if $\lim_{n, m \to \infty} G(u, u_n, u_m) = 0$ and the sequence $\{u_n\}$ is said to be G-convergent to u. Thus, if $u_n \to u$ in a G-metric space (G, X) then for any $\varepsilon > 0$, there exists a positive integer N such that

$$G(u, u_n, u_m) < \varepsilon \forall n, m \geq N.$$

Definition 3. [9.11]**.** Let (G, X) be a G-metric space. A sequence $\{u_n\}$ in X is called G-Cauchy if for every $\varepsilon > 0$, there is a positive integer N such that $G(u_n, u_m, u_l) < \varepsilon, \forall n, m, l \geq N$, that is, if $G(u_n, u_m, u_l) \to 0$, as $n, m, l \to \infty$.

Definition 4. [9]. Let (G, X) be a G-metric space. A mapping $W : X \times X \times X \times (0, 1] \to X$ is said to be convex structure on (G, X) if for each $(u, v, w, \lambda) \in X \times X \times X \times (0, 1]$ and for all $x, y \in X$ the condition

$$G(x, y, W(u, v, w, \lambda)) \leq \frac{\lambda}{3}\{G(u, v, x) + G(u, v, y) + G(u, v, z)\}$$

holds. If W is convex structure on a G-metric space (G, X) then the triplet (X, G, W) is called a convex G-metric space.

Definition 5. [9.11]. A G-Metric space (G, X) is said to be G-complete (or complete G-metric space) if every Cauchy sequence in (G, X) is convergent in X.

Definition 6. [3]. Assuming that T is a self-map on a complete G-metric space (G, X). Let $u_{n+1} = f(T, u_n)$ be an iteration sequence that yields the sequence u_n of the points from X. So, the iteration procedure $u_{n+1} = f(T, u_n)$ is going to be weakly T-stable if $\{u_n\}$ converges to a fixed point u^* of T, and if $\{v_n\}$ is a sequence in X such that $\lim_{n\to\infty} G(v_{n+1}, f(Tv_n), a) = 0$ and sequence $\{G(v_n, Tv_n, a)\}$ a is bounded, then $\lim_{n\to\infty} v_n = u^*$.

Lemma 1. [3]. *Let* $\{k_n\}, \{l_n\}$ *be a non-negative sequence that satisfies* $k_{n+1} \leq hk_n + l_n \forall n \in N$, $0 \leq h < 1$, $\lim_{n\to\infty} l_n = 0$, *then* $\lim_{n\to\infty} k_n = 0$.

3. Main Results

In this section, we prove complete convex generalized metric version of Banach's contraction principle (Banach [2]) by means of Mann's iteration algorithm, complete convex generalized metric of Kannan type fixed point theorem (Chen et al. [3]), and the weak T-stability procedure respectively.

Theorem 1. *Suppose that* (X, G, W) *is a complete convex G-metric space and* $T : X \to X$ *is a contraction mapping; that is, there exists* $B \in [0, 1)$ *such that*

$$G(Tu, Tv, a) \leq BG(u, v, a) \forall u, v \in X.$$

Let us choose $u_0 \in X$ *in such a way that* $G(u_0, Tu_0, a) = M < \infty$ *and define* $u_n = W(u_{n-1}, Tu_{n-1}; \alpha_{n-1})$ *where* $0 \leq \alpha_{n-1} < 1$ *and* $n \in N$. *If* $B < 1$ *and* $0 < \alpha_{n-1} < \frac{1^{\frac{1}{4}-B}}{1-B}$ *for each* $n \in N$; *then T has a unique fixed point in X.*

Proof. For any $n \in N$, it is said that

$$G(u_n, u_{n+1}, a) = G(u_n W(u_n, Tu_n; \alpha_n), a) \leq (1-\alpha_n) G(u_n, Tu_n, a)$$

and

$$G(u_n, Tu_n, a) \leq G(u_n, Tu_n; Tu_{n-1}) + G(Tu_{n-1}, Tu_n, a)$$

$$\leq G(W(u_{n-1}, Tu_{n-1}; \alpha_{n-1}), Tu_{n-1}, Tu_{n-1}) + BG(u_{n-1}, u_n, a)$$

$$\leq \alpha_{n-1} G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + B(1-\alpha_{n-1}) G(u_{n-1}, Tu_{n-1}, a).$$

From definition (1) (iii) and (iv) we have

$$\leq (\alpha_{n-1} + B(1-\alpha_{n-1})) G(u_{n-1}, Tu_{n-1}, a)$$

$$\therefore G(u_n, Tu_n, a) \leq (\alpha_{n-1} + B(1-\alpha_{n-1})) G(u_{n-1}, Tu_{n-1}, a)$$

Let $\lambda_{n-1} = \alpha_{n-1} + B(1-\alpha_{n-1})$. By gathering this and the above inequality with the assumptions $B < 1$ and $0 < \alpha_{n-1} < \frac{1^{\frac{1}{4}-B}}{1-B} \forall n \in N$, we get

$$G(u_n, Tu_n, a) \leq \lambda_{n-1} G(u_{n-1}, Tu_{n-1}, a) < 1^{\frac{1}{4}-B} . BG(u_{n-1}, Tu_{n-1}, a). \quad (1)$$

This means that $\{G(u_n, Tu_n, a)\}$ is a decreasing sequence of non-negative reals. Therefore, $\exists \eta \geq 0$ Such that

$$\lim_{n \to \infty} G(u_n, Tu_n, a) = \eta.$$

We will show that $\eta = 0$. Suppose that $\eta > 0$. Letting $n \to \infty$ in inequality (1), we obtain

$$\eta \leq B.1^{\frac{1}{4}-B}\eta < \eta,$$

a contradiction. Hence we get $\eta = 0$. Furthermore, we have

$$G(u_n, u_{n+1}, a) \leq (1-\alpha_n) G(u_n, Tu_n, a) < G(u_n, Tu_n, a),$$

which shows that $\lim_{n\to\infty} G(u_n, u_{n+1}, a) = 0$. Now we will show that $\{u_n\}$ is a Cauchy sequence.

In fact, if $\{u_n\}$ is not a Cauchy sequence, so there exists $\varepsilon_0 > 0$ and the subsequences $\{u_{\theta(k)}\}$ and $\{u_{\tau(k)}\}$ of $\{u_n\}$, such that $\theta(k)$ is the smallest natural index with $\theta(k) > \tau(k) > k$,

$$G(u_{\theta(k)}, u_{\tau(k)}, a) \geq \varepsilon_0$$

and

$$G(u_{\theta(k)-1}, u_{\tau(k)}, a) < \varepsilon_0$$

Therefore, it is concluded that

$$\varepsilon_0 \leq G(u_{\theta(k)}, u_{\tau(k)}, a) \leq G(u_{\theta(k)}, v, v) + G(v, u_{\tau(k)}, a)$$

which implies that

$$\varepsilon_0 \leq \limsup_{k\to\infty} G(u_{\theta(k)}, u_{\tau(k)+1}, v).$$

Noticing that

$$G(u_{\theta(k)}, u_{\tau(k)+1}, a) = G(W(u_{\theta(k)-1}, Tu_{\theta(k)-1}; \alpha_{\theta(k)-1}), u_{\tau(k)+1}, a)$$

$$\leq \alpha_{\theta(k)-1} G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) + (1-\alpha_{\theta(k)-1}) G(Tu_{\theta(k)-1}, u_{\tau(k)+1}, a)$$

$$\leq \alpha_{\theta(k)-1} G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) + (1-\alpha_{\theta(k)-1}) \{G(Tu_{\theta(k)-1}, Tu_{\tau(k)+1},$$

$$Tu_{r(k)+1}) + G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, a)\}$$

$$\leq \alpha_{\theta(k)-1} G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1}) + (1-\alpha_{\theta(k)-1}) \{BG(u_{\theta(k)-1}, u_{\tau(k)+1} u_{\tau(k)+1})$$

$$+ G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, a)\}$$

$$\leq (\alpha_{\theta(k)-1} + (1-\alpha_{\theta(k)-1})B)G(u_{\theta(k)-1}, u_{\tau(k)+1}, u_{\tau(k)+1})$$

$$+ (1-\alpha_{\theta(k)-1})G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, a)$$

$$\leq (\alpha_{\theta(k)-1} + (1-\alpha_{\theta(k)-1})B)\{G(u_{\theta(k)-1}, Tu_{\tau(k)+1}, Tu_{\tau(k)+1})$$

$$+ G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, u_{\tau(k)+1})\} + (1-\alpha_{\theta(k)-1})G(Tu_{\tau(k)+1}, u_{\tau(k)+1}, a)$$

We obtain

$$\varepsilon_0 \leq \limsup_{k\to\infty} G(u_{\theta(k)}, u_{\tau(k)+1}, a) \leq \varepsilon_0,$$

which is a contradiction. Hence, $\{u_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists $u^* \in X$ Such that $\lim_{n\to\infty} G(u_n, u^*, a) = 0$.

Next, we will verify that u^* is a fixed points of T. Note that

$$G(u^*, Tu^*, a) \leq G(u^*, u_n, u_n) + G(u_n, Tu^*, a).$$

Letting $n \to \infty$ we assume that $G(u^*, Tu^*, a) = 0$ which implies that $Tu^* = u^*$. Hence, u^* is a fixed point of T. Now we explain that T has a unique fixed point. Suppose that $v \in X$ is another fixed point, that is $Tv = v$. Then,

$$G(u^*, v, a) = G(Tu^*, Tv, a) \leq BG(u^*, v, a).$$

For some $B \in [0, 1)$, a contradiction. Hence $u^* = v$, which completes the proof.

Theorem 2. *Suppose that (X, G, W) is a complete convex G-metric space, and the mapping $T : X \to X$ be defined as*

$$G(Tu, Tv, a) \leq k(G(u, Tu, a) + G(v, Tv, a)) \forall u, v \in X, \tag{2}$$

and for some $k \in \left[0, \frac{1}{2}\right)$. Let us choose $u_0 \in X$ in such a way that $G(u_0, Tu_0, a) = M < \infty$ and we define $u_n = W(u_{n-1}Tu_{n-1}; \alpha_{n-1})$ for $n \in N$ and $\alpha_{n-1} \in \left(0, \frac{1}{2}\right]$. If $k \in \left[0, \frac{1}{4}\right]$, then T has a unique fixed point in X.

Proof. For any $n \in N$, we have

$$G(u_n, u_{n+1}, a) = G(u_n W(u_n, Tu_n; a_n), a) \leq (1 - \alpha_n) G(u_n, Tu_n, a) \tag{3}$$

and

$$G(u_n, Tu_n, a) = G(W(u_{n-1}, Tu_{n-1}; a_{n-1}), Tu_n, a)$$

$$\leq \alpha_{n-1} G(u_{n-1}, Tu_n, a) + (1 - \alpha_{n-1}) G(Tu_{n-1}, Tu_n, a)$$

$$\leq \alpha_{n-1}(G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + G(Tu_{n-1}, Tu_n, a)) + G(Tu_{n-1}, Tu_n, a)$$

$$\leq \alpha_{n-1}(G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + (\alpha_{n-1} + 1) G(Tu_{n-1}, Tu_n, a)$$

$$\leq \alpha_{n-1} G(u_{n-1}, Tu_{n-1}, Tu_{n-1}) + k(\alpha_{n-1} + 1)(G(u_{n-1}, Tu_n, a)$$

$$+ G(u_n, Tu_n, a))$$

Let $Tu_{n-1} = a$

$$\leq (\alpha_{n-1} + k\alpha_{n-1} + k) G(u_{n-1}, Tu_{n-1}, a) + (k\alpha_{n-1} + k) G(u_n, Tu_n, a)$$

i.e.,

$$(1 - (k\alpha_{n-1} + k)) G(u_n, Tu_n, a) \leq (\alpha_{n-1} + k\alpha_{n-1} + k) G(u_{n-1}, Tu_{n-1}, a)$$

Since

$$k\alpha_{n-1} + k \leq \frac{5}{4} k < \frac{5}{16} < 1,$$

then

$$G(u_n, Tu_n, a) \leq \frac{\alpha_{n-1} + k\alpha_{n-1} + k}{1 - (k\alpha_{n-1} + k)} G(u_{n-1}, Tu_{n-1}, a). \tag{4}$$

Denote $\lambda_{n-1} = \dfrac{\alpha_{n-1} + k\alpha_{n-1} + k}{1 - (k\alpha_{n-1} + k)}$ for $n \in N$. We assume that

$$\lambda_{n-1} = \frac{\alpha_{n-1} + k\alpha_{n-1} + k}{1 - (k\alpha_{n-1} + k)} < \frac{\frac{5}{4}}{1 - (k\alpha_{n-1} + k)} < \frac{\frac{5}{4}}{1 - \frac{5}{16}} - 1 = \frac{9}{11}.$$

From the above inequality, as well as inequality (4) with the assumptions of the theorem, we get

$$G(u_n, Tu_n, a) \leq \lambda_{n-1} G(u_{n-1}, Tu_{n-1}, a) < \frac{9}{11} G(u_{n-1}, Tu_{n-1}, a), \tag{5}$$

which implies that $\{G(u_n, Tu_n, a)\}$ is a decreasing sequence of non-negative reals. Hence, there exists $\eta \geq 0$ Such that

$$\lim_{n \to \infty} G(u_n, Tu_n, a) = \eta.$$

We will show that $\eta = 0$. suppose that $\eta > 0$. Letting $n \to \infty$ in inequality (5), we obtain that $\eta \leq \frac{9}{11} \eta < \eta$, a contradiction. Hence we get that $\eta > 0$; i.e.,

$$\lim_{n \to \infty} G(u_n, Tu_n, a) = 0.$$

Moreover, by inequality (3) we obtain

$$G(u_n, u_{n+1}, a) \leq (1 - \alpha_n) G(u_n, Tu_n, a) < G(u_n, Tu_n, a),$$

which implies that $\lim_{n \to \infty} G(u_n, u_{n+1}, a) = 0$. Now it will be shown that $\{u_n\}$ is a Cauchy sequence. In fact, if $\{u_n\}$ is not a Cauchy sequence, then there exist $\varepsilon_0 > 0$ and the subsequences $\{u_{\theta(l)}\}$ and $\{u_{\tau(l)}\}$ of $\{u_n\}$ such that $\theta(l)$ is the smallest natural index with $\theta(l) > \tau(l) > l$,

$$G(u_{\theta(l)}, u_{\tau(l)}, a) \geq \varepsilon_0$$

and

$$G(u_{\theta(l)-1}, u_{\tau(l)}, a) < \varepsilon_0$$

Therefore, it is concluded that

$$\varepsilon_0 \leq G(u_{\theta(l)}, u_{\tau(l)}, a) \leq G(u_{\theta(l)}, u_{\tau(l)+1}, u_{\tau(l)+1}) + G(u_{\theta(l)+1}, u_{\tau(l)}, a),$$

which implies that

$$\varepsilon_0 \leq \limsup_{l \to \infty} G(u_{\theta(l)}, u_{\tau(l)+1}, a)$$

Noticing that

$G(u_{\theta(l)}, u_{\tau(l)+1}, a) = G(W(u_{\theta(l)-1}, Tu_{\theta(l)-1}; \alpha_{\theta(l)-1}), u_{\tau(l)+1}, a)$

$\leq \alpha_{\theta(l)-1} G(u_{\theta(l)-1}, u_{\tau(l)+1}, a) + (1-\alpha_{\theta(l)-1}) G(Tu_{\theta(l)-1}, u_{\tau(l)+1}, a)$

$\leq \alpha_{\theta(k)-1} G(u_{\theta(l)-1}, u_{\tau(l)+1}, a) + (1-\alpha_{\theta(l)-1})\{G(Tu_{\theta(l)-1}, Tu_{\tau(l)+1}, Tu_{\tau(l)+1})$

$+ G(Tu_{\tau(l)+1}, u_{\tau(l)+1}, a)\} \leq \alpha_{\theta(l)-1} G(u_{\theta(l)-1}, u_{\tau(l)+1}, a)$

$+ (1-\alpha_{\theta(l)-1})\{kG(u_{\theta(l)-1} Tu_{\theta(l)-1}, a) + kG(u_{\tau(l)+1}, Tu_{(l)+1} a)$

$+ G(Tu_{\tau(l)+1}, u_{\tau(l)+1} a)\}$

$\leq \alpha_{\theta(l)-1} G(u_{\theta(l)-1}, u_{\tau(k)+1}, a) + (1-\alpha_{\theta(l)-1})(kG(u_{\theta(l)-1}, Tu_{\theta(l)-1}, a)$

$+ (k+1) G(u_{\theta(l)+1}, Tu_{\theta(l)+1}, a))$ (for some $k \in \left[0, \frac{1}{2}\right)$ satisfying (2))

$\leq \alpha_{\theta(l)-1}\{G(u_{\theta(l)-1}, u_{\tau(l)}, u_{\tau(l)}) + G(u_{\tau(l)}, u_{\tau(l)+1}, a)\}$

$+ (1-\alpha_{\theta(l)-1})\{kG(u_{\theta(l)-1}, Tu_{\theta(l)-1}, a) + (k+1) G(u_{\tau(l)+1}, Tu_{\tau(l)+1}, a)\}$

We obtain

$$\limsup_{l \to \infty} G(u_{\theta(l)}, u_{\tau(l)+1}, a) \leq \frac{\varepsilon_0}{4} < \varepsilon_0$$

which is a contradiction. Thus $\{u_n\}$ is a Cauchy sequence in X. By the completeness of X, it follows that there exists $u^* \in X$. Such that

$$\lim_{l \to \infty} G(u_n, u^*, a) = 0.$$

Now we will show that u^* is a fixed point of T. Since

$G(u^*, Tu^*, a) \leq G(u^*, u_n, u_n) + G(u, Tu^*, a)$

$\leq G(u^*, u_n, u_n) + G(u_n, Tu_n, Tu_n) + G(Tu_n, Tu^*, a)$

$\leq G(u^*, u_n, u_n) + G(u_n, Tu_n, Tu_n) + k\{G(u_n, Tu_n, a) + G(u^*, Tu^*, a)\}$

We conclude that

$$(1-k) G(u^*, Tu^*, a) \leq G(u^*, u_n, u_n) + (1+k)\left(\frac{9}{11}\right)^n G(u_n, Tu_n, a)$$

(for some $Tu_n = a$)

Consequently, we get that $G(u^*, Tu^*, a) = 0,$ so u^* is a fixed point of T.

In order to illustrate the uniqueness of the fixed point, suppose that $q \in X, q \neq u^*$, is another fixed point of T. Then $Tq = q$. However,

$$0 < G(u^*, q, a) = G(Tu^*, Tq, a) \leq kG(u^*, Tu^*, a) + kG(q, Tq, a) = 0,$$

a contradiction. Hence $u^* = q$ which completes the proof.

Theorem 3. *Under the assumptions of theorem* 1, *if, additionally*, $\lim_{n\to\infty} \alpha_n = 0$, *then Mann's iteration is weakly T-stable.*

Proof. By virtue of Theorem 1, we assume that u^* is a unique fixed point of T in X Assuming that $\{v_n\}$ is a sequence in X which satisfies $\limsup_{l\to\infty} G(v_{n+1}W, (v_n, Tv_n; \alpha_n), a) = 0$ and $\{G(v_n, Tv_n, a)\}$ is bounded. We obtain

$$G(v_{n+1}, u^*, a) \leq G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n : \alpha_n))$$

$$+ G(W(v_n, Tv_n; \alpha_n), u^*, a)$$

$$\leq G(v_{n+1}, W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + G(W(v_n, Tv_n; \alpha_n), Tv_n, Tv_n)$$

$$+ G(Tv_n, u^*, a)$$

$$\leq G(v_{n+1}, W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + \alpha_n G(v_n, Tv_n, Tv_n)$$

$$+ BG(v_n, u^*, a).$$

Noticing that $B < 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} G(v_{n+1}W(v_n Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) = 0$ and $(G(v_n Tv_n, Tv_n)\}$ is bounded, and taking into account Lemma 1, we get that

$$\lim_{n\to\infty} G(v_n, u^*, a) = 0,$$

which completes the proof.

Theorem 4. *Under all the assumptions of Theorem* 2, *if*, $\lim_{n\to\infty} \alpha_n = 0$ *and if the positive real number k from Theorem* 2 *satisfies, as well as the condition*

$\frac{k}{1-k} < 1$, *then Mann's iteration is weakly T-stable.*

Proof. From Theorem 2, it follows that T has a unique fixed point of u^* in X. Assume that $\{v_n\}$ is a sequence in X which satisfies

$$\lim_{l\to\infty} G(v_{n+1}W(v_n, Tv_n; \alpha_n), a) = 0.$$

and $\{G(v_n, Tv_n, a)\}$ is bounded. We obtain

$$G(v_n, u^*, a) \le G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n))$$

$$+ G(W(v_n, Tv_n; \alpha_n), u^*, a)$$

$$\le G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + G(W(v_n, Tv_n; \alpha_n), Tv_n, Tv_n)$$

$$+ G(Tv_n, u^*, a)$$

$$\le G(v_{n+1}W(v_n, Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) + \alpha_n G(v_n, Tv_n, Tv_n) + G(Tv_n, u^*, a)$$

Noticing that $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} G(v_{n+1}W(v_n Tv_n; \alpha_n), W(v_n, Tv_n; \alpha_n)) = 0$ and $\{G(v_n, Tv_n, Tv_n)\}$ is bounded, and taking into account Lemma 1, we get that

$$\lim_{n\to\infty} G(v_n, u^*, a) = 0,$$

which completes the proof.

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